Central sequence C^*-algebras and absorption of the Jiang-Su algebra

(Joint work with Eberhard Kirchberg)

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Outline

Let A be a unital C^*-algebra, and let ω be a free (ultra) filter on $\mathbb{N}$. Consider the central sequence C^*-algebra $A_\omega \cap A'$, where

$$A_\omega = \ell^\infty(A)/c_\omega(A), \quad c_\omega(A) = \big\{(x_n) \in \ell^\infty(A) \mid \lim_\omega \|x_n\| = 0\big\}.$$

What do we know about central sequence C^*-algebra $A_\omega \cap A'$?

Theorem (Kirchberg, 1994)

If A is a unital Kirchberg algebra (i.e., A is unital simple purely infinite separable and nuclear) and if ω is a free ultrafilter on $\mathbb{N}$, then $A_\omega \cap A'$ is simple and purely infinite.
In particular, $\mathcal{O}_\infty \hookrightarrow A_\omega \cap A'$, which entails that $A \cong A \otimes \mathcal{O}_\infty$.

Fact: $A \cong A \otimes \mathcal{Z} \iff \exists$ unital *-homomorphism $\mathcal{Z} \to A_\omega \cap A'$ for some free filter ω.

Example

If A is unital and approximately divisible, then $\bigotimes_{k=1}^\infty (M_2 \oplus M_3)$ maps unitally into $A_\omega \cap A'$. Hence $\mathcal{Z} \hookrightarrow A_\omega \cap A'$, so $A \cong A \otimes \mathcal{Z}$.

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Fact: If M is a II_1 von Neumann factor and if ω is a free ultrafilter, then $M^\omega \cap M'$ is either a II_1 von Neumann algebra or it is abelian.

If the former holds, then M is said to be a *McDuff factor*, and in this case $\mathcal{R} \hookrightarrow M^\omega \cap M'$ which entails that $M \cong M\bar{\otimes}\mathcal{R}$.

Theorem (Strengthened version of a theorem of Matui-Sato)

Let A be a unital separable C^*-algebra with a faithful trace τ. Let $M = \pi_\tau(A)''$, and let ω be a free ultrafilter on $\mathbb{N}$. Then the canonical map

$$A_\omega \cap A' \to M^\omega \cap M'$$

is surjective.
In particular, if A is non-elementary, unital, simple, nuclear and stably finite, then a quotient of $A_\omega \cap A'$ contains a subalgebra isomorphic to $\mathcal{R}$.

Matui and Sato proved the theorem above under the additional assumptions that A is simple and nuclear.

Idea of proof: The inclusion $A \to M$ induces a *-homomorphism $\Phi\colon A_\omega \to M^\omega$ which is *surjective* (by Kaplanski's density theorem).

Let $\pi_\omega\colon \ell^\infty(A) \to A_\omega$ be the quotient mapping and put $\widetilde{\Phi} = \Phi \circ \pi_\omega\colon \ell^\infty(A) \to M^\omega$.

Enough to show that if $b = (b_1, b_2, \dots) \in \ell^\infty(A)$ is such that $\widetilde{\Phi}(b) \in M^\omega \cap M'$, then $\exists c = (c_1, c_2, \dots) \in \ell^\infty(A)$ st $\widetilde{\Phi}(c) = \widetilde{\Phi}(b)$ and $\pi_\omega(c) \in A_\omega \cap A'$.

Put $D = C^*(A, b) \subseteq \ell^\infty(A)$ and put $J = \mathrm{Ker}(\widetilde{\Phi}|_D)$. Let $(e^{(k)}) \subseteq J$ be an asymptocially central approximate unit for J. Note that $ba - ab \in J$ for all $a \in A$. Hence, for all $a \in A$:

$$\begin{aligned} 0 &= \lim_{k\to\infty} \|(1-e^{(k)})(ba-ab)(1-e^{(k)})\| \\ &= \lim_{k\to\infty} \|(1-e^{(k)})b(1-e^{(k)})a - a(1-e^{(k)})b(1-e^{(k)})\|. \end{aligned}$$

We can therefore take $c_n = (1-e_n^{(k_n)})b_n(1-e_n^{(k_n)})$ for a suitable sequence (k_n).

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\begin{aligned}
0 &= \lim_{k\to\infty} \|(1 - e^{(k)})(ba - ab)(1 - e^{(k)})\| \\
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We can therefore take $c_n = (1 - e_n^{(k_n)})b_n(1 - e_n^{(k_n)})$ for a suitable sequence (k_n).

Example

There exist non-elementary, unital, simple, separable, nuclear (stably finite) C^*-algebras A that do not absorb the Jiang-Su algebra. E.g.:

- Villadsen's examples of simple AH-algebras with strongly perforated K_0-groups or with stable rank > 1.
- The example of a simple unital nuclear separable C^*-algebra with a finite and an infinite projection, [R], (which also provided a counterexample to the Elliott conjecture).
- Toms' refined counterexamples to the Elliott conjecture (which are AH-algebras).
- Many others!

For any of the C^*-algebras mentioned above, $\mathcal{Z}$ does not embed unitally into $A_\omega \cap A'$. For the stably finite ones, we still have a surjection $A_\omega \cap A' \to \mathcal{R}^\omega \cap \mathcal{R}'$, so $A_\omega \cap A'$ is not small (or abelian).

Proposition (Kirchberg (Abel Proceedings))

Let A and D be unital separable C^*-algebras, and let ω be a free filter on $\mathbb{N}$. If there is a unital *-hom $D \to A_\omega \cap A'$, then there is a unital *-hom

$$\bigotimes_{n=1}^{\infty} D \to A_\omega \cap A'$$

(where $\otimes = \otimes_{\max}$).

Lemma

If $A_\omega \cap A'$ has no character, then $\exists$ unital separable $D \subseteq A_\omega \cap A'$ st D has no character.

Corollary

If A is separable and $A_\omega \cap A'$ has no character, then $\exists$ unital C^*-algebra D with no characters and a unital *-homomorphism $\bigotimes_{n=1}^{\infty} D \to A_\omega \cap A'$.

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Hence, if there is a unital *-homomorphism $D \to A_\omega \cap A'$ (where D has no character), then there is a unital *-homomorphism

$$A \otimes \left(\bigotimes_{n=1}^{\infty} D \right) \to A_\omega, \quad \text{st. } a \otimes 1 \mapsto a, \ (a \in A).$$

Theorem (Dadarlat–Toms)

Let D be a unital C^*-algebra. If $\bigotimes_{k=1}^{\infty} D$ contains a unital subhomogeneous C^*-algebra without characters, then $\mathcal{Z} \hookrightarrow \bigotimes_{k=1}^{\infty} D$.

Hence: $A \cong A \otimes \mathcal{Z}$ if and only if $A_\omega \cap A'$ contains a unital subhomogeneous C^*-algebra without characters.

Fact: $\exists\ I(2,3) \to A_\omega \cap A'$ unital *-hom (and hence $\mathcal{Z} \hookrightarrow A_\omega \cap A'$) if $\exists a, b \in A_\omega \cap A'$ positive contractions st

$$a \sim b, \qquad a \perp b, \qquad 1 - a - b \precsim (a - \varepsilon)_+,$$

i.e., if there exists *-hom $CM_2 \to A_\omega \cap A'$ with "large image".

Hence, if there is a unital *-homomorphism $D \to A_\omega \cap A'$ (where D has no character), then there is a unital *-homomorphism

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Question

Suppose that A is a unital separable C^*-algebra st $A_\omega \cap A'$ has no characters (for some ultrafilter ω). Does it follow that $A_\omega \cap A'$ contains a unital copy of $\mathcal{Z}$ (so that $A \cong A \otimes \mathcal{Z}$)?

By the result of Dadarlat–Toms, this question is equivalent to the question if $\bigotimes_{n=1}^{\infty} D$ contains a unital copy of a subhomogeneous C^*-algebra without characters whenever D is a unital C^*-algebra without characters.

Definition

A unital C^*-algebra is said to have the *splitting property* if there are positive full elements $a, b \in A$ with $a \perp b$.

Note: A has the splitting property $\implies$ A has no characters.

The opposite implication is false in general, but it may be true if $A = \bigotimes_{n=1}^{\infty} D$ for some unital D. I don't know.

Lemma

If $A_\omega \cap A'$ has the splitting property, then there is a full *-homomorphism $CM_2 \to A_\omega \cap A'$.

Using results of [L. Robert + R] about divisibility properties for C^*-algebras we obtain:

Proposition

Let A be a unital separable C^*-algebra and let ω be a free ultrafilter on $\mathbb{N}$.

1. *If $A_\omega \cap A'$ has no characters, then A has the strong Corona Factorization Property.*
2. *If $A_\omega \cap A'$ has the splitting property, then $\exists N_k \in \mathbb{N}$ st*
   1. $\forall k \geq 2\ \forall y \in \mathrm{Cu}(A)\ \exists x \in \mathrm{Cu}(A) : kx \leq y \leq N_k x$.
   2. *Let $x, y \in \mathrm{Cu}(A)$. If $N_k x \leq ky$ for some $k \geq 1$, then $x \leq y$.*
3. *If $A_\omega \cap A'$ has the splitting property and A is simple, then A is either stably finite or purely infinite.*

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Corollary

There exist non-elementary, unital, simple, separable, nuclear C^*-algebras A st $A_\omega \cap A'$ has a character (and, at the same time, a sub-quotient $\cong \mathcal{R}$).

Outline

A year ago, the following remarkable result was proved:

Theorem (Matui–Sato)

Let A be a unital, separable, simple, non-elementary, stably finite, nuclear C^*-algebra, and suppose that $\partial_e T(A)$ is finite. Then the following are equivalent:

1. $A \cong A \otimes \mathcal{Z}$,
2. *A has strict comparison (i.e., $\mathrm{Cu}(A)$ is almost divisible),*
3. *Every cp map $A \to A$ can be excised in small central sequences,*
4. *A has property (SI).*

We get back to the properties mentioned in (3) and (4).

Note that if A is not stably finite, then $T(A) = \emptyset$ and (2) implies that A is purely infinite. Hence A is a Kirchberg algebra and $A \cong A \otimes \mathcal{O}_\infty \cong A \otimes \mathcal{Z}$.

It would be desirable to remove the condition that $\partial_e T(A)$ is finite!

A unital C^*-algebra with $T(A) \neq \emptyset$. Define

$$\|a\|_{2,\tau} = \tau(a^*a)^{1/2}, \qquad \|a\|_2 = \sup_{\tau \in T(A)} \|a\|_{2,\tau}, \quad a \in A.$$

Define $\|\cdot\|_2$ on A_ω by

$$\|\pi_\omega(a_1, a_2, a_3, \dots)\|_2 = \lim_\omega \|a_n\|_2,$$

where $\pi_\omega \colon \ell^\infty(A) \to A_\omega$ is the quotient map. Set

$$J_A = \{x \in A_\omega : \|a\|_2 = 0\} \ \lhd \ A_\omega.$$

Definition (Matui–Sato)

A unital simple C^*-algebra A is said to have *property (SI)* if for all positive contractions $e, f \in A_\omega \cap A'$ such that

$$e \in J_A, \qquad \sup_k \|1 - f^k\|_2 < 1,$$

there is $s \in A_\omega \cap A'$ with $fs = s$ and $s^*fs = e$.

Proposition

Let A be a separable, simple, unital, stably finite C^*-algebra with property (SI). TFAE:

1. $A \cong A \otimes \mathcal{Z}$,
2. $\exists$ *unital* **-homomorphism* $M_2 \to (A_\omega \cap A')/J_A$.
3. $\forall$ *UHF-algebras* B $\exists$ *unital* **-hom* $B \to (A_\omega \cap A')/J_A$.

Fact: $(2) + (\mathrm{SI}) \implies \exists$ unital *-hom $I(2,3) \to A_\omega \cap A' \implies (1)$.

Theorem

If A is a non-elementary, unital, simple, separable, exact, stably finite C^*-algebra st

1. $\pi_\tau(A)''$ *is McDuff factor for all* $\tau \in \partial_e T(A)$.
2. $\partial_e T(A)$ *is (weak* **) closed in* $T(A)$ *(i.e.,* $T(A)$ *is a Bauer simplex).*
3. $\partial_e T(A)$ *has finite covering dimension,*

then there is a unital **-homomorphism* $M_2 \to (A_\omega \cap A')/J_A$.

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then there is a unital **-homomorphism* $M_2 \to (A_\omega \cap A')/J_A$.

Results similar to the ones above and below have been obtained independently by Andrew Toms, Stuart White and Wilhelm Winter.

Corollary

Let A be a non-elementary, unital, simple, separable, exact, stably finite C^*-algebra st

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3. $\partial_e T(A)$ *has finite covering dimension.*
4. A *has property (SI)*

Then $A \cong A \otimes \mathcal{Z}$.

- Note that $A \cong A \otimes \mathcal{Z}$ implies (1), but not (2) and (3).
- It is not known if $A \cong A \otimes \mathcal{Z}$ implies (4).

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Definition (Matui–Sato)

A cp map $\varphi\colon A \to A \subseteq A_\omega$ can be *excised in small central sequences* if for all positive contractions $e, f \in A_\omega \cap A'$ with

$$e \in J_A, \qquad \sup_k \|1 - f^k\|_2 < 1,$$

there exists $s \in A_\omega$ st

$$fs = s, \qquad s^* as = \varphi(a)e, \qquad a \in A.$$

Proposition (Matui–Sato)

Let A be a unital simple C^*-algebra.

1. *If $\mathrm{id}_A\colon A \to A$ can be excised in small central sequences, then A has property (SI).*
2. *If A is simple, separable, unital and nuclear, and if A has strict comparison, then id_A can be excised in small central sequences.*

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Definition

Let A be a unital, simple, stably finite C^*-algebra. Then A has *local weak comparison* if there exists a constant $\gamma = \gamma(A)$ st for all positive element $a, b \in A$:

$$\gamma \cdot \sup_{\tau \in QT(A)} d_\tau(a) < \inf_{\tau \in QT(A)} d_\tau(b) \implies a \precsim b.$$

A has strict comparison $\iff$ $\mathrm{Cu}(A)$ is weakly unperforated $\implies$ $\mathrm{Cu}(A)$ has m-comparison for some $m < \infty$ (in the sense of Winter) $\implies$ A has local weak comparison.

Proposition

Let A be a unital, simple, stably finite C^*-algebra.

1. *If A has local weak comparison, then every nuclear cp $\varphi\colon A \to A$ can be excised in small central sequences.*
2. *If A is nuclear and has local weak comparison, then A has property (SI).*

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2. *If A is nuclear and has local weak comparison, then A has property (SI).*

Corollary

Let A be a non-elementary, stably finite, simple, separable, unital and nuclear C^*-algebra. Suppose that $\partial_e T(A)$ is closed in $T(A)$ and that $\partial_e T(A)$ has finite covering dimension. Then the following are equivalent:

1. $A \cong A \otimes \mathcal{Z}$,
2. *A has local weak comparison,*
3. *A has strict comparison ($\iff \mathrm{Cu}(A)$ is weakly unperforated).*

Question

Are (1), (2) and (3) above equivalent for all non-elementary, stably finite, simple, separable, unital and nuclear C^*-algebra?

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A bit about the proof. We want to find a unital **-homomorphism* $M_2 \to (A_\omega \cap A')/J_A$.

Proposition

Let B *be a unital* C^**-algebra, and let* $\varphi_1, \varphi_2, \dots, \varphi_m \colon M_2 \to B$ *be cpc order zero maps with commuting images.*

1. *If* $\varphi_1(1) + \varphi_2(2) + \cdots + \varphi_m(1) \leq 1$, *then there is a cpc order zero map* $\psi \colon M_2 \to B$ *such that*

$$\psi(1) = \varphi_1(1) + \varphi_2(2) + \cdots + \varphi_m(1).$$

2. *If* $\varphi_1(1) + \varphi_2(2) + \cdots + \varphi_m(1) = 1$, *then* $\psi \colon M_2 \to B$ *from (i) is a* **-homomorphism.*

Hence it suffices to find cp order zero maps

$$W_1, W_2 \dots, W_m \colon M_2 \to (A_\omega \cap A')/J_A$$

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Let now $\tau \in \partial_e T(A)$.

Apply the fact that $A_\omega \cap A' \to M_\tau^\omega \cap M_\tau{}'$ is onto and the assumption that M_τ is McDuff to find:

$$\begin{aligned} \varphi\colon &\quad M_2 \to M_\tau^\omega \cap M_\tau{}' && \text{(unital }^*\text{-homomorphism)} \\ V_0\colon &\quad M_2 \to A_\omega \cap A' && \text{(ucp lift of } \varphi) \\ V = (V_1, V_2, V_3, \dots)\colon &\quad M_2 \to \ell^\infty(A) && \text{(ucp lift of } V_0) \end{aligned}$$

Lemma

The ucp maps $V_n\colon M_2 \to A$ satisfy:

1. $\lim_\omega \tau\big(V_n(b^*b) - V_n(b)^* V_n(b)\big) = 0$ *for all* $b \in M_2$.
2. $\lim_\omega \|[a, V_n(b)]\| = 0$ *for all* $a \in A$ *and all* $b \in M_2$.

We must glue these maps (one for each trace) together!

We have a natural ucp map $\mathcal{T} \colon A \to C(\partial_e T(A))$ given by

$$\mathcal{T}(a)(\tau) = \tau(a), \qquad a \in A, \ \tau \in \partial_e T(A).$$

This induces a ucp map $\mathcal{T}_\omega \colon A_\omega \to C(\partial_e T(A))_\omega$

Proposition

If A is a unital separable C^*-algebra, for which $\partial_e T(A)$ is closed in $T(A)$, and if $\mathcal{A}$ denotes the multiplicative domain of $\mathcal{T}_\omega$, then $\mathcal{A} \subseteq (A_\omega \cap A') + J_A$, and

$$\mathcal{T}_\omega|_{\mathcal{A}} \colon \mathcal{A} \to C(\partial_e T(A))_\omega$$

is a *-isomorphism.

It follows that if $f_1, \dots, f_n \subseteq C(\partial_e T(A))$ are pairwise orthogonal positive contractions, $\varepsilon > 0$ and $F \subset A$ is finite, then there are pairwise orthogonal contractions $a_1, \dots, a_n \in A$ such that

$$\|\mathcal{T}(a_j) - f_j\|_\infty < \varepsilon, \quad \|\mathcal{T}(a_j^2) - f_j^2\|_\infty < \varepsilon, \quad \|[a, a_j]\| < \varepsilon, \quad a \in F.$$

We have a natural ucp map $\mathcal{T} \colon A \to C(\partial_e T(A))$ given by

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Suppose we are given:

- $\varepsilon > 0$ and $F \subseteq A$ finite,
- $V_1, V_2, \dots, V_k \colon M_2 \to A$ ucp maps,
- $U_1, U_2, \dots, U_k \subseteq \partial_e T(A)$ open, pairwise disjoint,
- $f_1, f_2, \dots, f_k \in C(\partial_e T(A))^+$ contractions; $\mathrm{supp}(f_j) \subseteq U_j$,
- $a_1, a_2, \dots, a_k \in A$ pairwise orthogonal positive contractions

such that

- $\tau\big(V_j(b^*b) - V_j(b)^*V_j(b)\big) < \varepsilon$ for all contractions $b \in M_2$ and all $\tau \in U_j$,
- $\|[a, V_j(b)]\| < \varepsilon$ for all contractions $b \in M_2$ and all $a \in F$,
- $\|[a, a_j]\| < \varepsilon$ for all $a \in F \cup \{$images of balls of the V_j's$\}$,
- $\|\mathcal{T}(a_j) - f_j\| < \varepsilon$ and $\|\mathcal{T}(a_j^2) - f_j^2\| < \varepsilon$

Then

$$W(b) = \sum_{j=1}^{k} a_j^{1/2} V_j(b) a_j^{1/2}, \qquad b \in M_2$$

defines a cp "tracially almost order zero" map $M_2 \to A$ with $W(1) = \sum_{j=1}^{m} a_j$.

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