Differential Geometry of Singular Spaces and Reduction of Symmetries Lecture 3

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Stratified subcartesian spaces

A stratification of a subcartesian space S is a partition of S by a locally finite family $\mathfrak{M}$ of locally closed connected submanifolds M, called strata of $\mathfrak{M}$, which satisfy the following

Frontier Condition. For $M, M' \in \mathfrak{M}$, if $M' \cap \overline{M} \neq \emptyset$, then either $M' = M$ or $M' \subset \overline{M} \backslash M$.

We showed that every subcartesian space S admits a partition $\mathfrak{O}$ by orbits of the family $\mathfrak{X}(S)$ of all vector fields on S, which we denote by $\mathfrak{O}$. It is of interest to see under what conditions this partition of S is a stratification.

The partition $\mathfrak{O}$ of a subcartesian space S by orbits of the family $\mathfrak{X}(S)$ of all vector fields on S satisfies Frontier Condition.

Proof. Let O and O' be orbits of $\mathfrak{X}(S)$. Suppose $x \in O' \cap \overline{O}$ with $O' \neq O$. We first show that $O' \subset \overline{O}$. Note that the orbit O is invariant under the family of one-parameter local groups of local diffeomorphisms of S generated by vector fields. Since, $x \in \overline{O}$, it follows that, for every vector field X on S, $\exp(tX)(x)$ is in $\overline{O}$ if it is defined. But, O' is the orbit of $\mathfrak{X}(S)$ through x. Hence, $O' \subset \overline{O}$.

Stratifications of S can be partially ordered by inclusion. If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are two stratifications of S, we say that $\mathfrak{M}_1$ is a refinement of $\mathfrak{M}_2$ and write $\mathfrak{M}_1 \geq \mathfrak{M}_2$, if, for every $M_1 \in \mathfrak{M}_1$, there exists $M_2 \in \mathfrak{M}_2$ such that $M_1 \subseteq M_2$. We say that $\mathfrak{M}$ is a minimal (coarsest) stratification of S if it is not a refinement of a different stratification of S. If S is a manifold, then the minimal stratification of S consists of a single manifold $M = S$.

If $(S, \mathfrak{M})$ is a stratified subcartesian space and N is a manifold, the product $S \times N$ is stratified by the family $\mathfrak{M}_{S\times N} = \{M \times N \mid M \in \mathfrak{M}\}$. If U is an open subset of a stratified space $(S, \mathfrak{M})$, we can consider a family $\mathfrak{M}_U = \{M \cap U \mid U \in \mathfrak{M}\}$. In general, $\mathfrak{M}_U$ need not be a stratification of U.

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A stratification $\mathfrak{M}$ of a subcartesian space S is locally trivial if, for every $M \in \mathfrak{M}$ and each $x \in M$,
(i) there exists an open neighbourhood U of x in S such that $\mathfrak{M}_U$ is a stratification of U,
(ii) there exists a subcartesian stratified space $(S', \mathfrak{M}')$ with a distinguished point $y \in S'$ such that the singleton $\{y\} \in \mathfrak{M}'$, and
(iii) there is an isomorphism $\varphi : (U, \mathfrak{M}_U) \to ((M \cap U) \times S', \mathfrak{M}'_{(M \cap U) \times S'})$ such that $\varphi(x) = (x, y)$.
Let $\mathfrak{M}$ be a stratification of a subcartesian space S.

Definition

We say that $\mathfrak{M}$ admits local extension of vector fields if, for each $M \in \mathfrak{M}$, for each vector field X_M on M and for each point $x \in M$, there exists a neighbourhood V of x in M, and a vector field X on S such that $X_{|V} = X_{M|V}$. In other words, the vector field X is an extension to S of the restriction of X_M to V.

Theorem

Every locally trivial stratification of a subcartesian space S admits local extensions of vector fields.

Proof. Let X_M be a vector field on $M \in \mathfrak{M}$. Since M is locally trivial, given $x_0 \in M$, there exists a neighbourhood U of x_0 in M, a stratified differential space $(S', \mathfrak{M}')$ with a distinguished point $y \in S'$ such that the singleton $\{y_0\} \in \mathfrak{M}'$, and an isomorphism $\varphi : U \longrightarrow (M \cap U) \times S'$of stratified subcartesian spaces such that $\varphi(x_0) = (x_0, y_0)$.
Let $\exp(tX_M)$ be the local one-parameter group of local diffeomorphisms of M generated by X_M, and let $X_{(M\cap U)\times S'}$ be a derivation of $\mathcal{C}^\infty((M \cap U) \times S')$ defined by

$$(X_{(M\cap U)\times S'}h)(x, y) = \frac{d}{dt}h(\exp(tX_M)(x), y)_{|t=0},$$

for every $h \in \mathcal{C}^\infty((M \cap U) \times S')$ and each $(x, y) \in (M \cap U) \times S'$. Since $X_{(M\cap U)\times S'}$ is defined in terms of a local one-parameter group $(x, y) \mapsto (\exp(tX_M)(x), y)$ of diffeomorphisms, it is a vector field on $(M \cap U) \times S'$.

We can use the inverse of the diffeomorphism $\varphi : U \to (M \cap U) \times S'$ to push-forward $X_{(M\cap U)\times S'}$ to a vector field $X_U = (\varphi^{-1})_* X_{(M\cap U)\times S'}$ on U. Choose a function $f_0 \in \mathcal{C}^\infty(S)$ with support in U and such that $f(x) = 1$ for x in some neighbourhood U_0 of x_0 contained in U. Let X be a derivation of $\mathcal{C}^\infty(S)$ extending $f_0 X_U$ by zero outside U. In other words, for every $f \in \mathcal{C}^\infty(S)$, if $x \in U$, then $(Xf)(x) = f_0(x)(X_U f)(x)$, and if $x \notin U_0$, then $(Xf)(x) = 0$. Clearly, X is a vector field on S extending the restriction of X_M to $M \cap U_0$. $\square$

Theorem

Let $\mathfrak{M}$ be a stratification of a subcartesian space S admitting local extensions of vector fields. The partition $\mathfrak{O}$ of S by orbits of the family $\mathfrak{X}(S)$ of all vector fields on S is a stratification of S, and $\mathfrak{M}$ is a refinement of $\mathfrak{O}$. Moreover, if $\mathfrak{M}$ is minimal, then $\mathfrak{M} = \mathfrak{O}$.

Proof. Let $\mathfrak{M}$ be a stratification of S admitting local extensions of vector fields. Since every vector field X_M on a manifold $M \in \mathfrak{M}$ extends locally to a vector field on S and M is connected, it follows that M is contained in an orbit $O \in \mathfrak{O}$.

Every orbit $O \in \mathfrak{O}$ is a union of strata of $\mathfrak{M}$. Since $\mathfrak{M}$ is locally finite, for each $x \in O$, there exists a neigbourhood V of x in S which intersects only a finite number of strata $M_1, ..., M_k$ of $\mathfrak{M}$. Hence, V intersects only a finite number of orbits in $\mathfrak{O}$. Moreover, since strata of $\mathfrak{M}$ form a partition of S, it follows that $V = \bigcup_{i=1}^{k} M_i \cap V$.

Consider $x \in M_1$. Since M_1 is locally closed there exists a neighbourhood U of x contained in V, and such that $M_1 \cap U$ is closed in U. We can relabel the manifolds $M_1, ..., M_k$ so that $O \cap U = \bigcup_{i=1}^{l} M_i \cap U$ for some $l \leq k$. Without loss of generality we may assume that $x \in \overline{M}_i$ for each $i = 2, ..., l$. We want to see if $O \cap U$ is closed in U.

Suppose we have a sequence (y_k) in $O \cap U$ convergent to $y \in U$. Since $O \cap U$ is a finite union of disjoined manifolds, there must be a subsequence of (y_k) contained in one of them. Without loss of generality we may assume that each $y_k \in M_i$ for some $i = 1, ..., l$. We want to show that the limit $y = \lim_{k\to\infty} y_k \in O \cap U$. If $y \in M_i$, then $y \in M_i \cap U \subseteq O \cap U$. If $y \in \overline{M}_i \backslash M_i$, then $y \in M_j$ for some $j = 1, ..., k$. By assumption, $y \in U$ and U intersects only the strata that have x in their closure. If $M_j \subseteq O$ then $y \in O \cap U$. Therefore, $y \notin O \cap U$ implies that M_j is not contained in O. By a construction in the proof of Sussmann's Theorem, $\exp_x \mathbf{X}(W)$ is an m dimensional locally closed submanifold of S. Let U_0 be an open neighbourhood of x in U such that $U_0 \cap \exp_x \mathbf{X}(W)$ is closed in U_0.

As before, we consider a sequence (y_k) in $M_i \cap U_0 \cap \exp_x \mathbf{X}(W) \subseteq O \cap U_0$, which converges to $y \in M_j \cap U_0$. Since $M_j \not\subseteq O$, it follows that $y \notin U_0 \cap \exp_x \mathbf{X}(W) \subseteq U_0 \cap O$. This contradicts the fact that $U_0 \cap \exp_x \mathbf{X}(W)$ is closed in U_0 Therefore, $O \cap U$ is closed in U. Since x is an arbitrary point of the orbit O, it follows that O is locally closed.

We have shown that the partition $\mathfrak{O}$ of S by orbits of the family $X(S)$ of all vector fields on S is locally finite and that each orbit in $\mathfrak{O}$ is locally closed. Also, we showed earlier that that $\mathfrak{O}$ is a stratification of S. By construction, every stratum of the original stratification $\mathfrak{M}$ is contained in a stratum of $\mathfrak{O}$. This implies that $\mathfrak{M} \geq \mathfrak{O}$. If $\mathfrak{M}$ is minimal, then $\mathfrak{M} = \mathfrak{O}$.
□

Theorem

The space P/G of orbits of a proper action of a Lie group G on a manifold P is a minimally stratified space that admits local extensions of vector fields.

Proof. Minimal stratification (Bierstone). Local extension of vector fields (Lusala - Śniatycki).

Symplectic reduction

- A symplectic form on a manifold P is a closed and non-degenerate 2-form on P. Non-degeneracy of ω implies that for every $f \in \mathcal{C}^{\infty}(P)$, there exists a unique vector field X_f such that

$$X_f \lrcorner \omega = -df,$$

called the Hamiltonian vector field of f.

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- Let G be a connected Lie group, and

$$\Phi : G \times P \to P : (g, p) \mapsto \Phi_g(p) = gp$$

  be an symplectic action of G on P with an Ad^*-equivariant momentum map $J : P \to \mathfrak{g}^*$. For each $\xi \in \mathfrak{g}$, the action on P of the one-parameter subgroup $\exp t\xi$ of G is given by translations along the integral curves of X_{J_ξ}, where $J_\xi = \langle J \mid \xi \rangle$. is the momentum corresponding to ξ.

Poisson algebra

- The assignment $f \mapsto X_f$ gives a linear map of the space $\mathcal{C}^\infty(P)$ of smooth functions on P into the space $\mathfrak{X}(P)$ of smooth vector fields on P. If P is connected, the kernel of this map consists of constant functions on P.

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- The symplectic form ω on P induces a bracket on $\mathcal{C}^\infty(P)$, called the Poisson bracket, such that for each $f_1, f_2 \in \mathcal{C}^\infty(P)$,

$$\{f_1, f_2\} = -X_{f_1} f_2 = X_{f_2} f_1 = -\omega(X_{f_1}, X_{f_2}).$$

  The Poisson bracket is bilinear, antisymmetric, acts as a derivation, and satisfies the Jacobi identity.

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- The action Φ of G on (P, ω) gives rise to the action $G \times \mathcal{C}^\infty(P) \to \mathcal{C}^\infty(P) : (g, f) \mapsto \Phi_g^* f$. Since the action of G on P is symplectic, it follows that its action on $\mathcal{C}^\infty(P)$ is Poisson. That is, it preserves the Poisson bracket. For each $g \in G$ and $f_1, f_2 \in \mathcal{C}^\infty(P)$, $\Phi_g^*\{f_1, f_2\} = \{\Phi_g^* f_1, \Phi_g^* f_2\}$.

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- The space $\mathcal{C}^\infty(P)^G$ of G-invariant smooth functions on P is a Poisson subalgebra of P.

Poisson reduction

- We assume here that the action of G on P is proper. We denote by $R = P/G$ the space of G-orbits on P and by $\rho : P \longrightarrow R$ the orbit map. The differential structure of R is

$$\mathcal{C}^\infty(R) = \{f : R \longrightarrow \mathbb{R} \mid \rho^* f \in \mathcal{C}^\infty(P)\}.$$

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- Given $f \in \mathcal{C}^\infty(R)$, let $X_f \in \mathrm{Der}\, \mathcal{C}^\infty(R)$ be defined by $X_f(h) = \{h, f\}$ for each $h \in \mathcal{C}^\infty(R)$. We refer to X_f as the Poisson derivation of f.

Poisson reduction

- We assume here that the action of G on P is proper. We denote by $R = P/G$ the space of G-orbits on P and by $\rho : P \to R$ the orbit map. The differential structure of R is

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- We denote by $\mathfrak{P}(R)$ the family of all Poisson derivations of $C^\infty(R)$.

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- Orbits of $\mathfrak{P}(R)$ are smooth manifolds immersed in strata of R.
- Let Q be the orbit of $\mathfrak{P}(R)$ through $x \in R$. For each $f \in \mathcal{C}^\infty(R)$, the restriction $X_{f|Q}$ of the Poisson vector field of f to Q is a vector field on Q, and $TQ = \{X_f(x) \mid x \in Q,\ f \in \mathcal{C}^\infty(R)\}$.

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Theorem

Each orbit Q of the family $\mathfrak{P}(R)$ of Poisson vector fields on R is a symplectic manifold with a unique symplectic form ω_Q on Q such that

$$\omega_Q(X_{f_1|Q}, X_{f_2|Q}) = -\{f_1, f_2\}_{|Q}$$

for every $f_1, f_2 \in \mathcal{C}^\infty(R)$. Moreover, for each $p \in \rho^{-1}(Q)$,

$$\omega_Q(X_{f_1|Q}, X_{f_2|Q})(\rho(p)) = \omega(X_{\rho^* f_1}, X_{\rho^* f_2})(p).$$

Level sets of the momentum map

Consider the family

$$\mathfrak{E}(P) = \{X_f \mid f \in \mathcal{C}^\infty(P)^G\}$$

of Hamiltonian vector fields of G-invariant smooth functions on P.

Theorem

For each $p \in P$, the orbit through p of the family $\mathfrak{E}(P)$ is contained in the set $P_{G_p} = \{x \in P \mid G_x = G_p\}$, where G_p is the isotropy group of p.

Proof. For $f \in \mathcal{C}^\infty(P)^G$, let $\exp tX_f$ denote the local one-parameter group of local diffeomorphisms generated by the Hamiltonian vector field X_f of f. The G-invariance of X implies that for each $g \in G$, $\Phi_g \circ \exp tX_f = (\exp tX_f) \circ \Phi_g$. Let $x = (\exp tX_f)(p)$, and $g \in G_x$. Then $x = \Phi_g x$ implies $(\exp tX_f)(p) = (\Phi_g \circ (\exp tX_f))(p)$. Hence,

$$p = ((\exp tX_f)^{-1} \circ \Phi_g \circ (\exp tX_f))(p) = ((\exp tX_f)^{-1} \circ (\exp tX_f) \circ \Phi_g)(p) =$$

and $g \in G_p$. Thus $G_x \subseteq G_p$. In a similar way, we can show that $G_p \subseteq G_x$. Hence, $G_x = G_p$, which ensures that the orbit of X_f through p is

Theorem

Assume that the action of G on P is proper. Then, for each $p \in P$,

$$\mathfrak{E}(P)_p = \ker_p dJ \cap T_p P_{G_p},$$

and the orbit of $\mathfrak{E}(P)$ through p is the connected component of $J^{-1}(J(p))) \cap P_{G_p}$ that contains p.
*(**ii**) For each compact subgroup H of G, connected components of P_H are symplectic manifolds.*
*(**iii**) If particular, if $p \in P_H$, $\mu = J(p)$ and L is the connected component of P_H that contains p, then the connected component of $J^{-1}(\mu) \cap L$ that contains p is a manifold and its tangent bundle is spanned by Hamiltonian vector fields of G-invariant functions.*

This theorem, due to Ortega and Ratiu, is at the foundation of their theory of optimal reduction.

We showed above orbits of the family $\mathfrak{P}(R)$ of Poisson vector fields on R are symplectic manifolds. In the theorem below, we show that they are projections to R of connected components of level sets of J with submanifolds of P with a fixed isotropy group.

Theorem

Assume that the action of a connected Lie group G on a symplectic manifold (P, ω) is Hamiltonian and proper. Given $p_0 \in P$, let $\mu = J(p_0)$ and $H = G_{p_0}$ be the isotropy group of p_0. The connected component K of $J^{-1}(\mu) \cap P_H$ is a submanifold of P, and the projection $Q = \rho(K)$ coincides with the orbit of $\mathfrak{P}(R)$ through $\rho(p_0)$. In particular, the symplectic form ω_Q satisfies the condition

$$\rho_K^* \omega_Q = \omega_K,$$

where $\rho_K : K \to Q$ is the restriction of the orbit map $\rho : P \to R$ to domain K and codomain Q, and ω_K is the pull-back of ω by the inclusion map $K \hookrightarrow P$.

Review of the stratification structure.

The proper action of G on P defines the orbit type stratifcation $\mathfrak{M}$ of P, whose strata are connected components of local manifolds

$$P_H = \{p \in P \mid G_p \text{ is conjugate to } H\},$$

where H is a compact subgroup of G. Note that this stratification is not minimal.

The orbit type stratification $\mathfrak{N}$ of $R = P/G$ is the projection to R of the stratification $\mathfrak{M}$ of P by the orbit map ρ. For each stratum $M \in \mathfrak{M}$, the projection $N = \rho(M)$ is a stratum of $\mathfrak{N}$. The stratification N of R coincides with the partition $\mathfrak{O}$ of R by orbits of the family $\mathfrak{X}(R)$ of all vector fields on R.

Theorem

We assume that the action of G on P is proper and denote by $\mathfrak{M}$ and $\mathfrak{N}$ orbit type stratifications of P and $R = P/G$, respectively.
*(**i**) For each $\mu \in \mathfrak{g}^*$, the family of sets*

$$\mathfrak{M}_\mu = \{\textit{connected components of } J^{-1}(\mu) \cap M \mid M \in \mathfrak{M}\}$$

is a stratification of the level set $J^{-1}(\mu)$. The inclusion map $J^{-1}(\mu) \hookrightarrow P$ is a morphism of stratified spaces.
*(**ii**) Connected components of the sets $\rho(J^{-1}(\mu) \cap M) = \rho(J^{-1}(\mu)) \cap N$, where $N = \rho(M)$, are symplectic orbits of the family $\mathfrak{P}(R)$ of Poisson vector fields on R.*
*(**iii**) The family of sets*

$$\mathfrak{N}_\mu = \{\textit{connected components of } \rho(J^{-1}(\mu)) \cap N \mid N \in \mathfrak{N}\}$$

is a stratification of $\rho(J^{-1}(\mu))$ with symplectic strata. The restriction $\rho_{|J^{-1}(\mu)}$ of ρ to $J^{-1}(\mu)$ is a morphism of stratified spaces.

Up to now we considered the differential structure of $J^{-1}(\mu)$ given by its inclusion in P and the structure of $\rho(J^{-1}(\mu))$ embedded in the orbit space $R = P/G$.

In Hamiltonian mechanics we often perform the reduction procedure by investigating first the structure of the quotient $J^{-1}(\mu)/G_\mu$, where

$$G_\mu = \{g \in G \mid Ad_g^* \mu = \mu\}$$

is the isotropy group of μ. Since $J : P \to \mathfrak{g}^*$ is continuous, it follows that $J^{-1}(\mu)$ is a closed subset of P. The local compactness of P implies that $J^{-1}(\mu)$ is locally compact. Moreover, the action G_μ on $J^{-1}(\mu)$ is proper because the action of G on P is proper. The general results on quotient spaces prove only that $J^{-1}(\mu)/G_\mu$ is a locally compact differential space with the quotient space topology and the differential structure

$$\mathcal{C}^\infty(J^{-1}(\mu)/G_\mu) = \{f \in C^0(J^{-1}(\mu)/G_\mu) \mid \rho_\mu^* f \in \mathcal{C}^\infty(J^{-1}(\mu))\},$$

where

$$\rho_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$$

is the orbit map. However, we can prove much more.

Theorem

There exists a diffeomorphism $I : J^{-1}(\mu)/G_\mu \rightarrow \rho(J^{-1}(\mu))$ *such that the following diagram*

$$
\begin{array}{ccccc}
J^{-1}(\mu) & & \stackrel{i}{\longrightarrow} & & P \\
\rho_\mu \ \downarrow & & & & \downarrow \ \rho \\
J^{-1}(\mu)/G_\mu & & & & R \\
& I \searrow & & \nearrow j & \\
& & \rho(J^{-1}(\mu)) & &
\end{array},
$$

where $i : J^{-1}(\mu) \longrightarrow P$ *and* $j : \rho(J^{-1}(\mu)) \longrightarrow R$ *denote the inclusion maps, commutes.*

Theorem

The stratification of $\rho(J^{-1}(\mu))$ gives rise to a stratification of $J^{-1}(\mu)/G_\mu$ such that the orbit map $\rho_{\mu:} : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$ is a morphism of stratified spaces.

This gives the flavour of type of results we can obtain in sigular reduction. We could continue with the desciption of reduction of coadjoint orbits, but we would not contribute anything new.