Permanence properties for crossed products and fixed point algebras of finite groups

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In this talk, we are interested in permanence properties for crossed products and fixed point algebras by finite groups. For the most part, we consider the following loosely related properties:

- The ideal property.
- The projection property.
- Topological dimension zero.
- Pure infiniteness for nonsimple C^*-alg.

Topological dimension zero

Definition (*L. G. Brown-Pedersen*, 2009**):**

A C^*-alg. A is said to have *topological dimension zero* if Prim(A) = a basis of compact-open sets.

• (*Bratteli-Elliott*, J.F.A. 1978): If X = Prim(A) for some $A = C^*$-alg. + sep., then: X = Prim(B) for some AF alg. $B \Leftrightarrow A$ has *topological dimension zero*.

Definition (*Kirchberg-Rørdam*):

A C^*-alg. A is said to be *purely infinite* if:

(1) A has no characters (or, equivalently, no non-zero abelian quotients), and

(2) $\forall a, b \in A^+$ such that $a \in \overline{AbA} \Rightarrow \exists \{x_n\} \subset A$ such that $a = \lim_{n\to\infty} x_n^* b x_n$.

Remark:

The study of purely infinite C^*-alg. was motivated by *Kirchberg*'s classification of the sep., nuclear C^*-alg. that tensorially absorb the Cuntz algebra $\mathcal{O}_\infty$ up to stable isomorphism by an ideal related KK-theory.

The ideal property

Definition:

A C^*-alg. A is said to have the *ideal property (i.p.)* if each (closed, two-sided) ideal of A is generated (as an ideal) by its projections.

Some remarks and results:

• $A =$ simple + unital $\Rightarrow A =$ i.p.

• RR$(A) = 0 \Rightarrow A =$ i.p.

• (*Rørdam-Sierakowski*): Let (A, G, α) be a C^*-dynamical system, where $G =$ discrete amenable group and the action of G on $\widehat{A}$ is free. Then $A =$ i.p. $\Rightarrow C^*(G, A, \alpha) =$ i.p.

• (*P.-Phillips*, 2004): Let $\alpha: G \to Aut(A)$ be an action of a finite group on A with the Rokhlin property. Then $A =$ i.p. $\Rightarrow C^*(G, A, \alpha) =$ i.p.

• (*Cuntz-Echterhoff-Li*): If R is a ring of integers in a number field $\Rightarrow$ the semigroup C^*-alg. $C_r^*(R \rtimes R^\times) =$ i.p. (+ purely infinite + RR$(C_r^*(R \rtimes R^\times)) \neq 0$)

• *(K. Stevens)*: Classification of a certain class of AI alg. + i.p.

• *(P.)*: Classification of the AH alg. + i.p. + s.d.g., up to a shape equivalence.

• *(P.)*: Several characterizations of the i.p. for an arbitrary AH alg.

• *(P.)*: If $A = AH$ alg. + i.p. + s.d.g. Then:

(1) sr$(A) = 1$;

(2) $K_0(A) =$ Riesz group + weakly unperforated (in the sense of Elliott).

• *(Gong-Jiang-Li-P.)*: If $A = AH$ alg. + i.p. + no dim. growth. $\Rightarrow A$ can be rewritten as an AH alg. with (special) local spectra of dim ≤ 3.

• *(P.-Rørdam*, J.F.A. 2000): i.p. $\otimes$ i.p. $\neq$ i.p. (even in the sep. case). If at least one of the "factors" is exact, then we have "equality".

• *(P.-Rørdam*, Crelle's Journal 2007): Let $A = C^*$-alg. + sep. + purely infinite. T.F.A.E.:

(1) $A =$ i.p.;

(2) $A =$ topological dimension zero.

• *(P.-Rørdam*, Crelle's Journal 2007): Let $A = C^*$-alg. + sep. T.F.A.E.:

(1) $A \otimes \mathcal{O}_2 =$ i.p.;

(2) $\mathrm{RR}(A \otimes \mathcal{O}_2) = 0$;

(3) $A =$ topological dimension zero.

Definition:

A C^*-alg. A is said to be an AH algebra (AH alg.), if A is the inductive limit C^*-alg. of:

$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \xrightarrow{\phi_{3,4}} \cdots \xrightarrow{\phi_{n-1,n}} A_n \xrightarrow{\phi_{n,n+1}} \cdots$$

with $A_n = \bigoplus_{i=1}^{t_n} P_{n,i} M_{[n,i]}(C(X_{n,i}))P_{n,i}$, where the local spectra $X_{n,i}$ = finite, connected CW complexes, $t_n, [n,i] \in \mathbb{N}$ and each $P_{n,i} \in \mathcal{P}(M_{[n,i]}(C(X_{n,i})))$.

Definition (*P.*, 2002):

A C^*-alg. A is said to have the *projection property (p.p.)* if any ideal of A has an increasing approximate identity consisting of projections.

- (*P.*): If $A = AH$ alg., then: $A =$ p.p. $\Leftrightarrow A =$ i.p.
- (*P.*): i.p. $\not\Rightarrow$ p.p. (even in the sep. case).

ROKHLIN ACTIONS OF FINITE GROUPS

Definition:

Let $\mathcal{C}$ be a class of C^*-alg. A *strong local $\mathcal{C}$-algebra* is a C^*-alg A such that for every finite set $S \subset A$ and every $\varepsilon > 0$, there is a C^*-alg. $B \in \mathcal{C}$ and a homomorphism $\varphi\colon B \to A$ (not necessarily injective) such that $dist(a, \varphi(B)) < \varepsilon$ for all $a \in S$. We also say that A *can be locally approximated by $\mathcal{C}$*.

Theorem (*Osaka-Phillips*):

Let $A =$ unital C^*-algebra, let $G =$ finite group, and let $\alpha\colon G \to Aut(A)$ be an action with the Rokhlin property. Then $C^*(G, A, \alpha)$ can be *locally approximated by the class of matrix algebras over corners of A*.

Proposition (*P.-Phillips*):

Let A = purely infinite unital C^*-alg., let G = finite group, and let $\alpha \colon G \to Aut(A)$ be an action with the Rokhlin property. Then $C^*(G, A, \alpha)$ and A^α = purely infinite unital C^*-alg.

Theorem (*P.-Phillips*):

Let $\mathcal{C}$ be the class of unital (sep. nuclear) C^*-alg. that are direct limits of sequences of finite direct sums of Kirchberg C^*-alg. satisfying the UCT. Let $A \in \mathcal{C}$, let G = finite group, and let $\alpha \colon G \to Aut(A)$ be an action with the Rokhlin property. Then $C^*(G, A, \alpha)$ and A^α are both in $\mathcal{C}$.

Remark (*P.-Phillips*):

The above result implies (using also a theorem of *Dadarlat-P.* (J.F.A. 2005) a *classification result* for crossed products and fixed point algebras of Rokhlin actions of finite groups on algebras in $\mathcal{C}$ by a topological invariant.

Definition (*Carrion-P.*):

A C^*-alg. A is a WB algebra if for any ideal $I \subset A$ that is generated by its projections, the extension

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

is *quasidiagonal*, that is, there is an approximate identity for I consisting of projections $(p_\lambda)_{\lambda\in\Lambda}$ (not necessarily countable or increasing) such that $\lim \|p_\lambda a - a p_\lambda\| = 0$ for all $a \in A$.

Remark (*P.-Phillips*):

Note that: AH alg. $\subset GAH$ alg. $\subset LB$ alg., and in the unital case we have LB alg. $\subset WB$ alg.

Proposition (*P.-Phillips*):

Let $A =$ unital WB algebra + i.p., let $G =$ finite group, and let $\alpha\colon G \to Aut(A)$ be an action with the Rokhlin property. Then $C^*(G, A, \alpha)$ and $A^\alpha =$ unital WB alg. + i.p.

STRONGLY POINTWISE OUTER ACTIONS AND THE IDEAL AND PROJECTION PROPERTIES

Definition (*Phillips*):

An action $\alpha\colon G \to Aut(A)$ is said to be *strongly pointwise outer* if, for every $g \in G \setminus \{1\}$ and any two α_g-invariant ideals $I \subset J \subset A$ with $I \neq J$, the automorphism of J/I induced by α_g is outer.

Definition (*Sierakowski*):

Let $\alpha\colon G \to Aut(A)$ be an action of a discrete group G on a C^*-alg. A. We say that A *separates the ideals in the reduced crossed product* $C^*_{\mathrm{r}}(G, A, \alpha)$ (or in $C^*(G, A, \alpha)$ when G is amenable) if each ideal of $C^*_{\mathrm{r}}(G, A, \alpha)$ has the form $C^*_{\mathrm{r}}(G, I, \alpha)$ for some α-invariant ideal $I \subset A$.

Theorem (*P.-Phillips*):

Let $G =$ finite group, let $A = C^*$-alg., and let $\alpha\colon G \to Aut(A)$ be a strongly pointwise outer action. Then A separates the ideals in $C^*(G, A, \alpha)$.

Corollary (*P.-Phillips*):

Let G = finite group, let A = unital C^*-alg., and let $\alpha\colon G \to Aut(A)$ be an action with the Rokhlin property. Then A separates the ideals in $C^*(G, A, \alpha)$.

Corollary (*P.-Phillips*):

Crossed products by strongly pointwise outer actions of finite groups preserve the i.p. and the p.p.

Remark (*P.-Phillips*):

We do not know of any example of any action at all of a finite group on a C^*-alg. = i.p. such that the crossed product does not have the i.p. Similarly, we do not know of any example of any action at all of a finite group on a C^*-alg. = p.p. such that the crossed product does not have the p.p.

Question: Does the i.p. pass to fixed point alg. of actions of finite groups?

Answer (*P.-Phillips*): No.

Question: Does the p.p. pass to fixed point alg. of actions of finite groups?

Answer (*P.-Phillips*): No.

Remark (*P.-Phillips*):

In fact, we produce an example of a pointwise outer (but not strongly pointwise outer) action of $\mathbb{Z}/2\mathbb{Z}$ on a C^*-alg. = p.p. such that the fixed point algebra does not even have the i.p.

Question (*Carrion-P.*, 2008): Let $A = C^*$-alg., let $n \in \mathbb{N}$, and suppose that $M_n(A) =$ i.p. Does it follow that $A =$ i.p.?

Answer (*P.-Phillips*): No.

Question: Let $A = C^*$-alg., let $n \in \mathbb{N}$, and suppose that $M_n(A) =$ p.p. Does it follow that $A =$ p.p.?

Answer (*P.-Phillips*): No.

Remark (*P.-Phillips*):

In fact, we construct a $A = C^*$-alg. such that $M_2(A) =$ p.p. but $A \neq$ i.p.

TOPOLOGICAL DIMENSION ZERO

Definition (*P.-Rørdam*):

An ideal I in a C^*-alg. A is said to be *compact* if whenever $(I_\lambda)_{\lambda \in \Lambda}$ is an increasing net of ideals in A such that $I = \overline{\bigcup_{\lambda \in \Lambda} I_\lambda}$, then there is $\lambda \in \Lambda$ such that $I = I_\lambda$.

Theorem (*P.-Phillips*):

Let $\alpha \colon G \to Aut(A)$ be an action of a finite group G on a C^*-algebra A. Suppose that A^α = topological dimension zero. Suppose also that whenever $I \subset A^\alpha$ is a compact ideal, then $\overline{AIA} \cap A^\alpha$ = compact ideal in A^α. Then A = topological dimension zero.

Theorem (*P.-Phillips*):

Let $\alpha\colon G \to Aut(A)$ be an action of a finite abelian group G on a C^*-alg. A. Suppose that A = topological dimension zero. Then $C^*(G, A, \alpha)$ and A^α = topological dimension zero.

Proposition (*P.-Phillips*):

Let $\alpha\colon G \to Aut(A)$ be an action of a finite group G on a C^*-alg. A. Assume that A separates the ideals in $C^*(G, A, \alpha)$. Suppose that A = topological dimension zero. Then $C^*(G, A, \alpha)$ and A^α = topological dimension zero.

Let $\alpha\colon G \to Aut(A)$ be an action of $G =$ *discrete* group on a C^*-alg. A. Let $E\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be the canonical conditional expectation. It is immediate that if $I \subset A$ is an α-invariant ideal, then

$$E(C^*_{\mathrm{r}}(G, I, \alpha)) = I. \tag{1}$$

It follows that for α-invariant ideals $I_1, I_2 \subset A$, we have

$$I_1 \subset I_2 \text{ if and only if } C^*_{\mathrm{r}}(G, I_1, \alpha) \subset C^*_{\mathrm{r}}(G, I_2, \alpha). \tag{2}$$

Lemma (*P.-Phillips*):

Let $\alpha\colon G \to Aut(A)$ be an action of $G =$ *discrete* group on a C^*-alg. A. Suppose A separates the ideals in $C^*_{\mathrm{r}}(G, A, \alpha)$. Let $I = \alpha$-invariant ideal of A. If $I =$ compact, then $C^*_{\mathrm{r}}(G, I, \alpha) =$ compact.

Proof. Let $(J_\lambda)_{\lambda \in \Lambda} =$ an increasing net of ideals in $C^*_{\mathrm{r}}(G, A, \alpha)$ such that

$$C^*_{\mathrm{r}}(G, I, \alpha) = \overline{\bigcup_{\lambda \in \Lambda} J_\lambda}.$$

By hypothesis, there are α-invariant ideals I_λ such that $J_\lambda = C^*_{\mathrm{r}}(G, I_\lambda, \alpha)$ for all $\lambda \in \Lambda$. By (2), we have $I_\lambda \subset I$ for all $\lambda \in \Lambda$, and moreover $(I_\lambda)_{\lambda \in \Lambda}$ is increasing. By (1) and because $E =$ continuous, we have

$$I = E(C^*_{\mathrm{r}}(G, I, \alpha)) = E\left(\overline{\bigcup_{\lambda \in \Lambda} C^*_{\mathrm{r}}(G, I_\lambda, \alpha)}\right)$$

$$\subset \overline{E\left(\bigcup_{\lambda \in \Lambda} C^*_{\mathrm{r}}(G, I_\lambda, \alpha)\right)} = \overline{\bigcup_{\lambda \in \Lambda} E(C^*_{\mathrm{r}}(G, I_\lambda, \alpha))} = \overline{\bigcup_{\lambda \in \Lambda} I_\lambda} \subset I.$$

Thus $I = \overline{\bigcup_{\lambda \in \Lambda} I_\lambda}$. Since $I =$ compact, there is $\lambda \in \Lambda$ such that $I = I_\lambda$. Then $C^*_{\mathrm{r}}(G, I, \alpha) = C^*_{\mathrm{r}}(G, I_\lambda, \alpha) = J_\lambda$. This shows that $C^*_{\mathrm{r}}(G, I, \alpha)$ is compact.

Proof of the Proposition. We first consider $C^*(G, A, \alpha)$.

It is not difficult to see that a C^*-alg. $D =$ topological dimension zero if and only if every ideal in D is the closure of the union of an increasing net of compact ideals.

So let $J =$ an arbitrary ideal in $C^*(G, A, \alpha)$. By hypothesis, there is an α-invariant ideal $I \subset A$ such that $J = C^*(G, I, \alpha)$. Since $A =$ topological dimension zero, there is an increasing net $(I_\lambda)_{\lambda \in \Lambda}$ of compact ideals of A such that $I = \overline{\bigcup_{\lambda \in \Lambda} I_\lambda}$. For $\lambda \in \Lambda$, define $L_\lambda = \sum_{g \in G} \alpha_g(I_\lambda)$. Then $(L_\lambda)_{\lambda \in \Lambda} =$ increasing net of α-invariant ideals and $\overline{\bigcup_{\lambda \in \Lambda} L_\lambda} = I$. Since a finite union of compact sets is compact, it follows that $L_\lambda =$ compact for all $\lambda \in \Lambda$. The ideals $C^*(G, L_\lambda, \alpha) =$ compact by the above Lemma. By (2), these ideals are increasing and satisfy

$$\overline{\bigcup_{\lambda \in \Lambda} C^*(G, L_\lambda, \alpha)} = C^*(G, I, \alpha).$$

This completes the proof for $C^*(G, A, \alpha)$.

The result for A^α now follows from the fact that topological dimension zero passes to hereditary subalgebras and a result of *Rosenberg* saying that if $\alpha \colon G \to Aut(B)$ is an action of $G =$ *compact* group on a C^*-alg. B, then B^α is isomorphic to a corner of $C^*(G, B, \alpha)$.

Corollary (*P.-Phillips*):

Let $\alpha\colon G \to Aut(A)$ be a strongly pointwise outer action of a finite group G on a C^*-alg. A. Suppose that $A =$ topological dimension zero. Then $C^*(G, A, \alpha)$ and $A^\alpha =$ topological dimension zero.

PURELY INFINITE C*-ALGEBRAS WITH FINITE PRIMITIVE SPECTRUM

Theorem (*P.-Phillips*):

Let $\alpha\colon G \to Aut(A)$ be an action of a finite group G on a C^*-alg. A. Assume that $A =$ finitely many α-invariant ideals. Then Prim($C^*(G, A, \alpha)$) = finite. Moreover, if in addition $A =$ purely infinite, then $C^*(G, A, \alpha) =$ purely infinite.

Corollary (*P.-Phillips*):

Let $\alpha\colon G \to Aut(A)$ be an action of a finite group G on a C^*-alg. A. If $A =$ purely infinite + finitely many α-invariant ideals, then $C^*(G, A, \alpha) =$ i.p.

Corollary (*P.-Phillips*):

Let $\alpha\colon G \to Aut(A)$ be an action of a finite group G on a C^*-alg. A. Assume that $A =$ finitely many α-invariant ideals. Then Prim(A^α) = finite. Moreover, if in addition $A =$ purely infinite, then $A^\alpha =$ purely infinite.

Theorem (*P.-Phillips*):

Let A = purely infinite C^*-alg. Suppose there is an ordinal κ and a composition series $(I_\lambda)_{\lambda \leq \kappa}$ for A such that $\text{Prim}(I_{\lambda+1}/I_\lambda)$ = finite for all $\lambda < \kappa$. Let G = finite group, and let $\alpha\colon G \to Aut(A)$ be any action of G on A. Then $C^*(G, A, \alpha)$ and A^α = purely infinite + composition series in which all the subquotients have finite primitive ideal spaces.

Proposition (*P.-Phillips*):

Let $\alpha\colon G \to Aut(A)$ be an action of a finite group G on a C^*-alg. A. Suppose that there is a set $\mathcal{I}$ of ideals in A, each of which is purely infinite and has finite primitive ideal space, with the following property. For every finite subset $S \subset A$ and every $\varepsilon > 0$, there is $I \in \mathcal{I}$ such that $dist(a, I) < \varepsilon$ for all $a \in S$. Then $C^*(G, A, \alpha)$ and A^α = purely infinite + i.p.