# Lecture II: Stochastic volatility modeling in energy markets

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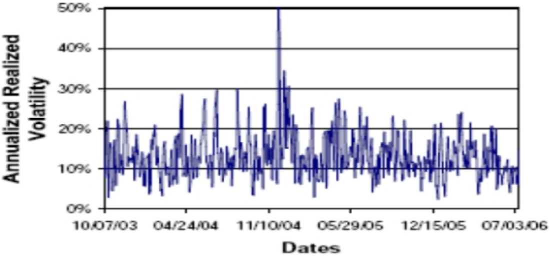
# Overview

1. Motivate and introduce a class of stochastic volatility models
2. Empirical example from UK gas prices
3. Comparison with the Heston model
4. Forward pricing
5. Discussion of generalizations to cross-commodity modelling

# Stochastic volatility model

# Motivation




- Annualized volatility of NYMEX sweet crude oil spot
  - Running five-day moving volatility
  - Plot from Hikspoors and Jaimungal 2008
- Stochastic volatility with fast mean-reversion

- Signs of stochastic volatility in financial time series
  - Heavy-tailed returns
  - Dependent returns
  - Non-negative autocorrelation function for squared returns
- Energy markets
  - Mean-reversion of (log-)spot prices
  - seasonality
  - Spikes
  - ... so, how to create reasonable stochastic volatility models?

# The stochastic volatility model

- Simple one-factor Schwartz model
  - but with stochastic volatility

$$S(t) = \Lambda(t) \exp(X(t)), \quad dX(t) = -\alpha X(t)\, dt + \sigma(t)\, dB(t)$$

- $\sigma(t)$ is a stochastic volatility (SV) process
  - Positive
  - Fast mean-reversion
- $\Lambda(t)$ deterministic seasonality function (positive)

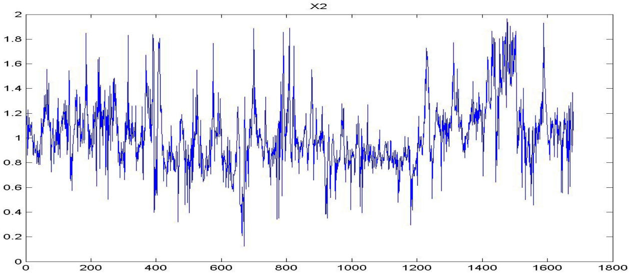
- Motivated by Barndorff-Nielsen and Shephard (2001): *n*-factor volatility model

$$\sigma^2(t) = \sum_{j=1}^{n} \omega_j Y_j(t)$$

where

$$dY_j(t) = -\lambda_j Y_j(t)\, dt + dL_j(t)$$

- $\lambda_j$ is the speed of mean-reversion for factor $j$
- $L_j$ are Lévy processes with only positive jumps
  - subordinators being driftless
  - $Y_j$ are all positive!
- The positive weights $\omega_j$ sum to one




- Simulation of a 2-factor volatility model
- Path of $\sigma^2(t)$

# Stationarity of the log-spot prices

- After de-seasonalizing, the log-prices become stationary

$$X(t) = \ln S(t) - \ln \Lambda(t) \sim \text{stationary}\,, \qquad t \to \infty$$

- The limiting distribution is a variance-mixture
  - Conditional normal distributed with zero mean

$$\ln S(t) - \ln \Lambda(t)|_{Z=z} \sim \mathcal{N}(0, z)$$

- $Z$ is characterized by $\sigma^2(t)$ and the spot-reversion $\alpha$

- Explicit expression the cumulant (log-characteristic function) of the stationary distribution of $X(t)$:

$$\psi_X(\theta) = \sum_{j=1}^{n} \int_0^{\infty} \psi_j \left( \frac{1}{2} \mathrm{i}\theta^2 \omega_j \gamma(u; 2\alpha, \lambda_j) \right) du$$

- $\psi_j$ cumulant of $L_j$
- The function $\gamma(u; a, b)$ defined as

$$\gamma(u; a, b) = \frac{1}{a-b} \left( \mathrm{e}^{-bu} - \mathrm{e}^{-au} \right)$$

- $\gamma$ is positive, $\gamma(0) = \lim_{u\to\infty} \gamma(u) = 0$, and has one maximum.

- Each term in the limiting cumulant of $X(t)$ can be written as the cumulant of centered normal distribution with variance

$$\widetilde{\psi}_X(\theta) = \int_0^\infty \psi_j\left(\theta\omega_j\gamma(u; 2\alpha, \lambda_j)\right) du$$

- One can show that $\widetilde{\psi}_X(\theta)$ is the cumulant of the stationary distribution of

$$\int_0^t \gamma(t-u; 2\alpha, \lambda_j)\, dL_j(u)$$

- Recall the constant volatility model $\sigma^2(t) = \sigma^2$
  - The Schwartz model
- Explicit stationary distribution

$$\ln S(t) - \ln \Lambda(t) \sim \mathcal{N}\left(0, \frac{\sigma^2}{2\alpha}\right)$$

- SV model gives heavy-tailed stationary distribution
  - Special cases: Gamma distribution, inverse Gaussian distribution....

# Probabilistic properties

- ACF of $X(t)$ is given as

$$\text{corr}(X(t), X(t+\tau)) = \exp(-\alpha\tau)$$

- No influence of the volatility on the ACF of log-prices
  - Energy prices have multiscale reversion
  - Above model is too simple, multi-factor models required

- Consider reversion-adjusted returns over $[t, t + \Delta)$

$$R_\alpha(t, \Delta) := X(t) - \mathrm{e}^{-\alpha\Delta} X(t-1) = \int_t^{t+\Delta} \sigma(s) \mathrm{e}^{-\alpha(t+\Delta-s)} \, dB(s)$$

- Approximately,

$$R_\alpha(t, \Delta) \approx \sqrt{\frac{1 - \mathrm{e}^{-2\alpha\Delta}}{2\alpha}} \sigma(t) \Delta B(t)$$

- $R_\alpha(t,\Delta)$ is a variance-mixture model

$$R_\alpha(t,\Delta)|\sigma^2(t) \sim \mathcal{N}(0, \frac{1-e^{-2\alpha\Delta}}{2\alpha}\sigma^2(t))$$

- Thus, knowing the stationary distribution of $\sigma^2(t)$, we can create distributions for $R_\alpha(t,\Delta)$
  - Based on empirical observations of $R_\alpha(t,\Delta)$, we can create desirable distributions from the variance mixture
- The reversion-adjusted returns are uncorrelated

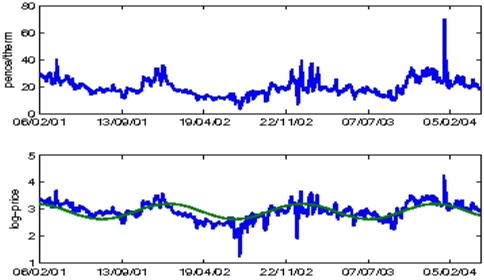
- ...but squared reversion-adjusted returns are correlated

$$\operatorname{corr}(R_\alpha^2(t+\tau,\Delta), R_\alpha^2(t,\Delta)) = \sum_{j=1}^{n} \widehat{\omega}_j e^{-\lambda_j \tau}$$

- $\widehat{\omega}_j$ positive constants summing to one, given by the second moments of $L_j$
- ACF for squared reversion-adjusted returns given as a sum of exponentials
  - Decaying with the speeds $\lambda_j$ of mean-reversions
- This can be used in estimation

# Empirical example: UK gas prices

- NBP UK gas spot data from 06/02/2001 till 27/04/2004
  - 806 records, weekends and holidays excluded
- Seasonality modelled by a sine-function for log-spot prices



- Estimate $\alpha$ by regressing $\ln \widetilde{S}(t+1)$ against $\ln \widetilde{S}(t)$

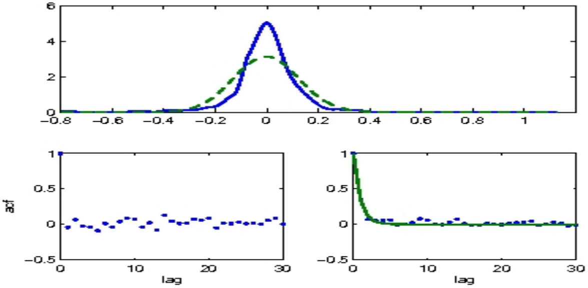
$$\widetilde{\alpha} = 0.127$$

- Regression has $R^2 = 78\%$
- Half-life: expected time until a shock is halfed in size

$$\text{half life} = \frac{\ln 2}{\widetilde{\alpha}}$$

- Half-life corresponding to 5.5 days

- Plot of residuals: histogram, ACF and ACF of squared residuals
  - Fitted speed of mean-reversion of volatility: $\widehat{\lambda} = 1.1$.



# The normal inverse Gaussian distribution

- The residuals are not reasonably modelled by the normal distribution
  - Peaky in the center, heavy tailed
- Motivated from finance, use the normal inverse Gaussian distribution (NIG)
  - Barndorff-Nielsen (1998)
- Four-parameter family of distributions
  - $a$: tail heaviness
  - $\delta$: scale (or volatility)
  - $\beta$: skewness
  - $\mu$: location

- Density of the NIG

$$f(x; a, \beta, \delta, \mu) = c \exp(\beta(x-\mu)) \frac{K_1\left(a\sqrt{\delta^2 + (x-\mu)^2}\right)}{\sqrt{\delta^2 + (x-\mu)^2}}$$

where $K_1$ is the modified Bessel function of the third kind with index one

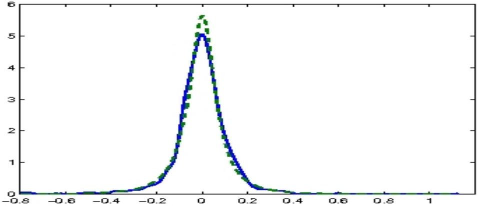
$$K_1(x) = \frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}x(z+z^{-1})\right)\, dz$$

- Explicit (log-)moment generating function

$$\phi(u) := \ln \mathbb{E}[e^{uL}] = u\mu + \delta\left(\sqrt{a^2 - \beta^2} - \sqrt{a^2 - (\beta + u)^2}\right)$$

- Fitted symmetric centered NIG using maximum likelihood

$$\widehat{a} = 4.83\,, \qquad \widehat{\delta} = 0.071$$



- Question: Does there exist SV driver $L$ such that residuals become NIG distributed?
- Answer is YES!
- There exists $L$ such that stationary distribution of $\sigma^2(t)$ is Inverse Gaussian distributed
  - Let $Z$ be normally distributed
  - The positive part of $1/Z$ is then Inverse Gaussian
- Conclusion:
  - Choose $L$ such that $\sigma^2(t)$ is Inverse Gaussian with specified parameters from the NIG estimation
  - Choose $\alpha$, $\lambda$ as estimated
  - Choose the seasonal function as estimated
  - Full specification of the SV volatility spot price dynamics

# The Heston Model: a comparison

- Heston's stochastic volatility: $\sigma^2(t) = Y(t)$,

$$dY(t) = \eta(\zeta - Y(t))\,dt + \delta\sqrt{Y(t)}\,d\widetilde{B}(t)$$

- $\widetilde{B}$ independent Brownian motion of $B(t)$
  - In general Heston, $\widetilde{B}$ correlated with $B$
  - Allows for leverage
- $Y$ recognized as the Cox-Ingersoll-Ross dynamics
  - Ensures positive $Y$

- The cumulant of stationary $Y$ is known (Cox, Ingersoll and Ross, 1981)

$$\psi_Y(\theta) = \zeta c \ln\left(\frac{c}{c - \mathrm{i}\theta}\right), \qquad c = 2\eta/\delta^2$$

- Cumulant of a $\Gamma(c, \zeta c)$-distribution
- Can obtain the same stationary distribution from our SV-model

- Choose a one-factor model $\sigma^2(t) = Y(t)$

$$dY(t) = -\lambda Y(t)\,dt + dL(t)$$

- $L(t)$ a compound Poisson process with exponentially distributed jumps with expected size $1/c$
- Choose $\lambda$ and the jump frequency $\rho$ such that $\rho/\lambda = \zeta c$
- Stationary distribution of $Y$ is $\Gamma(c, \zeta c)$.

- Question: what is the stationary distribution of $X(t)$ under the Heston model?
- Expression for the cumulant at time $t$

$$\psi_X(t,\theta) = \mathrm{i}\theta X(0)\mathrm{e}^{-\alpha t} + \ln \mathbb{E}\left[\exp\left(-\frac{1}{2}\theta^2 \int_0^t Y(s)\mathrm{e}^{-2\alpha(t-s)}\,ds\right)\right]$$

- An expression for the last expectation is unknown to us
  - The cumulant can be expressed as an affine solution
  - Coefficients solutions of Riccati equations, which are not analytically solvable
  - ...at least not to me....
- In our SV model the same expression can be easily computed

# Application to forward pricing

- Forward price at time $t$ an delivery at time $T$

$$F(t,T) = \mathbb{E}_Q\left[S(T) \mid \mathcal{F}_t\right]$$

- $Q$ an equivalent probability, $\mathcal{F}_t$ the information filtration
- Incomplete market
  - No buy-and-hold strategy possible in the spot
  - Thus, no restriction to have $S$ as $Q$-martingale after discounting

- Choose $Q$ by a Girsanov transform

$$dW(t) = dB(t) - \frac{\theta(t)}{\sigma(t)}\,dt$$

- $\theta(t)$ bounded measurable function
  - Usually simply a constant
  - Known as the *market price of risk*
- Novikov's condition holds since

$$\sigma^2(t) \geq \sum_{j=1}^{n} \omega_j Y_j(0) e^{-\lambda_j t}$$

- The $Q$ dynamics of $X(t)$, the deseasonalized log-spot price

$$dX(t) = (\theta(t) - \alpha X(t))\, dt + \sigma(t)\, dW(t)$$

- For simplicity it is supposed that there is no market price of volatility risk
  - No measure change of the $L_j$'s
- Esscher transform could be applied
  - Exponential tilting of the Lévy measure, preserving the Lévy property
  - Will make big jumps more or less pronounced
  - Scale the jump frequency

- Analytical forward price available (suppose one-factor SV for simplicity)

$$F(t,T) = \Lambda(T) H_\theta(t,T) \exp\left(\frac{1}{2}\gamma(T-t;2\alpha,\lambda)\sigma^2(t)\right) \times \left(\frac{S(t)}{\Lambda(t)}\right)^{\exp(-\alpha(T-t))}$$

- Recall the scaling function

$$\gamma(u;2\alpha,\lambda) = \frac{1}{2\alpha-\lambda}\left(\mathrm{e}^{-\lambda u} - \mathrm{e}^{-2\alpha u}\right)$$

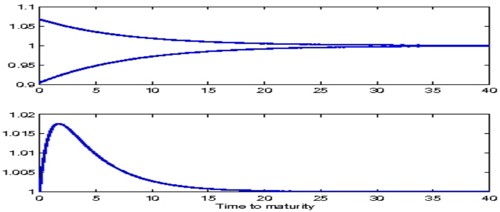
- $H_\theta$ is a risk-adjustment function

$$\ln H_\theta(t,T) = \int_t^T \theta(u) e^{-\alpha(T-s)}\,ds + \int_0^{T-t} \psi(-\mathrm{i}\frac{1}{2}\gamma(u;2\alpha,\lambda))\,du$$

- Here, $\psi$ being cumulant of $L$
- Note: Forward price may jump, although spot price is continuous
  - The volatility is explicitly present in the forward dynamics

- Recall $\gamma(0; 2\alpha, \lambda) = \lim_{u \to \infty} \gamma(u; 2\alpha, \lambda) = 0$
  - In the short and long end of the forward curve, the SV-term will not contribute
- Scale function has a maximum in $u^* = (\ln 2\alpha - \ln \lambda)/(2\alpha - \lambda)$
  - Increasing for $u < u^*$, and decreasing thereafter
  - Gives a hump along the forward curve
  - Hump size is scaled by volatility level $Y(t)$
- Many factors in the SV model gives possibly several humps
- Observe that the term $(S(t)/\Lambda(t))^{\exp(-\alpha(T-t))}$ gives
  - backwardation when $S(t) > \Lambda(t)$
  - Contango otherwise

- Shapes from the "deseasonalized spot"-term in $F(t, T)$ (top) and SV term (bottom)
- The hump is produced by the scale function $\gamma$
- Parameters chosen as estimated for the UK spot prices



# Forward price dynamics

$$\frac{dF(t,T)}{F(t-,T)} = \sigma(t)e^{-\alpha(T-t)}\,dW(t) + \sum_{j=1}^{n}\int_0^{\infty}\left\{e^{\omega_j\gamma(T-t;2\alpha,\lambda_j)z/2}-1\right\}\widetilde{N}_j(dz,dt)$$

- $\widetilde{N}$ compensated Poisson random measure of $L_j$
- Samuelson effect in $dW$-term. The jump term goes to zero as $t \to T$

# Comparison with the Heston model

- Forward price dynamics

$$F(t,T) = \Lambda(T) G_\theta(t,T) \exp\left(\xi(T-t)Y(t)\right) \left(\frac{S(t)}{\Lambda(t)}\right)^{\exp(-\alpha(T-t))}$$

where

$$\ln G_\theta(t,T) = \int_t^T \theta(u) e^{-\alpha(T-u)}\, du + \eta\zeta \int_0^{T-t} \xi(u)\, du$$

- $\xi(u)$ solves a Riccati equation

$$\xi'(u) = \delta\left(\xi(u) - \frac{\eta}{2\delta}\right)^2 - \frac{\eta^2}{4\delta} + \frac{1}{2}e^{-2\alpha u}$$

- Initial condition $\xi(0) = 0$
- It holds $\lim_{u\to\infty} \xi(u) = 0$ and $\xi$ has one maximum for $u = u^* > 0$
  - Shape much like $\gamma(u; 2\alpha, \lambda)$

# Extensions of the SV model

# Spikes and inverse leverage

- Spikes: sudden large price increase, which is rapidly killed off
  - sometimes also negative spikes occur
- Inverse leverage: volatility increases with increasing prices
  - Effect argued for by Geman, among others
  - Is it an effect of the spikes?

- Spot price model

$$S(t) = \Lambda(t) \exp\left( X(t) + \sum_{i=1}^{m} Z_i(t) \right)$$

where

$$dZ_i(t) = (a_i - b_i Z_i(t))\, dt + d\widetilde{L}_i(t)$$

- Spikes imply that $b_i$ are fast mean-reversions
- Typically, $\widetilde{L}_i$ are time-inhomogeneous jump processes, with only *positive* jumps
    - Negative spikes: must choose $\widetilde{L}_i$ having negative jumps

- Inverse leverage: Let $\widetilde{L}_i = L_i$ for one or more of the jump processes
- A spike in the spot price will drive up the vol as well
  - Or opposite, an increase in volatility leads to an increase (spike) in the spot
- Spot model analytically tractable
  - Stationary, with analytical cumulant
  - Probabilistic properties available
  - Forward prices analytical in terms of cumulants of the noises

# Cross-commodity modelling

- Suppose that $X(t)$ and $Z_i(t)$ are vector-valued Ornstein-Uhlenbeck processes
- The volatility structure follows Stelzer's multivariate extension of the BNS SV model

$$dX(t) = AX(t)\,dt + \Sigma(t)^{1/2}\,dW(t)$$

- $A$ is a matrix with eigenvalues having negative real parts
  - ...to ensure stationarity
- $\Sigma(t)$ is a matrix-valued process, $W$ is a vector-Brownian motion

- The volatility process:

$$\Sigma(t) = \sum_{j=1}^{n} \omega_j Y_j(t)$$

where

$$dY_j(t) = \left( C_j Y_j(t) + Y_j(t) C_j^T \right) dt + dL_j(t)$$

- $C_j$ are matrices with eigenvalues having negative real part
  - ...again to ensure stationarity
- $L_j$ are matrix-valued subordinators
- The structure ensures that $\Sigma(t)$ becomes positive definite

- Modelling approach allows for
  - Marginal modelling as above
  - Analyticity in forward pricing, say
  - Flexibility in linking different commodities
- However...not easy to estimate on data

# Conclusions

- Proposed an SV model for power/energy markets
- Discussed probabilistic properties, and compared with the Heston model
- Forward pricing, and hump-shaped forward curves
- Extensions to cross-commodity and multi-factor models
- Empirical example from UK gas spot prices

# Coordinates


- fredb@math.uio.no
- folk.uio.no/fredb
- www.cma.uio.no

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