Some Consequences of Martin's Conjecture

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Standard Borel Spaces

Definition

A *standard Borel space* *is a Polish space* X *equipped with its* σ*-algebra of Borel subsets.*

Some Examples

- $\mathbb{R}$, $[0,1]$, $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$, ...
- If σ is a sentence of $L_{\omega_1\omega}$, then

$$\mathrm{Mod}(\sigma) = \{\mathcal{M} = \langle \mathbb{N}, \cdots \rangle \mid \mathcal{M} \models \sigma\}$$

  is a standard Borel space.

Theorem (Kuratowski)

There exists a *unique* *uncountable standard Borel space up to isomorphism.*

Borel Maps and Equivalence Relations

Definition

Let X, Y be standard Borel spaces.

- *Then the map $\varphi : X \to Y$ is **Borel** iff $\mathrm{graph}(\varphi)$ is a Borel subset of $X \times Y$.*
- *Equivalently, $\varphi : X \to Y$ is Borel iff $\varphi^{-1}(B)$ is a Borel set for each Borel set $B \subseteq Y$.*

Definition

*If X is a standard Borel space, then a **Borel equivalence relation** on X is an equivalence relation $E \subseteq X^2$ which is a Borel subset of X^2.*

Countable Borel equivalence relations

Definition

The Borel equivalence relation E on the standard Borel space X is said to be countable iff every E-class is countable.

Standard Example

Let G be a countable (discrete) group and let X be a standard Borel G-space. Then the corresponding orbit equivalence relation E_G^X is a countable Borel equivalence relation.

Theorem (Feldman-Moore)

If E is a countable Borel equivalence relation on the standard Borel space X, then there exists a countable group G and a Borel action of G on X such that $E = E_G^X$.

Borel reductions

Definition

Let E, F be Borel equivalence relations on the standard Borel spaces X, Y respectively.

- *$E \leq_B F$ iff there exists a Borel map $f : X \to Y$ such that*

$$x \, E \, y \iff f(x) \, F \, f(y).$$

  In this case, f is called a Borel reduction *from E to F.*

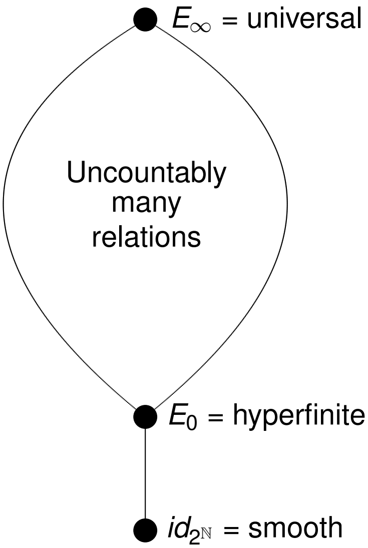
- *$E \sim_B F$ iff both $E \leq_B F$ and $F \leq_B E$.*
- *$E <_B F$ iff both $E \leq_B F$ and $E \nsim_B F$.*

Definition

More generally, $f : X \to Y$ is a Borel homomorphism *from E to F iff*

$$x \, E \, y \Longrightarrow f(x) \, F \, f(y).$$

Countable Borel equivalence relations



Definition

The Borel equivalence relation E is smooth iff $E \leq_B id_{2^{\mathbb{N}}}$.

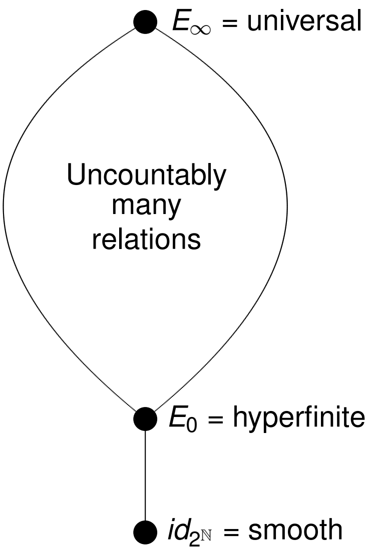
Definition

E_0 is the equivalence relation of eventual equality on $2^{\mathbb{N}}$.

Theorem (Adams-Kechris 2000)

There exist $2^{\aleph_0}$ many countable Borel equivalence relations up to Borel bireducibility.

Countable Borel equivalence relations



Definition

A countable Borel equivalence relation E is universal iff $F \leq_B E$ for every countable Borel equivalence relation F.

Theorem (JKL)

The orbit equivalence relation E_∞ of the shift action of the free group $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ is universal.

The measurable vs. Borel settings

Let G be a countable group and let X be a standard Borel G-space.

The Fundamental Question in the Borel setting

To what extent does the data $(\, X, E_G^X \,)$ "remember" the group G and its action on X?

Dirty Little Secret

We cannot possibly recover the group G from the data $(\, X, E_G^X \,)$ unless we add the hypotheses that:

- G acts freely on X; and
- there exists a G-invariant probability measure μ on X.

Essentially free relations

Definition

- *The countable Borel equivalence relation E on X is free iff there exists a countable group G with a free Borel action on X such that $E_G^X = E$.*
- *The countable Borel equivalence relation E is essentially free iff there exists a free countable Borel equivalence relation F such that $E \sim_B F$.*

Theorem (Thomas 2006)

The universal countable Borel equivalence relation E_∞ is not essentially free.

Strongly universal relations

Question (Thomas 2006)

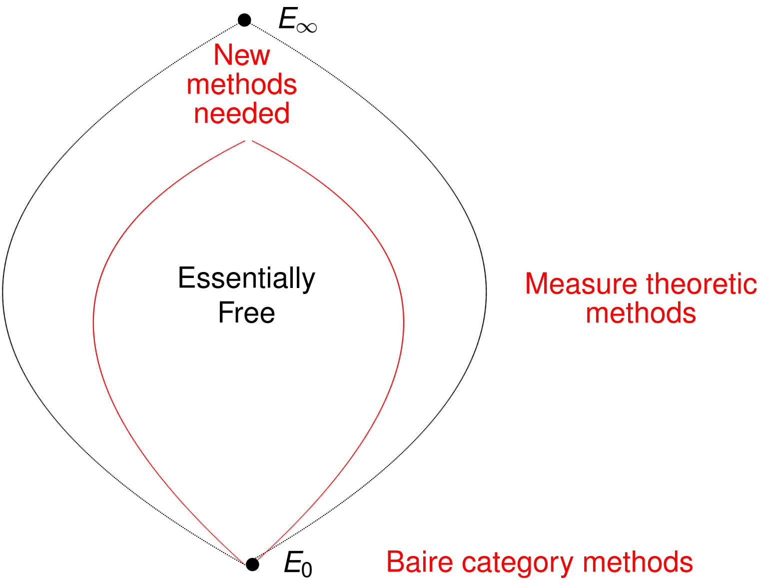
Does there exist a countable Borel equivalence relation E on a standard Borel space X such that:

- *there exists an E-invariant probability measure μ on X;*
- *whenever $Y \subseteq X$ is a Borel subset with $\mu(Y) = 1$, then $E \restriction Y$ is countable universal?*

Main Theorem (*MC*)

- *Let E be a countable Borel equivalence relation on the standard Borel space X and let μ be a (not necessarily E-invariant) Borel probability measure on X.*
- *Then there exists a Borel subset $Y \subseteq X$ with $\mu(Y) = 1$ such that $E \restriction Y$ is not universal.*

Countable Borel Equivalence Relations



Turing Reducibility

Convention

Throughout the powerset $\mathcal{P}(\mathbb{N})$ will be identified with $2^{\mathbb{N}}$ by identifying subsets of $\mathbb{N}$ with their characteristic functions.

Definition

If $x, y \in 2^{\mathbb{N}}$, then x is Turing reducible to y, written $x \leq_T y$, iff there exists a y-oracle Turing machine which computes x.

Remark

In other words, there is an algorithm which computes x modulo an oracle correctly answer questions of the form "Is $n \in y$?"

A Notion of Largeness

Definition

For each $z \in 2^{\mathbb{N}}$, the corresponding cone is $C_z = \{\, x \in 2^{\mathbb{N}} \mid z \leq_T x \,\}$.

- Suppose $z_n = \{\, a_{n,\ell} \mid \ell \in \mathbb{N} \,\} \in 2^{\mathbb{N}}$ for each $n \in \mathbb{N}$ and define

$$\oplus z_n = \{\, p_n^{a_{n,\ell}} \mid n, \ell \in \mathbb{N} \,\} \in 2^{\mathbb{N}},$$

  where p_n is the nth prime.

- Then $z_m \leq_T \oplus z_n$ for each $m \in \mathbb{N}$ and so $C_{\oplus z_n} \subseteq \bigcap_n C_{z_n}$.

Remark

It is well-known that if $C \subsetneq 2^{\mathbb{N}}$ is a proper cone, then C is both null and meager.

The Turing equivalence relation

Definition

The Turing equivalence relation $\equiv_T$ *on* $2^{\mathbb{N}}$ *is defined by*

$$x \equiv_T y \quad \textit{iff} \quad x \leq_T y \ \ \& \ \ y \leq_T x,$$

where $\leq_T$ *denotes Turing reducibility.*

Remark

- Clearly $\equiv_T$ is a countable Borel equivalence relation on $2^{\mathbb{N}}$.
- However, $\equiv_T$ is not essentially free and is not induced by the action of any countable subgroup of $\mathrm{Sym}(\mathbb{N})$.

Martin's Theorem

Theorem (Martin)

If $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then either X or $2^{\mathbb{N}} \setminus X$ contains a cone.

Remark

For later use, notice that if $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then the following are equivalent:

(i) X contains a cone.

(ii) For all $z \in 2^{\mathbb{N}}$, there exists $x \in X$ with $z \leq_T x$.

Ergodicity

Definition

Let G be a countable group and let X be a standard Borel G-space. Then the G-invariant probability measure μ is said to be ergodic *iff $\mu(A) = 0, 1$ for every G-invariant Borel subset $A \subseteq X$.*

Theorem

If μ is a G-invariant probability measure on the standard Borel G-space X, then the following statements are equivalent.

- *The action of G on (X, μ) is ergodic.*
- *If Y is a standard Borel space and $f : X \to Y$ is a G-invariant Borel function, then there exists a G-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $f \restriction M$ is a constant function.*

Ergodicity for Turing equivalence

Theorem (Folklore)

If $\varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel map, then there exists a cone C such that $\varphi \restriction C$ is a constant map.

Proof.

- For each $n \in \mathbb{N}$, there exists $\varepsilon_n \in \{0, 1\}$ such that $X_n = \{ x \in 2^{\mathbb{N}} \mid \varphi(x)(n) = \varepsilon_n \}$ contains a cone.
- Hence there exists a cone $C \subseteq \bigcap X_n$ and clearly $\varphi \restriction C$ is a constant map.

□

Proof of Martin's Theorem

- Suppose that $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset.
- Consider the two player Borel game $G(X)$

$$s(0) \quad s(1) \quad s(2) \quad s(3) \quad \cdots$$

  where I wins iff $s = (\, s(0)\, s(1)\, s(2) \cdots) \in X$.

- Then the Borel game $G(X)$ is determined. Suppose, for example, that $\sigma : 2^{<\mathbb{N}} \to 2$ is a winning strategy for I.
- Let $\sigma \leq_T t \in 2^{\mathbb{N}}$ and consider the run of $G(X)$ where
  - II plays $t = (\, s(1)\, s(3)\, s(5) \cdots)$
  - I responds with σ and plays $(\, s(0)\, s(2)\, s(4) \cdots)$.
- Then $s \in X$ and $s \equiv_T t$. Hence $t \in X$ and so $C_\sigma \subseteq X$.

Strong Ergodicity

Definition

- *Suppose that E, F are countable Borel equivalence relations on the standard Borel spaces X, Y and that μ is an E-invariant Borel probability measure on X.*
- *Then E is said to be F-ergodic iff for every Borel homomorphism $\varphi : X \to Y$ from E to F, there exists a Borel subset $Z \subseteq X$ with $\mu(Z) = 1$ such that φ maps Z into a single F-class.*

Example (Jones-Schmidt)

E_∞ is E_0-ergodic.

Strong Ergodicity for Turing equivalence

Definition

Let E be a countable Borel equivalence relation on the standard Borel space X. Then $\equiv_T$ is said to be E-m-ergodic iff for every Borel homomorphism $\varphi : 2^{\mathbb{N}} \to X$ from $\equiv_T$ to E, there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that φ maps C into a single E-class.

Target

Classify the countable Borel equivalence relations E such that $\equiv_T$ is E-m-ergodic.

Question

When is it "obvious" that $\equiv_T$ is not E-m-ergodic?

Weakly universal countable Borel equivalence relation

Definition

- *The Borel homomorphism $\varphi : X' \to X$ from E' to E is said to be a weak Borel reduction iff φ is countable-to-one. In this case, we write $E' \leq_B^w E$.*
- *A countable Borel equivalence relation E is said to be weakly universal iff $F \leq_B^w E$ for every countable Borel equivalence relation F.*

Some Examples

- If E is universal, then E is weakly universal.
- The Turing equivalence relation $\equiv_T$ is weakly universal.

Observation

If E is weakly universal, then $\equiv_T$ is not E-m-ergodic.

Strong Ergodicity for Turing equivalence

Strong Ergodicity Theorem (*MC*)

If E is any countable Borel equivalence relation, then exactly one of the following conditions holds:

(a) *E is weakly universal.*

(b) *$\equiv_T$ is E-m-ergodic.*

Remark

- There are currently no nonsmooth countable Borel equivalence relations E for which it has been proved that $\equiv_T$ is E-m-ergodic.
- In particular, it is not known whether $\equiv_T$ is E_0-m-ergodic, where E_0 denotes the eventual equality equivalence relation on $2^{\mathbb{N}}$.

The Kechris-Miller Theorem

Observation

Let E, F be countable Borel equivalence relations.

- *If $E \leq_B F$, then $E \leq_B^w F$.*
- *If $E \subseteq F$, then $E \leq_B^w F$.*

Theorem (Kechris-Miller)

If E, F are countable Borel equivalence relations on the uncountable standard Borel spaces X, Y respectively, then the following conditions are equivalent:

(i) $E \leq_B^w F$.

(ii) *There exists a countable Borel equivalence relation $S \subseteq F$ on Y such that $S \sim_B E$.*

The weak universality of Turing equivalence

Proposition (Kechris)

$\equiv_T$ is weakly universal.

Proof.

Identifying the free group $\mathbb{F}_2$ with a suitably chosen group of recursive permutations of $\mathbb{N}$, we have that $E_\infty \subseteq \equiv_T$. □

Remark

If $C = \{x \in 2^{\mathbb{N}} \mid z \leq_T x\}$ is a cone, then the map $y \mapsto y \oplus z$ is a weak Borel reduction from $\equiv_T$ to $\equiv_T \restriction C$ and hence $\equiv_T \restriction C$ is also weakly universal.

Martin's Conjecture

The Martin Conjecture (*MC*)

If $\varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ *is a Borel homomorphism from* $\equiv_T$ *to* $\equiv_T$*, then exactly one of the following conditions holds:*

(i) *There exists a cone* $C \subseteq 2^{\mathbb{N}}$ *such that* φ *maps* C *into a single* $\equiv_T$*-class.*

(ii) *There exists a cone* $C \subseteq 2^{\mathbb{N}}$ *such that* $x \leq_T \varphi(x)$ *for all* $x \in C$*.*

Theorem (Slaman-Steel)

Suppose that $\varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ *is a Borel homomorphism from* $\equiv_T$ *to* $\equiv_T$*. If* $\varphi(x) <_T x$ *on a cone, then there exists a cone* $C \subseteq 2^{\mathbb{N}}$ *such that* φ *maps* C *into a single* $\equiv_T$*-class.*

Some easy consequences of Martin's Conjecture

Theorem (*MC*)

If $\varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$, then exactly one of the following conditions holds:

(i) *There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that φ maps C into a single $\equiv_T$-class.*

(ii) *There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $\varphi \upharpoonright C$ is a weak Borel reduction from $\equiv_T \upharpoonright C$ to $\equiv_T$.*

Furthermore, in case (ii), if $D \subseteq 2^{\mathbb{N}}$ is any cone, then $[\,\varphi(D)\,]_{\equiv_T}$ contains a cone.

Some easy consequences of Martin's Conjecture

Corollary (*MC*)

- $\equiv_T <_B (\equiv_T \sqcup \equiv_T)$.
- *In particular, $\equiv_T$ is not countable universal.*

Corollary (*MC*)

If $A \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then $\equiv_T \restriction A$ is weakly universal iff A contains a cone.

Remark

There are currently no naturally occurring classes $D \subseteq 2^{\mathbb{N}}$ for which it is known that $\equiv_T \restriction D$ is not weakly universal.

Proof of the Strong Ergodicity Theorem (MC)

- Let E be any countable Borel equivalence relation.
- Since $E \leq_B^w \equiv_T$, we can suppose that $E \subseteq \equiv_T$.
- Suppose that $\varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to E and that φ does not map any cone to a single E-class.
- Then φ is also a Borel homomorphism from $\equiv_T$ to $\equiv_T$ and clearly φ does not map any cone to a single $\equiv_T$-class.
- Hence there exists a cone C such that $\varphi \restriction C$ is countable-to-one.
- Since $\equiv_T \restriction C$ is weakly universal and $(\equiv_T \restriction C) \leq_B^w E$, it follows that E is weakly universal.

Some applications of the Strong Ergodicity Theorem

Theorem (*MC*)

There exist uncountably many weakly universal countable Borel equivalence relations up to Borel bireducibility.

Definition

The countable group G is (weakly) action universal iff there exists a standard Borel G-space X such that E_G^X is (weakly) universal.

Theorem (*MC*)

If G is a countable group, then the following are equivalent.

(a) *G is weakly universal.*

(b) *The conjugacy relation on the space of subgroups of G is weakly universal.*

Borel Boundedness

Definition

If $c, d \in \mathbb{N}^{\mathbb{N}}$, then:

- *$c \leq^* d$ iff $c(n) \leq d(n)$ for all but finitely many $n \in \mathbb{N}$.*
- *$c =^* d$ iff both $c \leq^* d$ and $d \leq^* c$.*

Easy Observation

Suppose that E is a countable Borel equivalence relation on the standard Borel space X and that $\sigma : X \to \mathbb{N}^{\mathbb{N}}$ is any map. Then there exists a map $\psi : X/E \to \mathbb{N}^{\mathbb{N}}$ such that $\sigma(x) \leq^* \psi([x]_E)$ for all $x \in X$.

An application of Feldman-Moore

Lemma

Suppose that E is a countable Borel equivalence relation on the standard Borel space X and that $\sigma : X \to \mathbb{N}^{\mathbb{N}}$ is any map. Then there exists a Borel map $\psi : X \to \mathbb{N}^{\mathbb{N}}$ such that for all $x \in X$,

$$\sigma(y) \leq^* \psi(x) \quad \textit{for all } y \in [x]_E$$

Proof.

- By Feldman-Moore, we can realize E by a Borel action of a countable group $G = \{ \gamma_m \mid m \in \mathbb{N} \}$.
- Define $\psi(x)(n) = \max\{ \sigma(\gamma_m \cdot x)(n) \mid m \leq n \}$.

□

Borel Boundedness

Definition (Boykin-Jackson)

*The countable Borel equivalence relation E on the standard Borel space X is said to be **Borel-Bounded** iff for every Borel map $\theta : X \to \mathbb{N}^{\mathbb{N}}$, there exists a Borel homomorphism $\varphi : X \to \mathbb{N}^{\mathbb{N}}$ from E to $=^*$ such that $\theta(x) \leq^* \varphi(x)$ for all $x \in X$*

Theorem (Boykin-Jackson)

If E is hyperfinite, then E is Borel-Bounded.

Question (Boykin-Jackson)

Is Borel-Boundedness equivalent to hyperfiniteness?

Problem (Boykin-Jackson)

*Find an example of a countable Borel equivalence relation which is **not** Borel-Bounded.*

Solovay's Observation

Proposition

If (X, μ) *is a standard Borel probability space and* $\theta : X \to \mathbb{N}^{\mathbb{N}}$ *is a Borel map, then there exists a function* $h \in \mathbb{N}^{\mathbb{N}}$ *such that*

$$\mu(\{ x \in X \mid \theta(x) \leq^* h \}) = 1.$$

Proof.

For each $n \in \mathbb{N}$, there exists $h(n) \in \mathbb{N}$ such that

$$\mu(\{ x \in X \mid \theta(x)(n) > h(n) \}) \leq (1/2)^{n+1}.$$

By the Borel-Cantelli Lemma, we have that

$$\mu(\{ x \in X \mid \theta(x)(n) > h(n) \text{ for infinitely many } n \}) = 0.$$

□

An application of Martin's Conjecture

Theorem (*MC*)

The Turing equivalence relation $\equiv_T$ is not Borel-Bounded.

Corollary (*MC*)

If E is a weakly universal countable Borel equivalence relation, then E is not Borel-Bounded. In particular, E_∞ is not Borel-Bounded.

Proof.

By Boykin-Jackson, if E is Borel-Bounded and $F \leq_B^w E$, then F is also Borel-Bounded. □

Definition

Identifying each $r \in 2^{\mathbb{N}}$ *with the corresponding subset of* $\mathbb{N}$, *define the Borel map* $\theta : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ *by:*

- $\theta(r)$ *is the increasing enumeration of* $r \cap 2\mathbb{N}$, *if* $r \cap 2\mathbb{N}$ *is infinite;*
- $\theta(r)$ *is the zero function, otherwise.*

Observation

For each $h \in \mathbb{N}^{\mathbb{N}}$, *the* $\equiv_T$*-invariant Borel set*

$$D_h = \{ r \in 2^{\mathbb{N}} \mid (\exists s \in 2^{\mathbb{N}}) \; s \equiv_T r \text{ and } h < \theta(s) \}$$

contains a cone.

Proof of Theorem (*MC*)

- Suppose that $\varphi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $=^*$ such that $\theta(r) \leq^* \varphi(r)$ for all $r \in 2^{\mathbb{N}}$.
- Since $=^*$ is hyperfinite, it follows that $\equiv_T$ is $=^*$-*m*-ergodic.
- Hence there exists a cone C such that φ maps C into a single $=^*$-class; say, $[\, h\,]_{=^*}$.
- But then $C \cap D_h = \emptyset$, which is a contradiction.

Strongly universal relations

Question (Thomas 2006)

Does there exist a countable Borel equivalence relation E on a standard Borel space X such that:

- *there exists an ergodic E-invariant probability measure μ on X;*
- *whenever $Y \subseteq X$ is a Borel subset with $\mu(Y) = 1$, then $E \restriction Y$ is countable universal?*

Theorem (*MC*)

Let E be a countable Borel equivalence relation on the standard Borel space X and let μ be a (not necessarily E-invariant) Borel probability measure on X. Then there exists a Borel subset $Y \subseteq X$ with $\mu(Y) = 1$ such that $E \restriction Y$ is not weakly universal.

Proof of Theorem (*MC*)

- Let E be a countable Borel equivalence relation on the standard Borel space X and let μ be a Borel probability measure on X.
- Let $\theta : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the Borel map defined earlier.
- By the Feldman-Moore Theorem, there exists a Borel map $\psi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that if $r \equiv_T s$, then $\theta(s) \leq^* \psi(r)$.
- Let $\varphi : X \to 2^{\mathbb{N}}$ be a weak Borel reduction from E to $\equiv_T$ and let $\pi : X \to \mathbb{N}^{\mathbb{N}}$ be the Borel map defined by $\pi = \psi \circ \varphi$.
- Then there exists a function $h \in \mathbb{N}^{\mathbb{N}}$ such that the Borel set $Y = \{ x \in X \mid \pi(x) \leq^* h \}$ satisfies $\mu(Y) = 1$.
- Since the Borel set $Z = [\, \varphi(Y) \,]_{\equiv_T}$ satisfies $Z \cap D_h = \emptyset$, it follows that $\equiv_T \restriction Z$ is not weakly universal.
- Since $(E \restriction Y) \leq^w_B (\equiv_T \restriction Z)$, it follows that $E \restriction Y$ is not weakly universal.

Some Open Problems

Problem

Prove that $\equiv_T$ is E_0-m-ergodic.

Problem

- *Find a naturally occurring classes of degree $D \subseteq 2^{\mathbb{N}}$ such that $\equiv_T \restriction D$ is not weakly universal.*
- *For example, how about the classes of minimal degrees, hyperimmune-free degrees, ... ?*