Shift-minimal Groups

Workshop on Applications to Operator Algebras
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Robin D. Tucker-Drob

Department of Mathematics
California Institute of Technology
rtuckerd@caltech.edu

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Part I: Total Ergodicity.

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- For each $\sigma \in \Gamma$ we also have $\nu(\mathrm{Fix}(\sigma)) > 0$.

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- We have shown that the stabilizer subgroup Γ_x is locally finite for almost every $x \in X$

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Question

Is there a proof that all mixing actions are almost free which avoids Feit-Thompson?

Part II: Shift-minimality.

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Let $\Gamma \curvearrowright (X, \mu)$ be a non-trivial NA-ergodic action of Γ. Does there necessarily exist an amenable normal subgroup $N \leq \Gamma$ such that the stabilizer Γ_x is contained in N for almost every $x \in X$?

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It turns out that this is closely related to an open question concerning the reduced C^*-algebra of Γ.

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A countable group Γ is called *shift-minimal* if all of its non-trivial weak Bernoulli factors are free.

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- We obtain a representation $\int_X^{\oplus} \lambda_{\Gamma/\Gamma_x} \, d\mu$ of Γ on the Hilbert space $\mathcal{H} = \int_X^{\oplus} \ell^2(\Gamma/\Gamma_x) \, d\mu(x)$.

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- There is then an "obvious" vector such that the associate vector state τ is tracial. The action being non-free implies $\tau \neq \tau_\Gamma$. □

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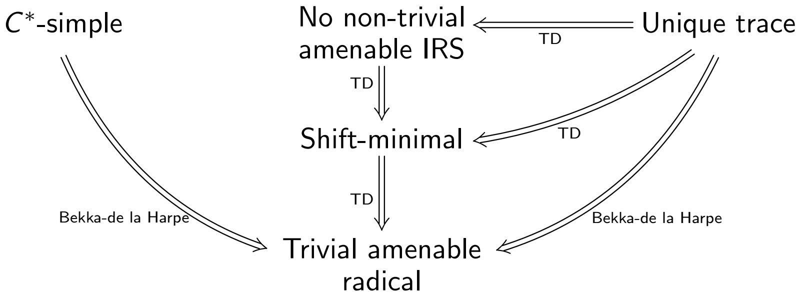
Question

If the first ℓ^2-Betti number of Γ is non-zero then is $C_r^*(\Gamma)$ simple with a unique tracial state?

Are they are all equivalent?



Results of T. Poznansky imply that these are all equivalent for Linear Groups.