# Introduction to Banach and Operator Algebras

## Lecture 6

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## Unitary Representations

In this lecture, we assume that all groups under consideration are discrete. Many results are still true for general locally compact groups.

Let $G$ be a discrete group. A unitary representation on a Hilbert space $H$ is a map

$$\pi_U : s \in G \to U_s \in \mathcal{U}(H), \text{ the unitary group in } B(H).$$

such that

$$U_s U_t = U_{st}.$$

In this case,

$$\pi_U : f = \sum \alpha_s \delta_s \in \ell_1(G) \to \pi_U(f) = \sum \alpha_s U_s \in B(H)$$

is a contractive unital *-homomorphism from $\ell_1(G)$ into $B(H)$, and $\pi(\ell_1(G)) = \{\pi(f) : f \in \ell_1(G)\}$ is a unital *-subalgebra in $B(H)$.

## Group C* and von Neumann Algebras

We let $C^*_\pi(G) = \pi(\ell_1(G))^{-\|\cdot\|}$ and $VN_\pi(G) = \pi(\ell_1(G))^{-s.o.t.}$ denote the group C*-algebra and group von Neumann algebra associated with the unitary representation $\pi$. In particular, for the left regular representation

$$\lambda : s \in G \to \lambda_s \in B(\ell_2(G)),$$

we get the reduced left group C*-algebra $C^*_\lambda(G)$ and the left group von Neumann algebra $VN_\lambda(G)$.

There is a universal representation

$$\pi_u : s \in G \to u_s = \oplus_\alpha U^\alpha_s \in B(\oplus_\alpha H_\alpha),$$

where the direct sum is taken over all non-equivalent classes of cyclic unitary representations. In this case, we can obtain the full group C*-algebra $C^*(G) = \pi_u(\ell_1(G))^{-\|\cdot\|}$.

It is known that there is a natural unital *-homomorphism

$$\pi_\lambda : C^*(G) \to C^*_\lambda(G)$$

from $C^*(G)$ onto $C^*_\lambda(G)$.

## Fourier Algebras $A(G)$

Let $A(G) = \{f : G \to \mathbb{C}$ such that $f(s) = \langle \lambda_s \xi | \eta \rangle\}$ be the space of all coeficient functions of the left regular representation $\lambda$. It was shown by Eymard in 1964 that $A(G)$ with the norm

$$\|f\|_{A(G)} = \inf\{\|\xi\|\|\eta\| : f(s) = \langle \lambda_s \xi | \eta \rangle\}$$

and pointwise multiplication

$$(fg)(s) = f(s)g(s)$$

is a commutative Banach algebra. We call $A(G)$ the Fourier algebra of $G$. It is known that we have the isometric identification $A(G) = VN_\lambda(G)_*$.

Therefore, if $G$ is an abelian group, then we have

$$C^*_\lambda(G) \cong C(\widehat{G}),\ VN_\lambda(G) \cong L_\infty(\widehat{G}),\ \text{and } A(G) \cong L_1(\widehat{G}).$$

## Fourier Stieltjes Algebras

We let $B(G) = \{f : G \to \mathbb{C}$ such that $f(s) = \langle u_s\xi|\eta\rangle\}$ be the space of all coefifcient functions of the universal unitary representation $\pi_u$ of $G$. Then $B(G)$ with the norm

$$\|f\|_{B(G)} = \{\|\xi\|\|\eta\| : f(s) = \langle u_s\xi|\eta\rangle\}$$

and the pointwise multiplication is a unital commutative Banach algebra. We call $B(G)$ the Fourier-Stieltjes algebra of $G$. In general, we have the isometric indetification

$$B(G) = C^*(G)^*.$$

A function $f : G \to \mathbb{C}$ is positive definite (or simply p.d.) if for any $s_1, \cdots, s_n \in G$, $[f(s_i^{-1}s_j)]$ is positive in $M_n(\mathbb{C})$.

Theorem: A function $f : G \to \mathbb{C}$ is p.d. if and only if $f(s) = \langle U_s\xi|\xi\rangle$ for some unitary representation $\pi_U$ of $G$.

Therefore, every p.d. function $f$ uniquely corresponds to a positive linear functional on $C^*(G)$.

## $B_\lambda(G)$

Moreover, we let $B_\lambda(G) = C^*_\lambda(G)^*$. Then the C*-quitent $\pi_\lambda : C^*(G) \to C^*_\lambda(G)$ induces an isometric inclusion

$$B_\lambda(G) \hookrightarrow B(G),$$

and by a standard duality argument, we have

$$C^*(G) \cong C^*_\lambda(G) \text{ if and only if } B(G) = B_\lambda(G).$$

In general, $A(G)$ and $B_\lambda(G)$ are two-sided ideals in $B(G)$ and we have the isometric inclusions

$$A(G) \hookrightarrow B_\lambda(G) \hookrightarrow B(G).$$

## Amenable Groups

A discrete group $G$ is amenable if $\ell_\infty(G)$ has a left invariant mean, i.e. there is a state $m : \ell_\infty(G) \to \mathbb{C}$ such that $m(s \cdot h) = m(h)$ for all $s \in G$ and $h \in \ell_\infty(G)$, where we let $s \cdot h(t) = h(s^{-1}t)$. Since $\ell_\infty(G)^* = \ell_1(G)^{**}$, this is equivalent to $\delta_s \star m = m$ for all $s \in G$.

Theorem: Let $G$ be a discrete group. TFAE:

(1) $G$ is amenable,

(2) There exists a net of $f_\alpha \geq 0$ in $\ell_1(G)$ such that $\|f_\alpha\|_1 = 1$ and $\|\delta_s \star f_\alpha - f_\alpha\|_1 \to 0$ for all $s \in G$,

(2') For every finite subset $F \subseteq G$ and $\varepsilon > 0$, there exists a $f \geq 0$ in $\ell_1(G)$ such that $\|f\|_1 = 1$ and $\|\delta_s \star f - f\|_1 < \varepsilon$ for all $s \in F$.

(3) $G$ satisfies the Følner condition, i.e. for any finite set $F \subseteq G$ and $\varepsilon > 0$, there exists a finite set $K \subseteq G$ such that $\frac{|s \cdot K \Delta K|}{|K|} < \varepsilon$ for all $s \in F$.

Theorem: Let $G$ be a discrete group. TFAE:

(1) $G$ is amenable,

(2) There exists a net of unit vectors $\xi_\alpha \in \ell_2(G)$ (with finite support) such that $\|\lambda_s\xi_\alpha - \xi_\alpha\|_2 \to 0$ for all $s \in G$,

(3) There exists a net of (positive definite) contractive $\varphi_\alpha \in A(G)$ (with finite support) such that $\varphi_\alpha(s) \to 1$ for all $s \in G$.

(4) $A(G)$ has a bounded appriximate identity,

(5) $C^*(G) = C^*_\lambda(G)$ or equivalently $B(G) = B_\lambda(G)$.

Outline of Proof: (1) $\Leftrightarrow$ (2) If $G$ is amenable, we get a net of positive functions $\{f_\alpha\}$ in (2) of previous theorem. Then $\xi_\alpha = f_\alpha^{\frac{1}{2}}$ is a net of unit vectors in $\ell_2(G)^+$ such that

$$\begin{aligned} \|\lambda_s\xi_\alpha - \xi_\alpha\|_2^2 &= \sum_{t\in G} |\xi_\alpha(s^{-1}t) - \xi_\alpha(t)|^2 \\ &\leq \sum_{t\in G} |\xi_\alpha(s^{-1}t) - \xi_\alpha(t)||\xi_\alpha(s^{-1}t) + \xi_\alpha(t)| \\ &= \sum_{t\in G} |f_\alpha(s^{-1}t) - f_\alpha(t)| = \|\delta_s \star f_\alpha - f_\alpha\|_1 \to 0. \end{aligned}$$

By an appropriate approximation, we can choose $\xi_\alpha$ with finite support.

On the other hand, if we have (2), then Then $f_\alpha = |\xi_\alpha|^2$ is a net of positive functions contained in $\ell_1(G)$ such that $\|f_\alpha\|_1 = 1$ and

$$\begin{aligned} \|\delta_s \star f_\alpha - f_\alpha\|_1 &= \sum_{t\in G} |f_\alpha(s^{-1}t) - f_\alpha(t)| \\ &= \sum_{t\in G} |\xi_\alpha(s^{-1}t) - \xi_\alpha(t)||\xi_\alpha(s^{-1}t) + \xi_\alpha(t)| \\ &\leq (\sum_{t\in G} |\xi_\alpha(s^{-1}t) - \xi_\alpha(t)|^2)^{\frac{1}{2}}(\sum_{t\in G} |\xi_\alpha(s^{-1}t) + \xi_\alpha(t)|^2)^{\frac{1}{2}} \\ &= \|\lambda_s\xi_\alpha - \xi_\alpha\|_2\|\lambda_s\xi_\alpha + \xi_\alpha\|_2 \leq 2\|\lambda_s\xi_\alpha - \xi_\alpha\|_2 \to 0. \end{aligned}$$

(2) $\Rightarrow$ (3) If we have (2), then $\varphi_\alpha(s) = \langle \lambda_s \xi_\alpha | \xi_\alpha \rangle$ is a net of positive definite contractive functions in $A(G)$ such that

$$|\varphi_\alpha(s) - 1| = |\langle \lambda_s \xi_\alpha - \xi_\alpha | \xi_\alpha \rangle| \to 0 \text{ for all } \; s \in G.$$

If $\xi_\alpha$ has a finite support, then so is $\varphi_\alpha$.

(3) $\Rightarrow$ (4) Suppose we have (3). We want to show that the net of contractive $\{\varphi_\alpha\}$ in $A(G)$ is an approximate identity of $A(G)$. Let us first note that each $\delta_s$ is contained in $A(G)$ since $\delta_s(t) = \langle \lambda_t \delta_e | \delta_s \rangle$. We also note that the linear span of $\{\delta_s : s \in G\}$ is norm dense in $A(G)$. So it sufficies to show that for all $s \in G$,

$$\|\varphi_\alpha \delta_s - \delta_s\|_{A(G)} = \|\varphi_\alpha(s)\delta_s - \delta_s\|_{A(G)} = |\varphi_\alpha(s) - 1| \|\delta_s\|_{A(G)} \to 0.$$

(4) $\Rightarrow$ (5) Suppose that $A(G)$ has a bounded approximate identity $\{\varphi_\alpha\}$. Reversing the above calculation, we can

$$|\varphi_\alpha(s) - 1| = |\varphi_\alpha(s) - 1| \|\delta_s\|_{A(G)} = \|\varphi_\alpha \delta_s - \delta_s\|_{A(G)} \to 0.$$

So $\varphi_\alpha(s) \to 1$ for all $s \in G$.

Let $\varphi \in B(G) = C^*(G)^*$. For any $s_1, \cdots, s_n \in G$, we have

$$\varphi(s_i) = 1 \cdot \varphi(s_i) = \lim_\alpha \varphi_\alpha(s_i)\varphi(s_i) = \lim_\alpha (\varphi_\alpha \varphi)(s_i)$$

Then for any $x = \sum a_i \pi_u(s_i) \in C^*(G)$, we have

$$\varphi(x) = \lim_\alpha (\varphi_\alpha \varphi)(x).$$

Since $\varphi_\alpha \varphi$ is a net of bounded elements in $A(G)$, we can conclude that for any $x \in \ker \pi_\lambda \subseteq C^*(G)$,

$$\varphi(x) = \lim_\alpha (\varphi_\alpha \varphi)(x) = \lim_\alpha (\varphi_\alpha \varphi)(\pi_\lambda(x)) = 0.$$

This shows that $ker \pi_\lambda = \{0\}$ and thus $\pi_\lambda$ is an isometric *-isomorphism from $C^*(G)$ onto $C^*_\lambda(G)$.

(5) $\Rightarrow$ (1) suppose $C^*(G) = C^*_\lambda(G)$. Then $A(G) \hookrightarrow B_\lambda(G) = B(G)$ is weak* dense in $B(G)$. For $1 \in B(G)$, we can find a net of unit vectors $\xi_\alpha \in \ell_2(G)^+$ such that for any $s \in G$,

$$1 = \lim_\alpha \langle \lambda_s \xi_\alpha | \xi_\alpha \rangle = \varphi_\alpha(s).$$

This implies (2), i.e.

$$\|\lambda_s \xi_\alpha - \xi_\alpha\|_2 = \|\lambda_s \xi_\alpha\|^2 + \|\xi_\alpha\|^2 - 2Re\langle \lambda_s \xi_\alpha | \xi_\alpha \rangle \to 0$$

for $s \in G$. So it follows from (1) $\Leftrightarrow$ (2) that $G$ is amenable.

## Completely Bounded and Completely Positive Maps

Let $A$ be a C*-algebra. Then for each $n \in \mathbb{N}$, there exists a unique C*-algebra norm on $M_n(A)$. Indeed, we can assume that $A \subseteq B(H)$. Then we can get a C*-algebra norm on $M_n(A)$ by the following identification

$$M_n(A) = \{[x_{ij}] : x_{ij} \in A\} \subseteq M_n(B(H)) = B(H^n).$$

If $T : x \in A \to T(x) \in B$ is a bounded linear map, then for each $n \in \mathbb{N}$, we obtain a bounded linear map $T_n : M_n(A) \to M_n(B)$ defined by

$$T_n([x_{ij}]) = [T(x_{ij}] \text{ for all } [x_{ij}] \in M_n(A).$$

$T$ is completely bounded (or simply cb) if $\|T\|_{cb} = \sup\{\|T_n\| : n \in \mathbb{N}\} < \infty$. $T$ is completely positive (or simply cp) if each $T_n : M_n(A) \to M_n(B)$ is positive.

Theorem: Every bounded/positive $T : A \to C(\Omega)$ (in particular, $T : A \to \mathbb{C}$) is cb/cp.

Theorem: If $T : A \to M_n(C(\Omega))$ is n-positive, then it is cp.

Theorem: A linear map $T : M_n(\mathbb{C}) \to B$ is cp if and only if for the matrix unit $\{e_{ij}\}$ of $M_n(\mathbb{C})$, $T_n([e_{ij}]) = [T(e_{ij})]$ is positive in $M_n(B)$.

**Stinespring/Arveson-Wittstock-Hahn-Banach Extension Theorem** Let $A \hookrightarrow B$ be C*-algebras and let $T : A \to B(H)$ be a cp/cb map. Then there exists a cp/cb map $\tilde{T} : B \to B(H)$ such that $\tilde{T}_{|A} = T$ and $\|\tilde{T}\|_{cb} = \|T\|_{cb}$.

**Theorem [Stinespring]:** Let $T : A \to B(H)$ be a cp map. Then there exist a Hilbert space $K$, a unital *-homomorphsim $\pi : A \to B(K)$, and a bounded linear map $V : H \to K$ such that

$$T(x) = V^*\pi(x)V$$

and $\|T\|_{cb} = \|V\|^2$.

**Theorem [Wittstock]:** Let $T : A \to B(H)$ be a cb map. Then there exist a Hilbert space $K$, a unital *-homomorphsim $\pi : A \to B(K)$, and bounded linear maps $V, W : H \to K$ such that

$$T(x) = V^*\pi(x)W$$

and $\|T\|_{cb} = \|V\|\|W\|$.

## C*-algebra Tensor Product and Nuclearity of C*-algebras

Let $A \subseteq B(H)$ and $B \subseteq B(K)$ be two C*-algebras. We can obtain a natural injective representation $A \otimes_{alg} B \subseteq B(H \otimes K)$. We define

$$A \otimes^{\text{min}} B = (A \otimes_{alg} B)^{-\|\cdot\|} \subseteq B(H \otimes K).$$

We define $A \otimes^{\text{max}} B$ to be the norm closure of $A \otimes_{alg} B$ under the norm

$$\|x\|_{\text{max}} = \text{sup}\{\|\pi_A \cdot \pi_B(x)\| = \|\sum \pi_A(x_i)\pi_B(y_i)\| \text{ if } x = \sum x_i \otimes y_i\},$$

where the supremum is taken over all representations : $\pi_A : A \to B(H)$ and $\pi_B : B \to B(H)$ with commutating range, i.e. $\pi_A(x)\pi_B(y) = \pi_B(y)\pi_A(x)$ for all $x \in A$ and $y \in B$. In general $\|\cdot\|_{\text{max}} \geq \|\cdot\|_{\text{min}}$ and the identity map on $a \otimes_{alg} B$ exntends to a C*-quotient map

$$A \otimes^{\text{max}} B \to A \otimes^{\text{min}} B.$$

A C*-algebra $A$ is nuclear (by C. Lance in the early 1970's) if for any C*-algebra $B$, we have the C*-isomorphism

$$A \otimes^{\text{max}} B = A \otimes^{\text{min}} B.$$

## Nuclear C*-algebras and Semidiscrete von Neumann Algebras

Theorem [Choi-Effros]; A C*-algebra $A$ is nuclear if and only if there exists two nets of cp and contractive maps $S_\alpha : A \to M_{n(\alpha)}$ and $T_\alpha : M_{n(\alpha)} \to A$ such that

$$\|T_\alpha \circ S_\alpha(x) - x\| \to 0 \text{ for all } x \in A.$$

A C*-algebra $A$ is said to have the CPAP if there exists a net of cp and contractive finite rank maps $T_\alpha : A \to A$ such that

$$\|T_\alpha(x) - x\| \to 0 \text{ for all } x \in A.$$

A von Neumann algebra $M$ is said to be semidiscrete if it has the weak* version of CPAP, i.e. there exists a net of weak* continuous cp and contractive finite rank maps $T_\alpha : M \to M$ such that

$$\langle T_\alpha(x) - x, \omega\rangle \to 0 \text{ for all } x \in M \text{ and } \omega \in M_*.$$

## Examples of Nuclear C*-algebras

Finite Dimensional C*-algebras $A = M_{n_1} \oplus \cdots \oplus M_{n_k}$,

Comm C*-algebra $C(\Omega)$

Rotation algebra $A_\theta$,

CAR algebra $A_{2^\infty}$,

Matrix algebras $M_n(A)$, inductive limit and $c_0$-direct sum of nulcear C*-algebras $A$ ...

**Theorem:** For discrete group $G$, we can easily prove that TFAE:

(1) $G$ is amenable,

(2) $C^*_\lambda(G)$ is nuclear,

(3) $C^*_\lambda(G)$ has the CPAP,

(4) $VN_\lambda(G)$ is semidiscrete.

**Outline of Proof:** (1) $\Rightarrow$ (2) Suppose that $G$ is amenable. It is known from the Følner condition that for any finite set $F$ in $G$ and $\varepsilon > 0$, there exists a finite subset $K_\alpha = K_{(F,\varepsilon)}$ in $G$ such that

$$\frac{|s \cdot K_\alpha \Delta K_\alpha|}{|K_\alpha|} < \varepsilon$$

for all $s \in F$.

Let $\iota_\alpha$ be the isometric inclusion $\ell_2(K_\alpha) \hookrightarrow \ell_2(G)$ and $P_\alpha : \ell_2(G) \to \ell_2(K_\alpha)$ be the projection. We obtain a complete contraction

$$S_\alpha : x \in C^*_\lambda(G) \to P_\alpha x \iota_\alpha \in B(\ell_2(K_\alpha)) = M_{n(\alpha)},$$

where $n(\alpha) = |K_\alpha|$ is the cardinality of $K_\alpha$.

Let $\{e^{\alpha}_{s,t}\}_{s,t\in K_\alpha}$ be the matrix unit of $B(\ell_2(K_\alpha))$. We can define a map

$$T_\alpha : e^{\alpha}_{s,t} \in B(\ell_2(K_\alpha)) = M_{n(\alpha)} \to \frac{\lambda_{st^{-1}}}{n(\alpha)} \in C^*_\lambda(G).$$

Now it is easy to verify that

$$e^{\alpha}_{s,s}\lambda_p(g)e^{\alpha}_{t,t} = \begin{cases} e^{\alpha}_{s,t} & \text{if } g = st^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for any $g \in G$, we have

$$S_\alpha(\lambda_g) = P_\alpha \lambda_g \iota_\alpha = \sum_{s,t\in K_\alpha} e^{\alpha}_{s,s}\lambda_g e^{\alpha}_{t,t} = \sum_{s\in K_\alpha \cap gK_\alpha} e^{\alpha}_{s,g^{-1}s},$$

and thus

$$T_\alpha \circ S_\alpha(\lambda_g) = \frac{|K_\alpha \cap gK_\alpha|}{n(\alpha)}\lambda_g.$$

It follows that

$$\|T_\alpha \circ S_\alpha(\lambda_g) - \lambda_g\| \le \frac{|F_\alpha \Delta gF_\alpha|}{n(\alpha)}\|\lambda_g\| < \varepsilon \qquad \text{for all } g \in E.$$

Therefore, we have $\|T_\alpha \circ S_\alpha(x) - x\| \to 0$ for every $x \in C^*_\lambda(G)$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Suppose that we have $C^*_\lambda(G)$ has the CPAP, i.e. there exists a net of cp finite rank contractions $T_\alpha : C^*_\lambda(G) \to C^*_\lambda(G) \subseteq B(\ell_2(G))$ such that $\|T_\alpha(x) - x\| \to 0$ for all $x \in C^*_\lambda(G)$.

Then we can get a net of functions $\{\varphi_\alpha\}$ on $G$ defined by

$$\varphi_\alpha(s) = \langle \lambda_s^* T_\alpha(\lambda_s)\delta_e|\delta_e\rangle = \langle T_\alpha(\lambda_s)\delta_e|\lambda_s\delta_e\rangle.$$

Since $T_\alpha$ are completely positive maps, each $\varphi_\alpha$ is a positive definite function contained in $B(G)$ and we have

$$\|\varphi_\alpha\|_{B(G)} = \varphi_\alpha(e) = \langle T_\alpha(1)\delta_e|\delta_e\rangle \leq \|T_\alpha(1)\| \leq 1.$$

Moreover, it is known by Haagerup that since each $T_\alpha$ is finite rank, then $\varphi_\alpha \in \ell_2(G) \subseteq A(G)$ with $\|\varphi_\alpha\|_{A(G)} = \|\varphi_\alpha\|_{B(G)} \leq 1$.

Finally, we see that for each $s \in G$, $T_\alpha(\lambda_s) \to \lambda_s$ in norm-topology implies that

$$\varphi_\alpha(s) = \langle T_\alpha(\lambda_s)\delta_e|\lambda_s\delta_e\rangle \to \langle \lambda_s\delta_e|\lambda_s\delta_e\rangle = 1.$$

This shows that the group $G$ is amenable.

Remark: It is quite often to consider the following proof of (2) $\Rightarrow$ (1).

Suppose that $C^*_\lambda(G)$ is nuclear. Then there exists two nets of cp and contractive maps $S_\alpha : C^*_\lambda(G) \to M_{n(\alpha)}$ and $T_\alpha : M_{n(\alpha)} \to C^*_\lambda(G)$ such that

$$\|T_\alpha \circ S_\alpha(x) - x\| \to 0 \text{ for all } x \in C^*_\lambda(G).$$

For each $\alpha$, we can obtain a cp extension $\tilde{S}_\alpha : B(\ell_2(G)) \to M_{n(\alpha)}$ of $S_\alpha$. Then we obtain a net of cp maps

$$\Phi_\alpha = T_\alpha \circ \tilde{S}_\alpha : B(\ell_2(G)) \to C^*_\lambda(G) \subseteq VN_\lambda(G).$$

Since $VN_\lambda(G)$ is a dual space, there exists a subnet of $\{\Phi_\alpha\}$ converging in the point-weak* topology to a cp map $\Phi : B(\ell_2(G)) \to VN_\lambda(G)$. In this case, we have $\Phi(x) = x$ for all $x \in C^*_\lambda(G)$ and

$$\Phi(\lambda_s x \lambda_t) = \lambda_s \Phi(x) \lambda_t$$

for all $x \in B(\ell_2(G))$. Let $\tau$ be the canonical trace on $VN_\lambda(G)$, then $\tau \circ \Phi(x)$ defines a state on $B(\ell_2(G))$. The restriction $m = \tau \circ \Phi_{|\ell_\infty(G)}$ is a left invariant mean on $\ell_\infty(G)$ since

$$m(s \cdot h) = \tau(\Phi(\lambda_s h \lambda_{s^{-1}})) = \tau(\lambda_s \Phi(h) \lambda_{s^{-1}}) = \tau(\Phi(h)) = m(h).$$

This shows that $G$ is amenable.

Theorem [Choi-Effros/Effros-Lance]: Let $A$ be a C*-algebra. TFAE:

(1) $A$ is nuclear,

(2) $A$ has the CPAP,

(3) $A^{**}$ is demidiscrete.

## References


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