Review talk about nonholonomic dynamics

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FOCUS PROGRAM ON GEOMETRY, MECHANICS AND DYNAMICS
the Legacy of Jerry Marsden
at The Fields Institute, Toronto

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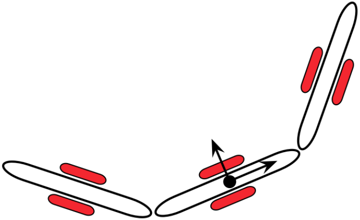
The birth of the theory of dynamics of nonholonomic systems occurred at the time when the universal and brilliant analytical formalism created by Euler and Lagrange was found, to general amazement, to be inapplicable to the very simple mechanical problems of rigid bodies rolling without slipping on a plane. Lindelöf's error, detected by Chaplygin, became famous and rolling systems attracted the attention of many eminent scientist of the time...

Neimark and Fufaev, 1972



What is nonholonomic mechanics?

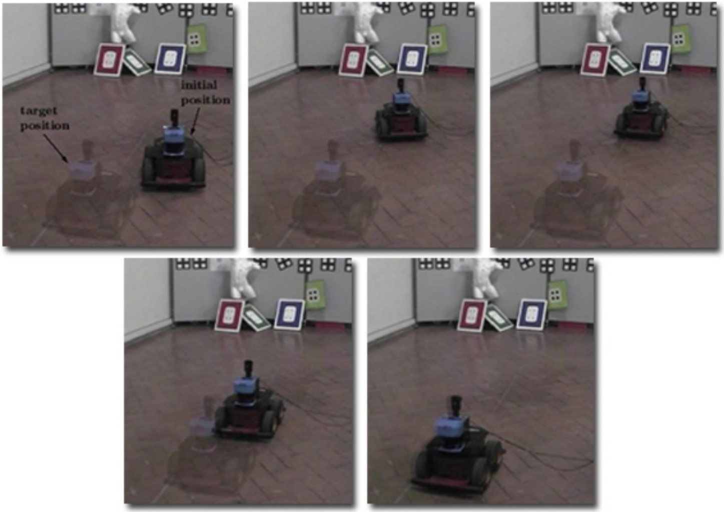
Nonholonomic constraints

(q^A), $1 \leqslant A \leqslant n$ coordinates on a configuration space Q
Constraints $\phi_i(q^A, \dot{q}^A, t) = 0$, $1 \leqslant i \leqslant m$

$$\begin{cases} \text{time-dependent (rheonomic)} \\ \underline{\text{time-independent}} \text{ (scleronomic)} \end{cases}$$

Holonomic or Nonholonomic

1. **Holonomic**. A holonomic constraint is derived from a constraint in configuration space.
   - **Example**: particle constrained to move on a sphere has the constraint $\sum_{A=1}^{n}(q^A)^2 = r^2, \quad q \cdot \dot{q} = 0.$
2. **Semi-Holonomic**. The constraint is integrable. These constraints typically imply conservation laws given by a foliation of Q by integral manifolds.
   - **Example**: vertical disk rolling on a straight line without slipping. Constraint $\dot{\phi} = \dot{x}$, implies $\phi = x + \text{constant}$.
3. **Nonholonomic**. The constraint is not integrable. Cannot be reduced to semi-holonomic constraints and does not impose restrictions on the configuration space.
   - **Example**: Nonholonomic particle $\phi = \dot{z} - y\dot{x} = 0$.


initial
position
target
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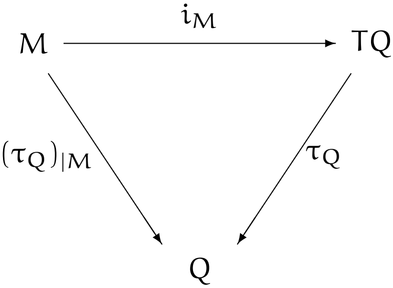
Geometrizing...

The constraints are globally described by a submanifold M of the velocity phase space TQ.



$$\begin{cases} \textbf{Linear constraints} : M \text{ is a vector subbundle of } TQ & \longrightarrow \mathcal{D} \\ \textbf{Affine constraints} : M \text{ is an affine subbundle of } TQ & \end{cases}$$

Lagrange-d'Alembert principle

J. L. Lagrange (1736–1813)

Jean Le Rond d'Alembert (1717–1783)

$\phi^i = \mu^i_A(q)\dot{q}^A, 1 \leqslant i \leqslant m$

Admissible infinitesimal virtual variation $\delta q^A \longrightarrow \mu^i_A \delta q^A = 0.$

Definition

A nonholonomic constraint is said to be ideal if the infinitesimal work of the constraint force vanishes for any admissible infinitesimal virtual displacement.

Lagrange-D'Alembert's equations

$$L : TQ \to \mathbb{R}$$

$(Q, L, \mathcal{D})$ a nonholonomic mechanical system

Lagrange-D'Alembert's equations

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^A} - \frac{\partial L}{\partial q^A} \right) \delta q^A = 0$$

with $\delta q \in \mathcal{D}$.

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^A} - \frac{\partial L}{\partial q^A} &= \lambda_i \mu^i_A \\ \mu_{iA}(q) \dot{q}^A &= 0 \end{aligned}$$

Nonlinear constraints. Chetaev's principle

P. Appell (1855–1930)

N.G. Chetaev (1902–1959)

P. Appell: Sur les liaisions exprimées par des realtions non linéaires entre les vitesses, *C. R. Acad. Sci. Paris* **152 (1911), 1197–1200.**

P. Appell: Exemple de mouvement d'un point assujetti à une liasion exprimée par une relation non linéaire entre les composantes de la vitesse, ***Rend. Circ. mat. Palermo*** **32 (1911), 48–50.**

$$\phi^i(t, q, \dot{q}) = 0, \quad 1 \leqslant i \leqslant m$$

Some physical properties of the constraints should impose restrictions to the set of possible values of the constraint forces.

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} - \frac{\partial L}{\partial q^A} &= \lambda_i \frac{\partial \phi^i}{\partial \dot{q}^A} \\ \phi^i(t, q, \dot{q}) &= 0 \end{aligned}$$

Simple examples show that Chetaev's rule cannot be used in general.

C.-M. Marle: **Various approaches to nonholonomic systems.** *Rep. Math. Phys.* **42 No 1/2 (1998), 211–229.**

Cendra, Hernán; Ibort, Alberto; de León, Manuel; Martín de Diego, David **A generalization of Chetaev's principle for a class of higher order nonholonomic constraints.** *J. Math. Phys.* **45 (2004), no. 7, 2785–2801.**

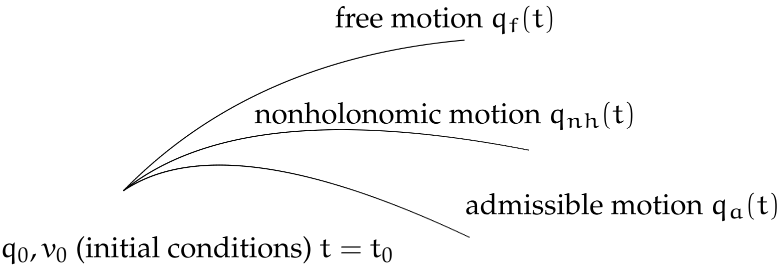
Cendra, H.; Grillo, S.: **Generalized nonholonomic mechanics, servomechanisms and related brackets.** *J. Math. Phys.* **47 (2006), no. 2, 022902, 29 pp.**

The principle of least constraint

"The motion of a system of particles connected together in any way, and whose motions are subject to arbitrary external restrictions , always takes place in the most complete agreement possible with free motion or under the weakest possible constraint. The measure of the constraint applied to the system at each elementary interval of time is the sum of products of the mass of each particle with the square of its departure from the free motion"

K. F. Gauss (1777–1855)

K. F. Gauss
Über ein neues Grundgesetz der Mechanik,
Journal de Crelle, Vol. IV (1829)



$$Z(\ddot{q}_a(t_0)) = \frac{1}{2}\frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B}(\ddot{q}_a^A - \ddot{q}_f^A)(\ddot{q}_a^B - \ddot{q}_f^B)_{|t_0}$$

measure of deviations of motions

Gauss's principle of least constraint

Among admissible motions the one that deviates least from the free motion is the nonholonomic motion

Hölder's principle

O. Hölder (1859–1937)

$$\mathcal{C}^2(x,y) = \{c : [0,1] \longrightarrow Q \mid c \text{ is } C^2, c(0) = x, \text{ and } c(1) = y\}.$$

$$\mathcal{V}_c = \{X \in T_c\mathcal{C}^2(x,y) \;/\; X(t) \in \mathcal{D}_{c(t)}, \forall t \in [0,1]\}$$

Action functional

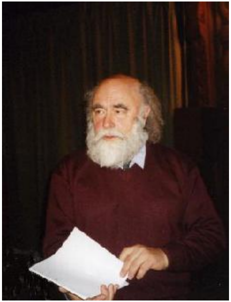
$$\begin{aligned} \mathcal{J} \quad : \quad & \mathcal{C}^2(x,y) \longrightarrow \mathbb{R} \\ & c \mapsto \int_0^1 L(\dot{c}(t))dt \end{aligned}$$

Hölder's principle

An admissible path c $(\dot{c}(t) \in \mathcal{D}_{c(t)})$, is a solution of the nonholonomic problem if

$$d\mathcal{J}(c)(X) = 0, \text{ for all } X \in \mathcal{V}_c$$

A. M. Vershik and L.D. Faddeev: Differential Geometry and Lagrangian mechanics with constraints. *Soviet Physics - Doklady* **17** (1) (1972), 34-36.



On the geometry of non-holonomic Lagrangian systems

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We present a geometric framework for non-holonomic Lagrangian systems in terms of distributions on the configuration manifold. If the constrained system is regular, an almost product structure on the phase space of velocities is constructed such that the constrained dynamics is obtained by projecting the free dynamics. If the constrained system is singular, we develop a constraint algorithm which is very similar to that developed by Dirac and Bergmann, and later globalized by Gotay and Nester. Special attention to the case of constrained systems given by connections is paid. In particular, we extend the results of Koiller for Čaplygin systems. An application to the so-called non-holonomic geometry is given. © *1996 American Institute of Physics.* [S0022-2488(96)02407-3]

I. INTRODUCTION

A non-holonomic Lagrangian system consists of a regular Lagrangian $L(q^A,\dot{q}^A)$ defined on the phase space of velocities TQ of a configuration manifold Q with local coordinates $(q^A), 1 \leqslant A \leqslant n = \dim Q$, subjected to constraints defined by m local functions $\phi_i(q^A,\dot{q}^A)$. That means that the only allowable velocities are those verifying that $\phi_i = 0$. We only consider the case of linear constraints, say those of the form $\phi_i(q^A,\dot{q}^A) = (\mu_i)_A(q)\dot{q}^A$. By applying a suitable Hamilton's principle, we arrive to the constrained Euler–Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial q^A} = -\lambda^i(\mu_i)_A,$$

where λ^i, $1 \leqslant i \leqslant m$, are some Lagrange multipliers to be determined (see, for instance, Valcovici,[1] Pars,[2] Neimark and Fufaev,[3] Vershik and Faddeev,[4] Saletan and Cromer,[5] Rumiantsev,[6] Pironneau,[7] Vershik and Gershkovich,[8] Massa and Pagani[9,10]). In some of them, a more general type of constraints was discussed. We notice that Hamilton's principle in the non-holonomic framework is not a variational principle. We remit to the excellent book by Rosenberg[11] for a detailed discussion on that subject.

In the last years, there is an increasing interest in non-holonomic mechanics, and other approaches from a geometrical point of view have appeared: Weber,[12] Pitanga,[13,14] Marle,[15] Massa and Pagani,[9,10] Bates and Śniatycki,[16] Giachetta,[17] Koiller,[18] Cariñena and Rañada,[19] Rañada,[20] Dazord,[21] Cariñena and Rañada,[22] Sarlet, Cantrijn and Saunders,[23,24] Sarlet,[25,26] de León and M. de Diego.[27–31]

Our approach is a globalization of the one by Cariñena and Rañada.[19] In order to globalize their picture, we will consider a distribution D of codimension m defined on Q. The constraints

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Unconstrained Lagrangian systems

$$\boxed{TQ}$$

$$\begin{array}{ccccc} TQ & & z \in T_x Q & & (q^A, \dot{q}^A) \\ \Big\downarrow \tau_Q & & \Big\downarrow & & \Big\downarrow \\ Q & & x & & (q^A) \end{array}$$

Vertical lift $z \in T_x Q$

$$\begin{array}{rcl} T_x Q & \longrightarrow & T_z TQ \\ X & \longmapsto & (X^v)_z \quad (t \to z + tX) \\ \text{(in coordinates) } X = X^A \frac{\partial}{\partial q^A} & \longmapsto & X = X^A \frac{\partial}{\partial \dot{q}^A} \end{array}$$

Liouville vector field Δ

$$\Delta(z) = (z^V)_z \qquad \Delta = \dot{q}^A \frac{\partial}{\partial \dot{q}^A}$$

Vertical endomorphism S

$$\begin{array}{rcl} T_z TQ & \longrightarrow & T_z TQ \\ Y & \longmapsto & \big(T\tau_Q(z)(Y)\big)^v_z \end{array}$$

$$S(X^A \frac{\partial}{\partial q^A} + \tilde{X}^A \frac{\partial}{\partial \dot{q}^A}) = X^A \frac{\partial}{\partial \dot{q}^A} \qquad S = \frac{\partial}{\partial \dot{q}^A} \otimes dq^A$$

$$L : TQ \longrightarrow \mathbb{R}$$

Poincaré-Cartan 1-form $\alpha_L = S^*(dL)$
Poincaré-Cartan 2-form $\omega_L = -d\alpha_L$
Energy function $E_L = \Delta L - L$

$$L \quad \text{is regular} \iff \left(\frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B}\right) \quad \text{is regular}$$

$i_{\Gamma_L}\omega_L = dE_L$ Γ_L **Euler-Lagrange vector field**

1. $S\Gamma_L = \Delta$ (Γ_L is a SODE)

$$\Gamma_L = \dot{q}^A \frac{\partial}{\partial q^A} + \Gamma^A \frac{\partial}{\partial \dot{q}^A}$$

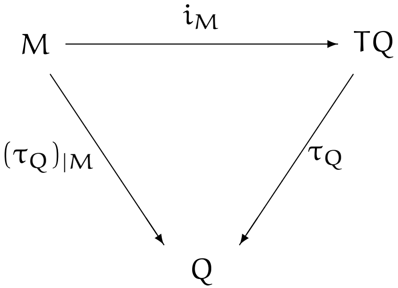
$$\begin{cases} \dfrac{dq^A}{dt} = \dot{q}^A \\ \\ \dfrac{d\dot{q}^A}{dt} = \Gamma^A(q^A, \dot{q}^A) \end{cases} \leftrightarrow \frac{d^2q^A}{dt^2} = \Gamma^A(q^A, \frac{dq^A}{dt})$$

2. The solutions of Γ_L are the solutions of Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial \dot{q}^A} = 0$$

Euler-Lagrange equations

Nonholonomic constraints



M submanifold of TQ.

$$\phi^1 = 0, \ldots, \phi^m = 0$$

Admissibility condition

$$\text{rank}\,\frac{\partial(\phi^1,\ldots,\phi^m)}{\partial(\dot{q}^1,\ldots,\dot{q}^n)} = m \Longleftrightarrow \forall x \in M, \quad \dim T_x M^o = \dim S^*(T_x M^o)$$

$$\dot{q}^i = \Psi^i(q^A, \dot{q}^a), \quad 1 \leqslant i \leqslant m, m+1 \leqslant a \leqslant n \text{ and } 1 \leqslant A \leqslant n$$

Linear constraints on the velocities

$$\phi^i(q^A, \dot{q}^A) = \mu^i_A(q)\dot{q}^A \longrightarrow vectorsubbundle$$

$M \equiv \mathcal{D} \to Q$

$\mathcal{D}$ distribution on Q, $r = \dim Q - m$.

$$\mathcal{D}^o = \langle \mu^i_A dq^A,\ 1 \leqslant i \leqslant m \rangle$$

Nonholonomic mechanics

$L : TQ \longrightarrow \mathbb{R}$ regular, regular distribution $\mathcal{D}$ on Q

$$\left\{ \begin{array}{l} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}^A}\right) - \dfrac{\partial L}{\partial q^A} = \lambda_i \mu^i_A \\ \Phi^i(q, \dot{q}) = \mu^i_A \dot{q}^A = 0 \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{rcl} (i_X \omega_L - dE_L)_{|M} & \in & S^*(TM^o) \\ X_{|M} & \in & TM \end{array} \right.$$

Compatibility condition

Compatibility condition: $F^{\perp_{\omega_L}} \cap TM = 0$

F distribution along M which annihilator is $S^*(TM^o)$ and $F^{\perp_{\omega_L}}$ its symplectic orthogonal

$$T_xTQ = F_x^{\perp_{\omega_L}} \oplus T_xM, \quad x \in M$$

$$\mathcal{P}_x : T_xTQ \longrightarrow T_xM\,,$$
$$\mathcal{Q}_x : T_xTQ \longrightarrow F_x^{\perp_{\omega_L}}\,.$$

$$\boxed{\Gamma_{L,M} = \mathcal{P}(\Gamma_{L|M})}$$

where

$$i_{\Gamma_L}\omega_L = dE_L$$

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NONHOLONOMIC REDUCTION*

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(*Received May 29, 1992 — Revised September 15, 1992*)

The geometric structure of Hamiltonian systems with nonholonomic constraints, reproduced under reduction by symmetry, is described.

1. Introduction

The theory of classical mechanical systems with constraints is an old and venerable topic dating back into the last century. In this theory one may view the constraints as a differential system on configuration space, which may or may not be integrable. In the latter case the constraints are said to be nonholonomic. These constraints are linear restrictions on velocities, and naturally occur in such mechanical problems as a penny rolling without sliding on a plane.

For the sake of simplicity, we will only consider such constraints in this paper. There are numerous difficulties involved with nonlinear constraints, not the least of which is deciding what the correct mechanical principles are that give the equations of motion. These difficulties are discussed at length in the book of Neimark and Fufaev [7], and the article by Weber [12]. More importantly for mechanics, there seems to be known mechanical system that necessitates such a theory.

The plan of this paper is to describe the structure of Hamiltonian systems with constraints in a way that facilitates understanding the reduction of the problem in the presence of symmetry. We look only at Hamiltonian systems that come from a Lagrangian system of the form kinetic minus potential where the Legendre transformation is a diffeomorphism, and the Lagrangian and the constraints are independent of time. We do this for the sake of simplicity of exposition. As the generalization to time-dependent constraints, terms linear in the velocities in the Lagrangian, magnetic terms in the symplectic structure etc. involves no new ideas and is only notationally more complex, it is not done here.

The reduction of these systems by symmetry is formulated in such a way that it looks as much as possible like a nonholonomic version of the standard theory of reduction of symplectic manifolds with symmetry. This is one good reason to work with the Hamiltonian formalism, but there is another, more subtle reason, and this is that the constraints in the

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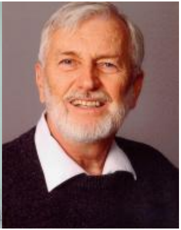


$$T_x TQ = (F_x \cap T_x M) \oplus (F_x \cap T_x M)^{\perp_{\omega_L}} \quad x \in M$$

$$\begin{aligned} \widetilde{\mathcal{P}}_x &: T_x TQ \longrightarrow (F_x \cap T_x M)\,, \\ \widetilde{\mathcal{Q}}_x &: T_x TQ \longrightarrow (F_x \cap T_x M)^{\perp_{\omega_L}}\,. \end{aligned}$$

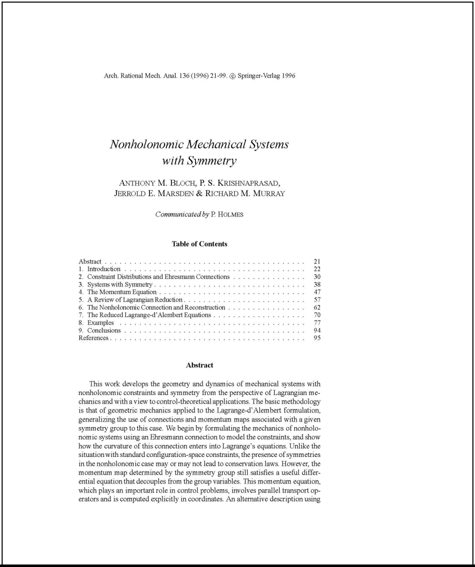
$$\boxed{\left(\dot{q}^A \frac{\partial \phi^i}{\partial \dot{q}^A} \right)_{|M} = 0 \Rightarrow \Gamma_{L,M} = \widetilde{\mathcal{P}}(\Gamma_{L|M})}$$

$\mathcal{H} = F \cap TM$, symplectic vector bundle on M

$$i_X(\omega_L)_{\mathcal{H}} = d_{\mathcal{H}} E_L$$

L. Bates, J. Śniatycki: *Nonholonomic reduction,* Reports on Mathematical Physics, **32** (1) (1992), 99-115.

Reduction of the dynamics of nonholonomic systems



Arch. Rational Mech. Anal. 136 (1996) 21-99. © Springer-Verlag 1996

Nonholonomic Mechanical Systems with Symmetry

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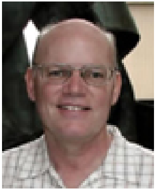
JERROLD E. MARSDEN & RICHARD M. MURRAY

Communicated by P. HOLMES

Table of Contents

Abstract

This work develops the geometry and dynamics of mechanical systems with nonholonomic constraints and symmetry from the perspective of Lagrangian mechanics and with a view to control-theoretical applications. The basic methodology is that of geometric mechanics applied to the Lagrange-d'Alembert formulation, generalizing the use of connections and momentum maps associated with a given symmetry group to this case. We begin by formulating the mechanics of nonholonomic systems using an Ehresmann connection to model the constraints, and show how the curvature of this connection enters into Lagrange's equations. Unlike the situation with standard configuration-space constraints, the presence of symmetries in the nonholonomic case may or may not lead to conservation laws. However, the momentum map determined by the symmetry group still satisfies a useful differential equation that decouples from the group variables. This momentum equation, which plays an important role in control problems, involves parallel transport operators and is computed explicitly in coordinates. An alternative description using

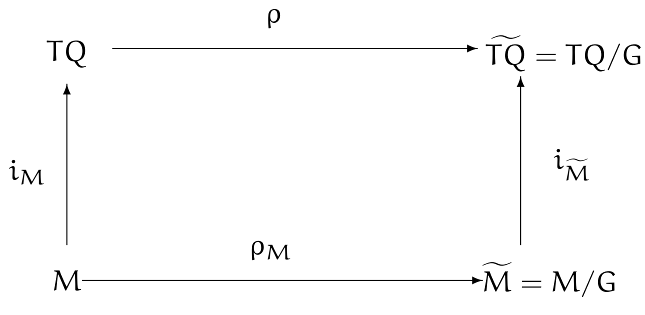


$$\Psi : G \times Q \longrightarrow Q \text{ free and proper}$$

$$\hat{\Psi} : G \times TQ \longrightarrow TQ \qquad \begin{cases} L \text{ is G-invariant} \\ M \text{ is G-invariant} \end{cases}$$

$$\alpha_L, \omega_L, E_L, \Gamma_L, F, \Gamma_{L,M} \text{ are G-invariant.}$$

Objective Taking into account the symmetries, reduce the number of degrees of freedom



$$\mathcal{V} = \ker T\rho, \qquad \mathcal{V}_x = T_x(Gx), x \in TQ$$

$$\mathcal{V}_x \subset T_x M, \; \forall x \in M$$

For unconstrained systems, Noether's theorem states that invariance of the Lagrangian implies a momentum conservation law, but, in nonholonomic mechanics is necessary to account the effect of constraint forces.

Let $J : TQ \longrightarrow \mathfrak{g}^*$ the canonical momentum map associated with the G-action

$$\Gamma_{L,M}(J_\xi) = -i_{\Gamma_{L,M}}\omega_L(\xi_M) = -\xi_M(E_L) + \beta(\xi_M) = \beta(\xi_M)$$

with $\beta \in F^0 = S^*(TM^0)$.

Horizontal symmetry: $\xi_M \in \Gamma(\mathcal{V} \cap F) \Rightarrow \Gamma_{L,M}(J_\xi) = 0$.

Classification of nonholonomic systems with symmetry

$$\begin{cases} \text{Principal or purely kinetical case} : \{0\} = \mathcal{V}_{|M} \cap F \\ \text{Case of Horizontal symmetries} : \mathcal{V}_{|M} \cap \mathcal{F} = \mathcal{V}_{|M} \\ \text{General case} : \{0\} \subsetneq \mathcal{V}_x \cap F_x \subsetneq \mathcal{V}_x \end{cases}$$

Bates and Śniatycki approach

$$\mathcal{U} = (F \cap TM) \cap (\mathcal{V} \cap F)^{\perp_{\omega_L}}, \qquad \text{vector subbundle of } TTQ_{|M}$$

$$\Gamma_{L,M} \in \mathcal{U}$$

$$\mathcal{U} \longrightarrow \widetilde{\mathcal{U}}$$

$$\omega_{\widetilde{\mathcal{U}}}, \qquad \widetilde{dE} = (d(\widetilde{E_L})_{\widetilde{M}})_{|\widetilde{\mathcal{U}}}$$

Proposition

[**Bates-Śniatycki** 1992]
The projection $\widetilde{\Gamma_{L,M}}$ of $\Gamma_{L,M}$ onto $\widetilde{M}$ is a section of $\widetilde{\mathcal{U}}$ satisfying the equation

$$i_{\widetilde{\Gamma_{L,M}}}\omega_{\widetilde{\mathcal{U}}} = \widetilde{dE}$$

The momentum equation

$$\mathfrak{g}^q = \{\xi \in \mathfrak{g} \ / \ \xi_{TQ}(v_q) \in F_{v_q}, \text{ for all } v_q \in T_qQ \cap M\}$$

$$\mathfrak{g}^M = \cup_{q \in Q} \mathfrak{g}^q \longrightarrow Q$$

Nonholonomic momentum map

The nonholonomic momentum map is the mapping $J^{nh} : TQ \longrightarrow (\mathfrak{g}^M)^*$ defined by

$$\langle J^{nh}(v_q), \xi \rangle = \alpha_L(\xi_{TQ})(v_q)$$

A global section $\widetilde{\xi}$ of the vector bundle $\mathfrak{g}^M \rightarrow Q$ induces a vector field Ξ on Q as follows

$$\Xi(q) = (\widetilde{\xi}(q))_Q(q) \in T_q Q$$

Momentum equation

$$\Gamma_{L,M}(J^{nh}_{\widetilde{\xi}}) = \Xi^c(L)$$

Principal case

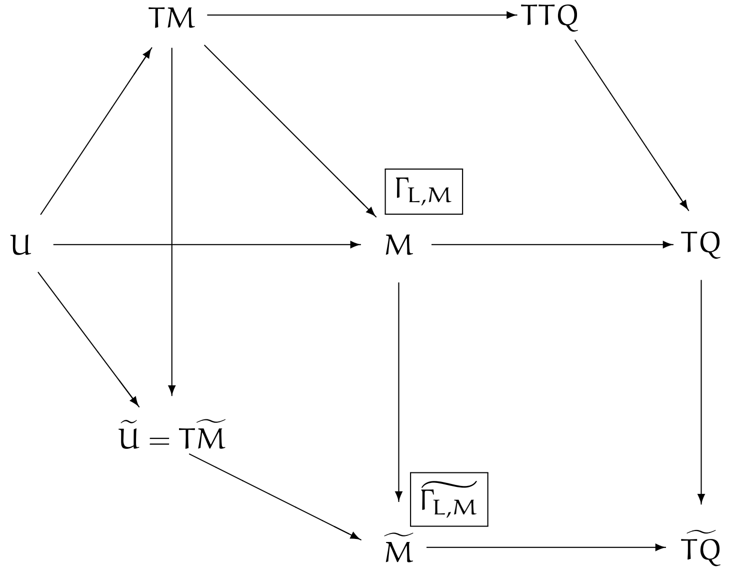
$$\{0\} = \mathcal{V}_{|M} \cap F, \qquad T_x M = \mathcal{V}_x + \mathcal{H}_x$$

$$TM = \mathcal{V}_{|M} \oplus \mathcal{H}$$

$$U = \mathcal{H}, \qquad \widetilde{U} = T\widetilde{M}$$

$$i_{\widetilde{\Gamma_{L,M}}} \widetilde{\omega_L} = d\widetilde{E_L}$$

$\widetilde{\omega_L}$ is an almost symplectic 2-form on $\widetilde{M}$.


TM
TTQ
Γ_{L,M}
U
M
TQ
Ũ = TM̃
Γ̃_{L,M}
M̃
T̃Q

Theorem

The reduced equation of motion can be written in the form

$$i_{\widetilde{\Gamma_{L,M}}}\widetilde{\omega} = d\widetilde{E_L} - \widetilde{\alpha}$$

where $\widetilde{\omega} = -d\widetilde{\alpha}_L$ is a symplectic form, $\widetilde{\alpha}_L$ is the projection on $\widetilde{M}$ of $\mathbf{h}^*(i_M^*\alpha_L)$ and $\tilde{\alpha}$ is the projection of $\alpha = i_{\Gamma_{L,M}}(\mathbf{h}^* d(i_M^*\alpha_L) - d\mathbf{h}^*(i_M^*\alpha_L))$. Moreover, we have

$$i_{\widetilde{\Gamma}_{L,M}}\widetilde{\alpha} = 0.$$

By a Čaplygin system
we mean a mechanical system given
by a lagrangian function $L : TQ \longrightarrow \mathbb{R}$
with a configuration manifold Q which is
a principal G-bundle, say $\pi : Q \longrightarrow Q/G$.

$$TQ = H \oplus \mathcal{W}$$

where $\mathcal{W}$ denotes the vertical bundle.

S.A. Čaplygin (1869–1942)

$$L^* : T(Q/G) \longrightarrow \mathbb{R} \qquad L^*(Y_{\tilde{q}}) = L((Y_q)^H)$$

$$i_{\tilde{\Gamma}_{L,M}} \omega_{L^*} = dE_{L^*} - \tilde{\alpha}.$$

$$\mathcal{V}_{|M} \cap F = \mathcal{V}_{|M}$$

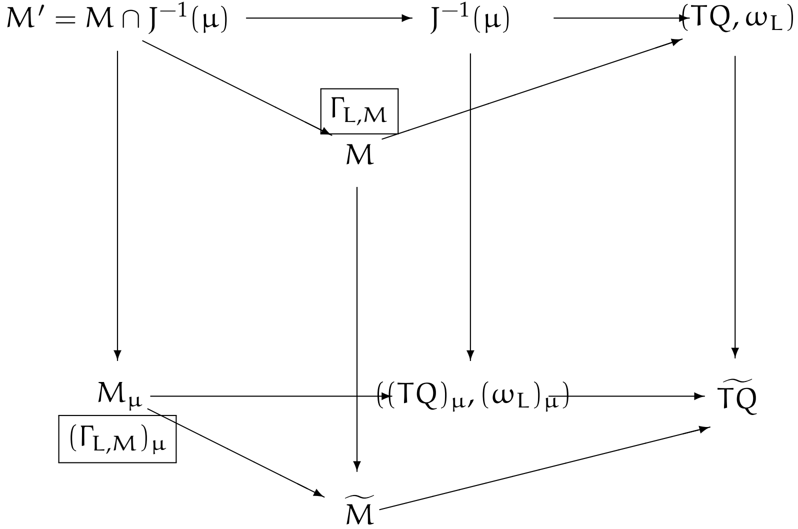
$$J^{nh} = J : TQ \longrightarrow \mathfrak{g}^* \qquad \Gamma_{L,M}(J_\xi) = 0 \text{ for all } \xi \in \mathfrak{g}$$

$$\pi_\mu : J^{-1}(\mu) \longrightarrow (TQ)_\mu = J^{-1}(\mu)/G_\mu$$

The projection $(\Gamma_{L,M})_\mu$ of the restriction of $\Gamma_{L,M}$ to $M' = M \cap J^{-1}(\mu)$ is a solution of the reduced equations of motion

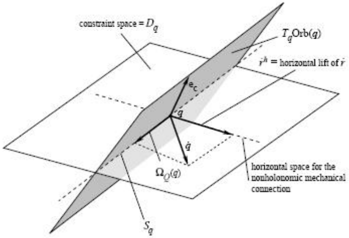
$$\begin{cases} i_{(\Gamma_{L,M})_\mu}(\omega_L)_\mu - d(E_L)_\mu \in F^o_\mu \\ (\Gamma_{L,M})_\mu \in TM_\mu \end{cases}$$

where $(E_L)_\mu$ is the reduced energy.


M′ = M ∩ J⁻¹(μ)
J⁻¹(μ)
(TQ, ω_L)
Γ_{L,M}
M
M_μ
((TQ)_μ, (ω_L)_μ)
T̃Q
(Γ_{L,M})_μ
M̃

The General case

$$\{0\} \subsetneq \mathcal{V}_x \cap F_x \subsetneq \mathcal{V}_x$$



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Admissibility condition

$$\dim TM^o = \dim S^*(TM^o)\,.$$

$$\begin{array}{ccc} TQ & \xrightarrow{\ \mathrm{Leg}\ } & T^*Q \\ & {\scriptstyle \tau_Q} \searrow \quad \swarrow {\scriptstyle \pi_Q} & \\ & Q & \end{array}
\qquad
\begin{array}{ccc} (q^A, \dot{q}^A) & \xrightarrow{\ \mathrm{Leg}\ } & \left(q^A, p_A = \dfrac{\partial L}{\partial \dot{q}^A}\right) \\ & {\scriptstyle \tau_Q} \searrow \quad \swarrow {\scriptstyle \pi_Q} & \\ & (q^A) & \end{array}$$

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$$
\begin{array}{ccc}
 & \text{Leg} & \\
E_L & \longrightarrow & H \\
\omega_L & \longrightarrow & \omega_Q \\
M & \longrightarrow & \bar{M} \\
\phi^i & \longrightarrow & \psi^i(q^A, p_A) = \phi^i(q^A, \frac{\partial H}{\partial p_A}) \\
S^*(TM^o) & \longrightarrow & \bar{F}^o
\end{array}
$$

Compatibility condition $(\mathcal{H}^{AB}) = (\partial^2 H/\partial p_A \partial p_B)$

$$\bar{F}^{\perp} \cap T\bar{M} = \{0\} \Longleftrightarrow (C^{ij}) = \left(\frac{\partial \psi^i}{\partial p_A}\mathcal{H}_{AB}\frac{\partial \psi^j}{\partial p_B}\right) \text{ is regular}$$

$$T\bar{M} \oplus \bar{F}^{\perp}_{|\bar{M}} = T_{\bar{M}}(T^*Q) \longrightarrow \begin{array}{l} P : T_{\bar{M}}(T^*Q) \longrightarrow T\bar{M} \\ Q : T_{\bar{M}}(T^*Q) \longrightarrow \bar{F}^{\perp}. \end{array}$$

$$\mu^i = \frac{\partial \Psi^i}{\partial p_A}\mathcal{H}_{AB}\, dq^B,\ 1 \leqslant i \leqslant m$$

$\bar{F}^{\perp}$ is generated by $Z^i, 1 \leqslant i \leqslant m, i_{Z^i}\omega_Q = \mu^i$

$$Z^i = -\frac{\partial \psi^i}{\partial p_A}\mathcal{H}_{AB}\frac{\partial}{\partial p_B}.$$

$$P = \mathrm{id} + C_{ij} Z^i \otimes d\psi^j \longrightarrow \boxed{P(X_H) = X_{H,\bar{M}}}$$

$$(\bar{F}\cap T\bar{M})\oplus(\bar{F}\cap T\bar{M})^{\perp} = T_{\bar{M}}(T^*Q) \longrightarrow \begin{array}{l} \mathcal{P} : T_{\bar{M}}(T^*Q) \longrightarrow (\bar{F}\cap T\bar{M}) \\ \mathcal{Q} : T_{\bar{M}}(T^*Q) \longrightarrow (\bar{F}\cap T\bar{M})^{\perp} . \end{array}$$

$$\mathcal{P} \;\; = \;\; \mathrm{id} - C_{ij}C_{i'j'}\{\psi^j, \psi^{j'}\}Z^i \otimes \mu^{i'} - C_{ij}X_{\psi^i} \otimes \mu^j + C_{ij}Z^i \otimes d\psi^j ,$$

$$\text{If } \Delta_{|M} \in TM \longrightarrow \boxed{\mathcal{P}(X_H) = X_{H,\bar{M}}} .$$

The nonholonomic bracket

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ON THE HAMILTONIAN FORMULATION OF NONHOLONOMIC MECHANICAL SYSTEMS

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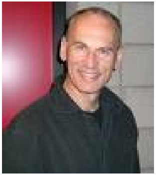
A simple procedure is provided to write the equations of motion of mechanical systems with constraints as Hamiltonian equations with respect to a "Poisson" bracket on the constrained state space, which does not necessarily satisfy the Jacobi identity. It is shown that the Jacobi identity is satisfied if and only if the constraints are holonomic.

1. Introduction

The theory of mechanical systems with nonholonomic constraints has a long history in classical mechanics; see e.g. the books by Neimark & Fufaev [14], Edelen [6], Rosenberg [16], Arnold [1] and the references quoted in there. In this literature, nonholonomic mechanical systems are described within the variational framework by Euler–Lagrange equations with extra terms corresponding to the constraint forces.

The present note is largely influenced by a recent paper of Bates & Śniatycki [4], see also Stanchenko [17], where it is shown that the dynamics of mechanical systems with nonholonomic constraints may be alternatively described within a Hamiltonian framework. However, the two-form with respect to which the Hamiltonian equations of motion (on a reduced state space, and without constraint forces) are defined is not necessarily closed, as may be demonstrated on simple examples. As a consequence, the resulting equations of motion, albeit of a Hamiltonian format, need not admit canonical coordinates and thus need not be transformable to the standard Hamiltonian equations: $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$, $i = 1, \ldots, n$. In the present note we will use, instead of the notion of a (not necessarily closed) two-form, the dual object of a "Poisson" bracket not necessarily satisfying the *Jacobi identity*. We will show in a



Alternative constructions of nonholonomic brackets

Let Λ be the bivector induced by ω_Q and η, ν two arbitrary sections of $T_{\bar{M}} T^*Q$

Bivector	**bracket**
$\Lambda_1(\eta, \nu) = \Lambda(P^*(\eta), P^*(\nu))$,	$\{f, g\}_1 = \Lambda_1(df, dg)$.
$\Lambda_2(\eta, \nu) = \Lambda(\mathcal{P}^*(\eta), \mathcal{P}^*(\nu)) = \omega_Q(\mathcal{P}(X_\eta), \mathcal{P}(X_\nu))$	$\{f, g\}_2 = \Lambda_2(df, dg)$.

Proposition

Along the constraint submanifold $\bar{M}$ we have that

$$\Lambda_1 = \Lambda_2 \qquad (\Lambda_{nh} \text{ and } \{\ ,\ \}_{nh}) .$$

Homogeneous case $\longrightarrow$ $\boxed{\dot{f} = X_{H,\bar{M}}(f) = \{f, H\}_{nh}\,.}$

Almost Poisson bracket on the constraint submanifold

$$\Lambda_{\bar{M}}(x)(\eta_{\bar{M}}(x), \nu_{\bar{M}}(x)) = \Lambda_{nh}(x)(\eta(x), \nu(x))$$

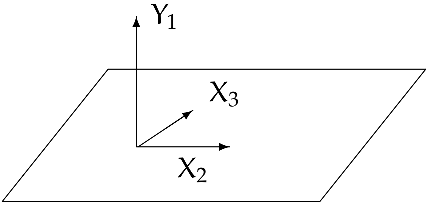
$$\boxed{\{f, g\}_{\bar{M}} = \Lambda_{\bar{M}}(df, dg) = \Lambda_{nh}(dF, dG) = \{F, G\}_{nh}}$$

Van der Shaft and Maschke construction

$$L = T - V, \quad \Phi^{\alpha} := \mu^{\alpha}_{A}(q)\dot{q}^{A} = 0, (A = 1, \ldots, m),$$

$$H = \frac{1}{2} g^{AB} p_A p_B + V(q) \qquad \psi^i = \mu^i_A \frac{\partial H}{\partial p_A}, \ 1 \leqslant i \leqslant m\,.$$

$$\begin{array}{ll} \tilde{p}_a = X^A_a p_A, & m+1 \leqslant a \leqslant n \\ \tilde{p}_i = Y^A_i p_A, & 1 \leqslant i \leqslant m \end{array}$$



$$\psi^i = C^{ij} \frac{\partial \tilde{H}}{\partial \tilde{p}_j} = \tilde{p}_i = 0\,.$$

$$
\begin{aligned}
\{q^A, q^B\} = 0 \quad & \{q^A, \tilde{p}_a\} = X_a^A \quad \{q^A, \tilde{p}_i\} = Y_i^A \\
& \{\tilde{p}_a, \tilde{p}_b\} = X_b^A p_B \frac{\partial X_a^B}{\partial q^A} - X_a^A p_B \frac{\partial X_b^B}{\partial q^A} \\
& \{\tilde{p}_i, \tilde{p}_b\} = X_b^A p_B \frac{\partial Y_i^B}{\partial q^A} - Y_i^A p_B \frac{\partial X_b^B}{\partial q^A} \\
& \{\tilde{p}_i, \tilde{p}_j\} = Y_j^A p_B \frac{\partial Y_i^B}{\partial q^A} - Y_i^A p_B \frac{\partial Y_j^B}{\partial q^A}
\end{aligned}
\quad \longrightarrow \begin{pmatrix} \{q^A, q^B\} & \{q^A, \tilde{p}_b\} & \{q^A, \tilde{p}_j\} \\ \{\tilde{p}_a, q^B\} & \{\tilde{p}_a, \tilde{p}_b\} & \{\tilde{p}_a, \tilde{p}_\beta\} \\ \{\tilde{p}_i, q^B\} & \{\tilde{p}_i, \tilde{p}_b\} & \{\tilde{p}_i, \tilde{p}_j\} \end{pmatrix}
$$

$$J_{\widetilde{M}}(q^i, \tilde{p}_a) = \begin{pmatrix} \{q^A, q^B\} & \{q^A, \tilde{p}_b\} \\ \{\tilde{p}_a, q^B\} & \{\tilde{p}_a, \tilde{p}_b\} \end{pmatrix}$$

$$\{q^A, q^B\}_{nh} = 0 \quad \{q^A, \tilde{p}_a\}_{nh} = X_a^A \quad \{q^A, \tilde{p}_i\}_{nh} = 0$$

$$\{\tilde{p}_a, \tilde{p}_b\}_{nh} = X_b^A p_B \frac{\partial X_a^B}{\partial q^A} - X_a^A p_B \frac{\partial X_b^B}{\partial q^A}, \quad \{\tilde{p}_i, \tilde{p}_b\}_{nh} = 0 \quad \{\tilde{p}_i, \tilde{p}_j\}_{nh} = 0\,.$$

$$\{f, g\}_{\widetilde{M}} = \left(\left(\frac{\partial f}{\partial q^A}\right)^T, \left(\frac{\partial f}{\partial \tilde{p}_a}\right)^T \right) J_{\widetilde{M}} \begin{pmatrix} \frac{\partial g}{\partial q^B} \\ \frac{\partial g}{\partial \tilde{p}_b} \end{pmatrix}$$

General case (no homogeneous)

$$\dot{f} = X_{H,\bar{M}}(f) = R_H(f) + \{f, H\}_{nh}$$

donde $R_H = P(X_H) - \mathcal{P}(X_H)$.

$$\begin{aligned}\dot{f} &= R_H(f) + \{f, H\}_{nh} \\ &= \left(-C_{ij} Z^i(H) P(X_{\psi^j})(f)\right) + \Big(\{f, H\} + C_{ij} Z^i(H)\{f, \psi^j\} \\ &-C_{ij} Z^i(f)\{H, \psi^j\} + C_{ij} C_{i'j'}\{\psi^j, \psi^{j'}\} Z^i(f) Z^{i'}(H)\Big)\end{aligned}$$

$$\dot{f} = R_H(f) + \{f, H_{\bar{M}}\}_{\bar{M}}$$

Standard Mechanical Lagrangian System

1. an n-dimensional configuration manifold Q,
2. a Riemannian metric $\mathcal{G}^{TQ}$ on Q describing the kinetic energy,
3. a function V on Q describing the potential energy,

$\mathcal{G}^{TQ}$ be a Riemannian metric on Q

$$\Downarrow$$

Levita-Civita connection $\nabla^{\mathcal{G}^{TQ}} : \mathfrak{X}(Q) \times \mathfrak{X}(Q) \to \mathfrak{X}(Q)$

$$\begin{aligned} 2\mathcal{G}^{TQ}(\nabla_X^{\mathcal{G}^{TQ}} Y, Z) &= X(\mathcal{G}^{TQ}(Y,Z)) + Y(\mathcal{G}^{TQ}(X,Z)) - Z(\mathcal{G}^{TQ}(X,Y)) \\ &\quad + \mathcal{G}^{TQ}(X,[Z,Y]) + \mathcal{G}^{TQ}(Y,[Z,X]) \\ &\quad - \mathcal{G}^{TQ}(Z,[Y,X]) \end{aligned}$$

for $X, Y, Z \in \mathfrak{X}(Q)$.

Alternatively, $\nabla^{\mathcal{G}^{TQ}}$ is determined by

$$[X, Y] = \nabla^{\mathcal{G}^{TQ}}_X Y - \nabla^{\mathcal{G}^{TQ}}_Y X \text{ (symmetry)}$$
$$X(\mathcal{G}^{TQ}(Y, Z)) = \mathcal{G}^{TQ}(\nabla^{\mathcal{G}^{TQ}}_X Y, Z) + \mathcal{G}^{TQ}(Y, \nabla^{\mathcal{G}^{TQ}}_X Z) \text{ (metricity) },$$

The solutions of the mechanical problem with Lagrangian $L : TQ \longrightarrow \mathbb{R}$:

$$L(v) = \frac{1}{2}\mathcal{G}^{TQ}(v, v) - V(\tau_{TQ}(v))$$

are the curves $\sigma : I \subset \mathbb{R} \to Q$ such that

$$\nabla^{\mathcal{G}^{TQ}}_{\dot\sigma(t)}\dot\sigma(t) + \mathrm{grad}_{\mathcal{G}^{TQ}} V(\sigma(t)) = 0.$$

$$\Updownarrow$$

Euler-Lagrange equations $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^A}\right) - \frac{\partial L}{\partial q^A} = 0$

Local description

Local coordinates (q^A) on Q, then

$$\mathcal{G}^{TQ} = (\mathcal{G}^{TQ})_{AB} dq^A \otimes dq^B$$

The Christoffel symbols of the connection $\nabla^{\mathcal{G}^{TQ}}$ are obtained from the following expression

$$\nabla^{\mathcal{G}^{TQ}}_{\partial_B} \partial_C = \Gamma^A_{BC} \partial_A .$$

The equations of motion are locally written as

$$\ddot{q}^C = -\Gamma^C_{AB} \dot{q}^A \dot{q}^B - (\mathcal{G}^{TQ})^{CA} \frac{\partial V}{\partial q^A}$$

The nonholonomic connection

Geodesics in non-holonomic geometry.

Von

J. L. Synge in Dublin (Irland).

Synopsis:

1. Introduction.
2. The constraint.
3. Equations of constrained geodesics derived from a variational principle.
4. Second form of the equations of constrained geodesics.
5. Parallel (Γ) propagation.
6. Geodesic stability for a general correspondence.
7. Isometric and normal correspondences.
8. Dynamical significance of the paper.

1. Introduction.

The stability of geodesics in Riemannian space was first discussed simultaneously and independently by Levi-Civita [1]) and myself [2]). The tensorial equation obtained may be written

$$\bar{\bar{\eta}}^r + G^r_{.msn} x^{m'} \eta^s x^{n'} = 0\,, \tag{1.1}$$

where η^r is the infinitesimal vector joining a point on the fundamental geodesic to the corresponding point of a neighbouring geodesic, $x^{r'} = \frac{dx^r}{ds}$, where x^r is the coordinate system, $G^r_{.msn}$ is the mixed curvature tensor

[1]) „Sur l'écart géodésique," Math. Annalen 97 (1926), p. 291—320. Cf. also Levi-Civita, „The Absolute Differential Calculus," English Translation (1927).

[2]) „On the Geometry of Dynamics," Phil. Trans. Roy. Soc., A, 226 (1926), p 31—106. That paper will be referred to as GD. The dynamical problem of „Stability in the action sense," discussed in GD., Chap. IX, is precisely the geometrical problem of the stability of geodesics in Riemannian space.

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The nonholonomic connection

1. an n-dimensional configuration manifold Q,
2. a Riemannian metric $\mathcal{G}^{TQ}$ on Q describing the kinetic energy,
3. a function V on Q describing the potential energy,
4. a distribution $\mathcal{D}$ of feasible velocities describing the linear velocity constraints

$\nabla^{\mathcal{G}^{TQ}}_{\dot\sigma(t)}\dot\sigma(t)+\mathrm{grad}_{\mathcal{G}^{TQ}}V(\sigma(t)) \in \mathcal{D}^{\perp}_{\dot c(t)}(t) \in \mathcal{D}_{c(t)}$, where $\mathcal{D}^{\perp}$ is the $\mathcal{G}^{TQ}$-orthogonal complement of $\mathcal{D}$

$$\mathcal{P} : TQ \to \mathcal{D}$$
$$\mathcal{Q} : TQ \to \mathcal{D}^{\perp}$$

Nonholonomic equations $\tilde{\nabla}^{\mathcal{G}^{TQ}}_X Y = \nabla^{\mathcal{G}^{TQ}}_X Y + (\nabla^{\mathcal{G}^{TQ}}_X \mathcal{Q})(Y)$

$$\tilde{\nabla}^{\mathcal{G}^{TQ}}_{\dot{c}(t)} \dot{c}(t) + \mathcal{P}\left(\mathrm{grad}_{\mathcal{G}^{TQ}} V(\tau_{\mathcal{D}}(\dot{c}(t)))\right) = 0$$

Proposition:

For all $Z \in \mathfrak{X}(Q)$ and $X, Y \in \mathcal{D}$ we have that

$$Z\left(\mathcal{G}^{TQ}(X, Y)\right) = \mathcal{G}^{TQ}(\tilde{\nabla}^{\mathcal{G}^{TQ}}_Z X, Y) + \mathcal{G}^{TQ}(X, \tilde{\nabla}^{\mathcal{G}^{TQ}}_Z Y)$$

but with TORSION!!!!!

Is it possible to derive a Levi-Civita connection for nonholonomic dynamics?

IDEA: Modify the Lie bracket!!

We induce, by restriction,

1. a bundle metric $\mathcal{G}^{\mathcal{D}} : \mathcal{D} \times_Q \mathcal{D} \to \mathbb{R}$
2. an induced bracket

$$[\![X, Y]\!]_{\mathcal{D}} = \mathcal{P}[i_{\mathcal{D}}(X), i_{\mathcal{D}}(Y)]$$

   where $X, Y \in \Gamma(\tau_{\mathcal{D}})$ (vector fields on Q taking values on $\mathcal{D}$).

3. anchor map $\rho_{\mathcal{D}} \doteq i_{\mathcal{D}} : \mathcal{D} \hookrightarrow TQ$

Skew-symmetric algebroids

An skew-symmetric algebroid structure on the vector bundle $\tau_{\mathcal{D}} : \mathcal{D} \to Q$ is a $\mathbb{R}$-linear bracket $[\![\cdot,\cdot]\!]_{\mathcal{D}} : \Gamma(\tau_{\mathcal{D}}) \times \Gamma(\tau_{\mathcal{D}}) \to \Gamma(\tau_{\mathcal{D}})$ on the space $\Gamma(\tau_{\mathcal{D}})$ and a vector bundle morphism $\rho_{\mathcal{D}} : \mathcal{D} \to Q$, the *anchor map*, such that:

1. $[\![\cdot,\cdot]\!]_{\mathcal{D}}$ is skew-symmetric, that is,

$$[\![X,Y]\!]_{\mathcal{D}} = -[\![Y,X]\!]_{\mathcal{D}}, \quad \text{for } X,Y \in \Gamma(\tau_{\mathcal{D}}).$$

2. If we also denote by $\rho_{\mathcal{D}} : \Gamma(\tau_{\mathcal{D}}) \to \mathfrak{X}(Q)$ the morphism of $C^\infty(Q)$-modules induced by the anchor map then

$$[\![X,fY]\!]_{\mathcal{D}} = f[\![X,Y]\!]_{\mathcal{D}} + \rho_{\mathcal{D}}(X)(f)Y, \quad \text{for } X,Y \in \Gamma(\mathcal{D}) \text{ and } f \in C^\infty(Q)$$

If the bracket $[\![\cdot,\cdot]\!]_{\mathcal{D}}$ satisfies the Jacobi identity, we have that the pair $([\![\cdot,\cdot]\!]_{\mathcal{D}}, \rho_{\mathcal{D}})$ is a *Lie algebroid structure* on the vector bundle $\tau_{\mathcal{D}} : \mathcal{D} \to Q$.

The "differential"

If $([\![\cdot,\cdot]\!]_{\mathcal{D}}, \rho_{\mathcal{D}})$ is an skew-symmetric algebroid structure on the vector bundle $\tau_{\mathcal{D}} : \mathcal{D} \to Q$ then we may define an *almost differential* $d^{\mathcal{D}}$ by

$$(d^{\mathcal{D}}\alpha)(X_0, X_1, \ldots, X_K) = \sum_{I=0}^{K} (-1)^I \rho_{\mathcal{D}}(X_I)(\alpha(X_0, \ldots, \hat{X}_I, \ldots, X_K)) + \sum_{I<J} (-1)^{I+J} \alpha([\![X_I, X_J]\!]_{\mathcal{D}}, X_0, X_1, \ldots, \hat{X}_I, \ldots, \hat{X}_J, \ldots, X_K)$$

for $\alpha \in \Gamma(\Lambda^K \tau_{\mathcal{D}^*})$ and $X_0, X_1, \ldots, X_K \in \Gamma(\tau_{\mathcal{D}})$.
In general $(d^{\mathcal{D}})^2 \neq 0$. Indeed, $([\![\cdot,\cdot]\!]_{\mathcal{D}}, \rho_{\mathcal{D}})$ is a Lie algebroid structure on the vector bundle $\tau_{\mathcal{D}} : \mathcal{D} \to Q$ if and only if $(d^{\mathcal{D}})^2 = 0$

Linear almost Poisson structures and Hamilton-Jacobi equation. Applications to nonholonomic Mechanics. *M. de León, J.C. Marrero, DMdD. Journal of Geometric Mechanics 2011.*

Some coordinate expressions

Suppose that (q^A) are local coordinates on Q and that $\{e_a\}$ is a local basis of the space of sections $\Gamma(\tau_{\mathcal{D}})$ then

$$[\![e_a, e_b]\!]_{\mathcal{D}} = \mathcal{C}^c_{ab} e_c, \quad \rho_{\mathcal{D}}(e_a) = (\rho_{\mathcal{D}})^A_a \frac{\partial}{\partial q^A}. \qquad (1)$$

The local functions $\mathcal{C}^c_{ab}, (\rho_{\mathcal{D}})^A_a \in C^\infty(Q)$ are called the *local structure functions* of the skew-symmetric algebroid $\tau_{\mathcal{D}} : \mathcal{D} \to Q$.
If $\{e^a\}$ ($e^a \in \Gamma(\tau_{\mathcal{D}^*})$, where $\tau_{\mathcal{D}^*} : \mathcal{D}^* \longrightarrow \mathbb{R}$) is the dual basis of $\{e_a\}$ then

$$\begin{aligned} d_{\mathcal{D}} F &= (\rho_{\mathcal{D}})^A_a \frac{\partial F}{\partial q^A} e^a, \\ d^{\mathcal{D}} \kappa &= \left\{ (\rho_{\mathcal{D}})^A_a \frac{\partial \kappa_a}{\partial q^A} - \frac{1}{2} \mathcal{C}^c_{ab} \kappa_c \right\} e^a \wedge e^b, \end{aligned}$$

where $F \in C^\infty(Q)$ and $\kappa \in \Gamma(\tau_{\mathcal{D}^*})$ where $\kappa = \kappa_b e^b$.

Given a section $X \in \Gamma(\tau_{\mathcal{D}})$ the curves $\sigma : I \subseteq \mathbb{R} \to Q$ which verify the equations:

$$\dot{\sigma} = \rho_{\mathcal{D}}(X) \circ \sigma$$

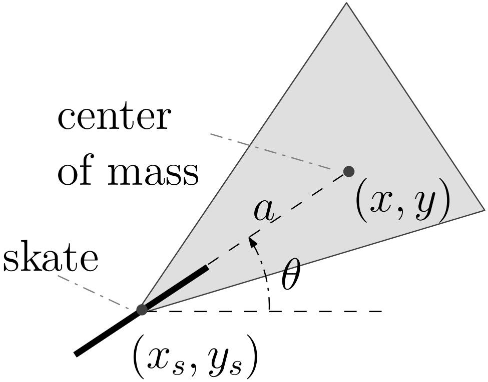
are called integral curves of the section X, that is, they are the integral curves of the associated vector field $\rho_{\mathcal{D}}(X) \in \mathfrak{X}(Q)$. Locally are written as

$$\dot{\sigma}^A = (\rho_{\mathcal{D}})^A_a X^a \circ \sigma \text{ or in other words } \dot{q}^A = (\rho_{\mathcal{D}})^A_a X^a(x)$$

where $X = X^a e_a$.

A Typical example: The Chaplygin sleigh.

As an example of nonholonomic system on a Lie algebra, we study the Chaplygin sleigh which describes a rigid body sliding on a plane. The body is supported in three points, two of which slides freely without friction while the third point is a knife edge.



A Typical example: The Chaplygin sleigh.

The configuration space before reduction is the Lie group $G = SE(2)$ of the Euclidean motions of the 2-dimensional plane $\mathbb{R}^2$.

We will need in the sequel to fix some notation about the Lie algebra $\mathfrak{se}(2)$. First of all its elements are matrices of the form

$$\hat{\xi} = \begin{pmatrix} 0 & \xi_3 & \xi_1 \\ -\xi_3 & 0 & \xi_2 \\ 0 & 0 & 0 \end{pmatrix}$$

and a basis of the Lie algebra $\mathfrak{se}(2) \cong \mathbb{R}^3$ is given by

$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have that

$$[E_3, E_1] = E_2, \quad [E_2, E_3] = E_1, \quad [E_1, E_2] = 0.$$

An element $\xi \in \mathfrak{se}(2)$ is of the form

The Chaplygin system is described by the kinetic Lagrangian function

$$\begin{array}{rccl} L: & \mathfrak{se}(2) & \longrightarrow & \mathbb{R} \\ & (v_1, v_2, \omega) & \longmapsto & \frac{1}{2}\Big[(J + m(a^2 + b^2))\omega^2 + mv_1^2 + mv_2^2 - 2bm\omega v_1 \end{array}$$

where m and J denotes the mass and moment of inertia of the sleigh relative to the contact point and (a, b) represents the position of the center of mass with respect to the body frame determined placing the origin at the contact point and the first coordinate axis in the direction of the knife axis.

A Typical example: The Chaplygin sleigh.

Additionally, the system is subjected to the nonholonomic constraint determined by the linear subspace of $\mathfrak{se}(2)$:

$$\mathcal{D} = \{(v_1, v_2, \omega) \in se(2) \mid v_2 = 0\}.$$

Instead of $\{E_1, E_2, E_3\}$ we take the basis of $\mathfrak{se}(2)$:

$$\{e_1 = E_3, e_2 = E_1, e_3 = -maE_3 - mabE_1 + (J + ma^2)E_2\}$$

which is a basis adapted to the decomposition $\mathcal{D} \oplus \mathcal{D}^\perp$;
$\mathcal{D} = \text{span}\,\{e_1, e_2\}$ and $\mathcal{D}^\perp = \text{span}\,\{e_3\}$.
In the induced coordinates (y^1, y^2) on $\mathcal{D}$ the restricted lagrangian is

$$l(y^1, y^2) = \frac{1}{2}\left[(J + m(a^2 + b^2))(y^1)^2 + m(y^2)^2 - 2bmy^1y^2\right] ,$$

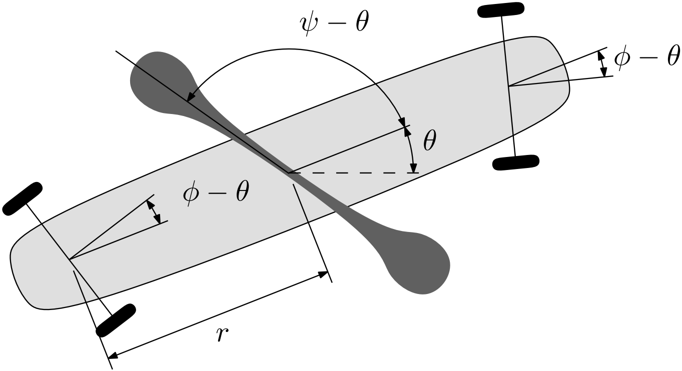
and moreover,

$$[e_1, e_2]_{\mathcal{D}} = \frac{ma}{J + ma^2} e_1 + \frac{mab}{J + ma^2} e_2 ,$$

Therefore, $\mathcal{C}^1_{12} = \frac{ma}{J+ma^2}$ and $\mathcal{C}^2_{12} = \frac{mab}{J+ma^2}$.

The snakeboard

As a mechanical system the snakeboard has as configuration space $Q = SE(2) \times T^2$ with coordinates $(x, y, \theta, \psi, \phi)$



The nonholonomic dynamics is described by

- The *Lagrangian*

$$L(q,\dot{q}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}(J+2J_1)\dot{\theta}^2 + \frac{1}{2}J_0(\dot{\theta}+\dot{\psi})^2 + J_1\dot{\phi}^2,$$

  where m is the total mass of the board, $J > 0$ is the moment of inertia of the board, $J_0 > 0$ is the moment of inertia of the rotor of the snakeboard mounted on the body's center of mass and $J_1 > 0$ is the moment of inertia of each wheel axles. The distance between the center of the board and the wheels is denoted by r. For simplicity, we assume that $J + J_0 + 2J_1 = mr^2$.

- The *nonholonomic constraints* induced by the non sliding condition in the sideways direction of the wheels:

$$-\dot{x}\sin(\theta+\phi) + \dot{y}\cos(\theta+\phi) - r\dot{\theta}\cos\phi = 0$$
$$-\dot{x}\sin(\theta-\phi) + \dot{y}\cos(\theta-\phi) + r\dot{\theta}\cos\phi = 0.$$

Observe that the Lagrangian is induced by the riemannian metric $\mathcal{G}$ on Q,

$$\mathcal{G} = m dx^2 + m dy^2 + m r^2 d\theta^2 + J_0 d\theta \otimes \psi + J_0 d\psi \otimes d\theta + J_0 d\psi^2 + 2J_1 d\phi^2.$$

The constraint subbundle $\tau_{\mathcal{D}} : \mathcal{D} \longmapsto Q$ is

$$\mathcal{D} = \text{span}\left\{ e_1 = \frac{\partial}{\partial \psi}, e_2 = \frac{\partial}{\partial \phi}, e_3 = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial \theta} \right\}.$$

where

$$\begin{aligned} a &= -r(\cos\phi\cos(\theta-\phi) + \cos\phi\cos(\theta+\phi)) = -2r\cos^2\phi\cos\theta \\ b &= -r(\cos\phi\sin(\theta-\phi) + \cos\phi\sin(\theta+\phi)) = -2r\cos^2\phi\sin\theta \\ c &= \sin(2\phi). \end{aligned}$$

In the induced coordinates $(x, y, \theta, \psi, \phi, y^1, y^2, y^3)$ on $\mathcal{D}$ the restricted lagrangian is

$$
\begin{aligned}
l(x, y, \theta, \psi, \phi, y^1, y^2, y^3) &= 2mr^2\cos^2\phi (y^3)^2 + J_0 c y^1 y^3 \\
&\quad + \frac{1}{2} J_0 (y^1)^2 + J_1 (y^2)^2.
\end{aligned}
$$

where now the nonholonomic constraints are rewritten as: $y^4 = 0$ and $y^5 = 0$. After some straightforward computations we deduce that

$$
\begin{aligned}
[e_1, e_2]_{\mathcal{D}} &= 0, \\
[e_1, e_3]_{\mathcal{D}} &= 0, \\
[e_2, e_3]_{\mathcal{D}} &= \frac{2mr^2\cos^2\phi}{mr^2 - J_0\sin^2\phi} e_1 - \frac{(mr^2 + \cos 2\phi)\tan\phi}{mr^2 - J_0\sin^2\phi} e_3.
\end{aligned}
$$

Levi-Civita connection on a skew-symmetric algebroid with a bundle metric

Let $\mathcal{G}^{\mathcal{D}} : \mathcal{D} \times_Q \mathcal{D} \to \mathbb{R}$ be a bundle metric on a skew-symmetric algebroid $(\mathcal{D}, [\![\cdot, \cdot]\!]_{\mathcal{D}}, \rho_{\mathcal{D}})$. Given this bundle metric we can construct a unique connection $\nabla^{\mathcal{G}^{\mathcal{D}}}$ on $\mathcal{D}$ which is torsion-less and metric with respect to $\mathcal{G}^{\mathcal{D}}$.

The Levita-Civita connection $\nabla^{\mathcal{G}^{\mathcal{D}}} : \Gamma(\tau_{\mathcal{D}}) \times \Gamma(\tau_{\mathcal{D}}) \to \Gamma(\tau_{\mathcal{D}})$ associated to the bundle metric $\mathcal{G}^{\mathcal{D}}$ is defined by the formula:

$$[\![X, Y]\!]_{\mathcal{D}} = \nabla^{\mathcal{G}^{\mathcal{D}}}_X Y - \nabla^{\mathcal{G}^{\mathcal{D}}}_Y X \text{ (symmetry)}$$
$$\rho_{\mathcal{D}}(X)(\mathcal{G}^{\mathcal{D}}(Y, Z)) = \mathcal{G}^{\mathcal{D}}(\nabla^{\mathcal{G}^{\mathcal{D}}}_X Y, Z) + \mathcal{G}^{\mathcal{D}}(Y, \nabla^{\mathcal{G}^{\mathcal{D}}}_X Z) \text{ (metricity)},$$

A $\rho_{\mathcal{D}}$-admissible curve is a curve $\gamma : I \longrightarrow \mathcal{D}$ such that

$$\frac{d(\tau_{\mathcal{D}} \circ \gamma)}{dt}(t) = \rho_{\mathcal{D}}(\gamma(t)) \, .$$

Given a potential function $V : Q \to \mathbb{R}$, the solutions of the mechanical problem with Lagrangian $L : \mathcal{D} \longrightarrow \mathbb{R}$:

$$L(v) = \frac{1}{2}\mathcal{G}^{\mathcal{D}}(v, v) - V(\tau_{\mathcal{D}}(v))$$

are the $\rho_{\mathcal{D}}$-admissible curves $\gamma : I \longrightarrow \mathcal{D}$ such that

$$\nabla^{\mathcal{G}^{\mathcal{D}}}_{\gamma(t)}\gamma(t) + \mathrm{grad}_{\mathcal{G}^{\mathcal{D}}} V(\tau_{\mathcal{D}}(\gamma(t))) = 0.$$

Given local coordinates (q^A, y^a) associated with the basis $\{e_a\}$ of sections of $\mathcal{D}$ the Equations can be written as

$$\begin{aligned} \dot{q}^A &= (\rho_{\mathcal{D}})^A_a y^a \\ \dot{y}^c &= -\Gamma^c_{ab} y^a y^b - (\mathcal{G}^{\mathcal{D}})^{cb} (\rho_{\mathcal{D}})^A_b \frac{\partial V}{\partial q^A} \ . \end{aligned}$$

where $(\mathcal{G}^{\mathcal{D}})^{ab}$ are the entries of the inverse matrix of $((\mathcal{G}^{\mathcal{D}})_{ab})$ $(\mathcal{G}^{\mathcal{D}} = \mathcal{G}^{\mathcal{D}}{}_{ab} e^a \otimes e^b)$.
The geodesics are just the integral curves of a vector field on D, called *the geodesic spray* $\xi_{\mathcal{G}^D}$, whose local expression is

$$\xi_{\mathcal{G}_{\mathcal{D}}} = (\rho_{\mathcal{D}})^A_c y^c \frac{\partial}{\partial q^A} - \Gamma^c_{ab} y^a y^b \frac{\partial}{\partial y^c}.$$

Reduction of Nonholonomic systems with Symmetries

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Reduction of Some Classical Non-Holonomic Systems with Symmetry

JAIR KOILLER

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Abstract

Two types of nonholonomic systems with symmetry are treated: (i) the configuration space is a total space of a G-principal bundle and the constraints are given by a connection; (ii) the configuration space is G itself and the constraints are given by left-invariant forms. The proofs are based on the method of quasicoordinates. In passing, a derivation of the Maurer-Cartan equations for Lie groups is obtained. Simple examples are given to illustrate the algorithmical character of the main results.

Contents

One of the interesting ocurrences of symmetry in mechanics is the rolling of a solid body without slipping along a two dimensional surface (possibly of a complex profile). The results of this process are studied by the mechanics of nonholonomic systems .... Recently, deep and interesting connections of this subject with Lie groups were discovered ....
A. T. FOMENKO, Visual and hidden Symmetry in geometry, in *Comp. Math. Appl.*, 17, 1989.

The symmetric product

The associated *symmetric product* is defined as follows:

$$\langle X : Y \rangle_{\mathcal{G}^{\mathcal{D}}} = \nabla^{\mathcal{G}^{\mathcal{D}}}_X Y + \nabla^{\mathcal{G}^{\mathcal{D}}}_Y X \,, \quad X, Y \in \Gamma(\tau_{\mathcal{D}}) \,.$$

$(\mathfrak{l}, \mathfrak{d})$ is a nonholonomic Lagrangian system on $\mathfrak{g}$, where $\mathfrak{l} : \mathfrak{g} \to \mathbb{R}$ is a Lagrangian function defined by $\mathfrak{l}(\xi) = \frac{1}{2}\langle \mathbb{I}\xi, \xi\rangle$, $\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^*$ is a symmetric positive definite inertia operator and $\mathfrak{d}$ is a vector subspace of $\mathfrak{g}$. We have the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{d}^{\perp},$$

where $\mathfrak{d}^{\perp} = \{\xi' \in \mathfrak{g} \,|\, \langle \mathbb{I}\xi', \xi\rangle = 0 \; \forall \xi \in \mathfrak{d}\}$. Take now an adapted basis to this decomposition $\{e_a, e_\alpha\}$ where $\mathfrak{d} = \operatorname{span}\{e_a\}$ and $\mathfrak{d}^{\perp} = \operatorname{span}\{e_\alpha\}$. Then, the *Euler-Poincaré-Suslov equations* for $(\mathfrak{l}, \mathfrak{d})$ are

$$\dot{y}^c = -\Gamma^c_{ab} y^a y^b,$$

where $\{y^a, y^\alpha\}$ are the global coordinates on $\mathfrak{g}$ induced by the basis $\{e_a, e_\alpha\}$.

Chaplygin sleigh

$$
\begin{aligned}
\dot{y}^1 &= \frac{ma}{J+ma^2} y^1 \left(b y^1 - y^2 \right) \\
\dot{y}^2 &= \frac{ma}{J+ma^2} y^1 \left((J + m(a^2+b^2)) y^1 - b y^2 \right).
\end{aligned}
$$

Nonholonomic systems on Atiyah algebroids associated with principal G-bundles.

$$\tau_A : A = \mathfrak{g} \times TM \longrightarrow M,$$

where $\mathfrak{g}$ is the Lie algebra of the Lie group G and M is a smooth manifold. The Lie bracket of the space $\Gamma(\tau_A)$ is characterized by the following condition

$$[\![(\xi, X), (\xi', X')]\!]_A = \left([\![\xi, \xi']\!]_{\mathfrak{g}}, [X, X']\right),$$

for $\xi, \xi' \in \mathfrak{g}$ and $X, X' \in \mathfrak{X}(M)$. The anchor map ρ_A is the canonical projection onto the second factor.

Suppose now that D is a vector subbundle of A over M of constant rank (the constraint bundle) such that

$$M \in x \longrightarrow D_V(x) := D(x) \cap (\mathfrak{g} \times \{0_{T_x M}\}) \subseteq \mathfrak{g} \times T_x M$$

is a vector subbundle of A. Then we can choose a local basis $\{\xi_a\}_{1 \leqslant a \leqslant r}$ of $\Gamma(\tau_{D_V})$, with $\xi_a : U \subseteq M \longrightarrow \mathfrak{g}$ smooth maps, and a local basis $\{X_A\} = \{\xi_a, (\eta_\alpha, Y_\alpha)\}$ of $\Gamma(\tau_D)$, with $\eta_\alpha : U \subseteq M \longrightarrow \mathfrak{g}$ and $Y_\alpha \in \mathfrak{X}(U)$.
Moreover, if (q^A) are local coordinates on $U \subseteq M$ and $Y_\alpha = Y_\alpha^i \frac{\partial}{\partial x^i}$, the *Lagrange-D'Alembert-Poincaré equations* are:

$$\begin{aligned} \dot{q}^A &= Y_\alpha^A y^\alpha, \\ \dot{y}^c &= -\Gamma_{ab}^c y^a y^b - \frac{\partial V}{\partial q^A} Y_\alpha^A (\mathcal{G}^D)^{c\alpha}, \\ \dot{y}^\alpha &= -\Gamma_{ab}^\alpha y^a y^b - \frac{\partial V}{\partial q^A} Y_\beta^A (\mathcal{G}^D)^{\alpha\beta}, \end{aligned}$$

where (q^A, y^c, y^α) are the corresponding local coordinates on $\mathcal{D}$.

Construction of nonholonomic integrators

- $L_d : Q \times Q \to \mathbb{R}$ a *regular discrete Lagrangian*
- *The constraint distribution* $\mathcal{D}$
- *The discrete constraint embedded submanifold* $\mathcal{M}_c$
  $i_{\mathcal{M}_c} : \mathcal{M}_c \to Q \times Q$ is an embedded submanifold of $Q \times Q$

Assumption

$$\dim \mathcal{M}_c = \dim \mathcal{D}$$

$(L_d, \mathcal{M}_c, \mathcal{D}) \equiv$ *a discrete nonholonomic Lagrangian system on* $Q \times Q$

$$(q_0, q_1) \in \mathcal{M}_c$$

$$((q_0, q_1), (q_1, q_2)) \text{ is a solution}$$

$$\Updownarrow$$

$$(q_1, q_2) \in \mathcal{M}_c$$

$$D_2 L_d(q_0, q_1) + D_1 L_d(q_1, q_2) = \lambda_\alpha \mu^\alpha(q_1)$$

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