Approximate identities and BSE norms for Banach function algebras

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Dedicated to Dona Strauss on the day of her 80th anniversary

Function algebras

Let K be a locally compact space. Then $C_0(K)$ is the algebra of all continuous functions on K that vanish at infinity.

We define

$$|f|_K = \sup\{|f(x)| : x \in K\} \quad (f \in C_0(K))\,,$$

so that $|\cdot|_K$ is the **uniform norm** on K and $(C_0(K), |\cdot|_K)$ is a commutative, semisimple Banach algebra.

A **function algebra** on K is a subalgebra A of $C_0(K)$ that separates strongly the points of K, in the sense that, for each $x, y \in K$ with $x \neq y$, there exists $f \in A$ with $f(x) \neq f(y)$, and, for each $x \in K$, there exists $f \in A$ with $f(x) \neq 0$.

Banach function algebras

A **Banach function algebra** (=BFA) on K is a function algebra A on K with a norm $\|\cdot\|$ such that $(A, \|\cdot\|)$ is a Banach algebra.

Always $\|f\| \geq |f|_K \quad (f \in A)$.

A BFA is a **uniform algebra** when the norm is equal to the uniform norm, and it is **equivalent to a uniform algebra** when the norm is equivalent to the uniform norm.

Characters and maximal ideals

Let A be a function algebra on K. For each $x \in K$, define

$$\varepsilon_x(f) = f(x) \quad (f \in A)\,.$$

Then each ε_x is a character on A. A Banach function algebra A on K is **natural** if all characters are evaluation characters, and then all maximal modular ideals of A have the form

$$M_x = \{f \in A : f(x) = 0\}$$

for some $x \in K$ (and we set $M_\infty = A$).

We shall also refer to J_∞, the ideal of functions in A of compact support, and J_x the ideal of functions in J_∞ that vanish on a neighbourhood of x. Then A is **strongly regular** if $\overline{J_x} = M_x$ for all $x \in K \cup \{\infty\}$.

Approximate identities

Let A be a commutative Banach algebra (=CBA). A net (e_α) in A is an **approximate identity** (= AI) for A if

$$\lim_\alpha ae_\alpha = a \quad (a \in A)\,;$$

an AI (e_α) is **bounded** if $\sup_\alpha \|e_\alpha\| < \infty$, and then $\sup_\alpha \|e_\alpha\|$ is the **bound**; an approximate identity is **contractive** if it has bound 1.

We refer to a BAI and a CAI, respectively, in these two cases.

A natural BFA A on K is **contractive** if M_x has a CAI for **each** $x \in K \cup \{\infty\}$.

Basic example Let K be locally compact. Then $C_0(K)$ is contractive.

Pointwise approximate identities

Let A be a natural Banach function algebra on a locally compact space K. A net (e_α) in A is a **pointwise approximate identity** (PAI) if

$$\lim_\alpha e_\alpha(x) = 1 \quad (x \in K)\,;$$

the PAI is **bounded**, with **bound** $m > 0$, if $\sup_\alpha \|e_\alpha\| \leq m$, and then (e_α) is a **bounded pointwise approximate identity** (BPAI); a bounded pointwise approximate identity of bound 1 is a **contractive pointwise approximate identity** (CPAI).

Clearly a BAI is a BPAI, a CAI is a CPAI.

Introduced by Jones and Lahr, 1977.

The algebra A is **pointwise contractive** if M_x has a CPAI for each $x \in K \cup \{\infty\}$.

Clearly a contractive BFA is pointwise contractive.

Some questions

Question 1 How many other contractive BFAs are there? Must a contractive BFA be a uniform algebra?

Question 2 Let A be a BFA that is not contractive. What is the minimum bound of BAIs (if any) in maximal modular ideals?

Question 3 Give some examples where there are CPAIs, but no CAIs or BAIs, or even no approximate identities. Give some examples of pointwise contractive BFAs that are not contractive, in particular find uniform algebras with this property.

[Jones and Lahr gave a complicated example of a BFA with a CPAI, but no AI.]

Factorization

Let A be a CBA with a BAI. Then A **factors** in the sense that each $a \in A$ can be written as $a = bc$ for some $b, c \in A$.

This is a (weak form of) **Cohen's factorization theorem**.

The converse is not true in general, even for uniform algebras, but it is true for various classes of maximal modular ideals in BFAs.

Question 4 When can we relate factorization to the existence of (pointwise) approximate identities? What is the relation between 'A has a BPAI (or CPAI)' and '$A = A^2$', especially for uniform algebras A?

Peak points

Let A be a function algebra on a compact K. A closed subset F of K is a **peak set** if there exists a function $f \in A$ with $f(x) = 1 \ (x \in F)$ and $|f(y)| < 1 \ (y \in K \setminus F)$; in this case, f **peaks** on F; a point $x \in K$ is a **peak point** if $\{x\}$ is a peak set, and a p-**point** if $\{x\}$ is an intersection of peak sets.

The set of p-points of A is denoted by $\Gamma_0(A)$; it is sometimes called the **Choquet boundary** of A.

[In the case where A is a BFA, a countable intersection of peak sets is always a peak set.]

Theorem Let A be a natural, contractive BFA on K. Then every point of K is a p-point. $\square$

The Šilov boundary

Let A be a BFA on a compact K.

A closed subset L of K is a **closed boundary** for A if $|f|_L = |f|_K \quad (f \in A)$; the intersection of all the closed boundaries for A is the **Šilov boundary**, $\Gamma(A)$.

Suppose that K is compact and that A is a natural uniform algebra on K. Then $\Gamma(A) = \overline{\Gamma_0(A)}$ and $\Gamma(A)$ is a closed boundary.

Suppose that K is compact and metrizable and that A is a natural Banach function algebra on K. Then the set of peak points is dense in $\Gamma(A)$. (HGD - thesis!)

Contractive uniform algebras

Theorem Let A be a **uniform** algebra on a compact space K, and take $x \in K$. Then the following conditions on x are equivalent:

(a) $\varepsilon_x \in \mathrm{ex}\{\lambda \in A' : \|\lambda\| = \lambda(1_K) = 1\}$;

(b) $x \in \Gamma_0(A)$;

(c) M_x has a BAI;

(d) M_x has a CAI.

Proof of (c) $\Rightarrow$ (d) (from DB).

M_x'' is a maximal ideal in A'', a closed subalgebra of $C(K)'' = C(\widetilde{K})$. A BAI in M_x gives an identity in M_x'', hence an idempotent in $C(\widetilde{K})$. The latter have norm 1. So there is a CAI in M_x. $\square$

Cole algebras

Definition Let A be a natural uniform algebra on a locally compact space K. Then A is a **Cole algebra** if $\Gamma_0(A) = K$.

Suppose that K is compact and metrizable. Then A is a Cole algebra if and only if every point of K is a peak point.

Theorem A uniform algebra is contractive if and only if it is a Cole algebra. $\square$

It was a long-standing conjecture, called the **peak-point conjecture**, that $C(K)$ is the only Cole algebra on K.

The first counter-example is due to Brian Cole in his thesis. An example of Basener gives a compact space K in $\mathbb{C}^2$ such that the uniform algebra $R(K)$ of all uniform limits on K of the restrictions to K of the functions which are rational on a neighbourhood of K, is a Cole algebra, but $R(K) \neq C(K)$.

Gleason parts for uniform algebras

Theorem Let A be a natural uniform algebra on a compact space K, and take $x, y \in K$. Then the following are equivalent:

(a) $\|\varepsilon_x - \varepsilon_y\| < 2$;

(b) there exists $c \in (0, 1)$ with $|f(x)| < c|f|_K$ for all $f \in M_y$. $\square$

Now define $x \sim y$ for $x, y \in K$ if x and y satisfy the conditions of the theorem. It follows that $\sim$ is an equivalence relation on K; the equivalence classes with respect to this relation are the **Gleason parts** for A. These parts form a decomposition of K, and each is σ-compact.

Every p-point is a one-point part.

Pointwise contractive uniform algebras

Theorem Let A be a natural uniform algebra on a compact space K, and take $x \in K$. Then the following are equivalent:

(a) $\{x\}$ is a one-point Gleason part;

(b) M_x has a CPAI;

(c) for each $y \in K \setminus \{x\}$, there is a sequence (f_n) in M_x such that $|f_n|_K \leq 1$ $(n \in \mathbb{N})$ and $f_n(y) \to 1$ as $n \to \infty$.

Thus A is pointwise contractive if and only if each singleton in K is a one-point Gleason part.
$\square$

Examples of uniform algebras

Example 1 The disc algebra $A(\overline{\mathbb{D}})$. Here $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Take $z \in \overline{\mathbb{D}}$. Then M_z has a BAI iff M_z has a CAI iff M_z has a BPAI iff M_z has a CPAI iff $\{z\}$ is a peak point iff $|z| = 1$. The open disc $\mathbb{D}$ is a single Gleason part. If $z \in \mathbb{D}$, then $M_z \neq M_z^2$ and M_z^2 is closed, and hence M_z has no approximate identity. $\square$

Example 2 Let A be a uniform algebra on a compact set K, and take $x \in K$. It is possible to have $x \in \Gamma(A)$, but such that M_x does not have a BPAI. Indeed, let $K = \overline{\mathbb{D}} \times \mathbb{I}$ and take A to be the **tomato can algebra**, the uniform algebra of all $f \in C(K)$ such that $z \mapsto f(z, 1)$, $\overline{\mathbb{D}} \to \mathbb{C}$, belongs to $A(\overline{\mathbb{D}})$. Then

$$\Gamma_0(A) = \{(z,t) \in K : 0 \leq t < 1\} \cup \{(z,1) \in K : z \in \mathbb{T}\}$$

and $\Gamma(A) = K$. The set $K \setminus \Gamma_0(A)$ is a part, and again M_x has a BPAI if and only if M_x has a CPAI if and only if x is a peak point. $\square$

More examples of uniform algebras

Example 3 For compact K in $\mathbb{C}$, $R(K) = C(K)$ iff $R(K)$ is pointwise contractive. $\square$

Example 4 Let H^∞ be the (non-separable) uniform algebra of all bounded analytic functions on $\overline{\mathbb{D}}$. The character space of H^∞ is large. Each point of the Šilov boundary $\Gamma(H^\infty)$ is a p-point, and hence a one-point part, but there are one-point parts that are not in $\Gamma(H^\infty)$. Here M_x factors iff $\{x\}$ is a one-point part. $\square$

Example 5 (D-Feinstein) We have a natural, separable, uniform algebra on a compact, metric space K such that **each** point of K is a one-point Gleason part, but $\Gamma(A) \subsetneq K$. Thus A is pointwise contractive, but not contractive. $\square$

More examples of uniform algebras

Example 6 Joel Feinstein constructed a separable, regular, natural uniform algebra A on a compact space K such that there is a two-point Gleason part, but all other points are one-point Gleason parts. In this example, each maximal ideal has a BPAI (with a uniform bound), but the algebra is not pointwise contractive. $\square$

Example 7 Stu. Sidney has examples of natural uniform algebras A on compact spaces K and points $x \in K \setminus \Gamma(A)$ such that $\{x\}$ is a one-point part, but M_x^2 is not dense in M_x. Hence M_x has a CPAI, but no approximate identity. $\square$

Open question Suppose that M_x factors (or just $M_x = M_x^2$). Is $\{x\}$ necessarily a one-point Gleason part?

Group algebras

Let Γ be a locally compact group. Then the **Fourier algebra** on Γ is $A(\Gamma)$. For $p > 1$, the **Herz–Figà-Talamanca algebra** is $A_p(\Gamma)$ Thus $A_p(\Gamma)$ is a self-adjoint, natural, strongly regular Banach function algebra on Γ.

BAIs and BPAIs in group algebras

Proposition (More is known to Brian Forrest, Tony Lau et al.)

Let Γ be a locally compact group, and take $p > 1$. Then the following are equivalent:

(a) Γ is amenable;

(b) $A_p(\Gamma)$ has a BAI (**Leptin**);

(c) $A_p(\Gamma)$ has a BPAI;

(d) $A_p(\Gamma)$ has a CAI. □

Bounds

Let Γ be an infinite locally compact group, and let M be a maximal modular ideal of $A(\Gamma)$.

It is standard that M has a BAI of bound 2. By a theorem of Delaporte and Derigetti, the number 2 is the minimum bound for such a BAI.

Theorem A lower bound for the bound of a BPAI in M is also 2. In particular, $A(\Gamma)$ is not pointwise contractive. $\square$

Segal algebras

Definition Let $(A, \|\cdot\|_A)$ be a natural Banach function algebra on a locally compact space K. A Banach function algebra $(B, \|\cdot\|_B)$ is an **abstract Segal algebra** (with respect to A) if B is an ideal in A and there is a net in B that is an approximate identity for both $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$.

Classical **Segal algebras** are abstract Segal algebras with respect to $L^1(G)$.

An example of a Segal algebra

Example Let G be a non-discrete LCA group with dual group Γ. Take $p \geq 1$, define

$$S_p(G) = \{f \in L^1(G) : \widehat{f} \in L^p(\Gamma)\},$$

and set

$$\|f\|_{S_p} = \max\left\{\|f\|_1, \left\|\widehat{f}\right\|_p\right\} \quad (f \in S_p(G)).$$

Then $(S_p(G), \star, \|\cdot\|_{S_p})$ is a Segal algebra with respect to $L^1(G)$ and a natural Banach function algebra on Γ. Since $S_p(G)^2 \subsetneq S_p(G)$, $S_p(G)$ does not have a BAI. However, by a result of Inoue and Takahari, $S_p(G)$ has a CPAI whenever G is also non-compact.

Thus $S_p(\mathbb{R})$ has a CPAI, but no BAI. $\square$

A construction

The following gives a cheap method of obtaining examples with CPAI, but no approximate identity.

Proposition Let $(A, \|\cdot\|)$ be a natural Banach function algebra on a locally compact space K. Suppose that $f_0 \in C_0(K) \setminus A$ is such that $ff_0 \in A$ and $\|ff_0\| \le \|f\|$ for each $f \in A \cup \{f_0\}$. Set $B = A \oplus \mathbb{C}f_0$, with

$$\|f + zf_0\| = \|f\| + |z| \quad (f \in A,\, z \in \mathbb{C})\,.$$

Then B is a natural Banach function algebra on K containing A as a closed ideal. Further, $B^2 \subset A$, and so B does not have an approximate identity.

Suppose that A has a CPAI or is pointwise contractive. Then B has a CPAI or is pointwise contractive, respectively. $\square$

An application

Example Consider $L^1(G)$ for a non-discrete, LCA group G, and take a singular measure $\mu_0 \in M(G)_{[1]} \setminus L^1(G)$ with $\mu_0 \star \mu_0 \in L^1(G)$. Since $L^1(G)$ is a closed ideal in $M(G)$, it follows that $f \star \mu_0 \in L^1(G)$ for each $f \in L^1(G)$. We regard μ_0 as an element of $C_0(\Gamma)$, where Γ is the dual group to G. Thus the conditions of the above proposition are satisfied. Set $B = L^1(G) \oplus \mathbb{C}\mu_0$.

In this case, $L^1(G)$ has a CAI and each maximal ideal of $L^1(G)$ has a BAI of bound 2. Thus B has a CPAI and each maximal ideal of B has a BPAI of bound 2. However, B does not have an approximate identity. $\square$

Banach sequence algebras

Let S be a non-empty set. The algebra of all functions on S of finite support is denoted by $c_{00}(S)$; the characteristic function of $s \in S$ is denoted by δ_s, so that $\delta_s \in c_{00}(S)\ (s \in S)$.

Definition Let S be a non-empty set. A **Banach sequence algebra** on S is a Banach function algebra A on S such that $c_{00}(S) \subset A$. Set $A_0 = \overline{c_{00}(S)}$. Then A is **Tauberian** if $A_0 = A$, so that A is strongly regular.

Example Let A be the space ℓ^p, where $p \geq 1$. Then A is a natural, self-adjoint, Tauberian Banach sequence algebra on $\mathbb{N}$, and A is an ideal in A''. Clearly A and each maximal modular ideal in A have approximate identities, but A does not have a BPAI. Here $A^2 = \ell^{p/2}$, and so A does not factor. $\square$

Two propositions

Proposition Let A be a Tauberian Banach sequence algebra. Then A is natural and A is an ideal in A''. $\square$

The following is a weak converse to the above.

Proposition (also Blecher-Read) Let A be a Banach function algebra such that A is an ideal in A'' and A has a BPAI. Then A also has a BAI, with the same bound. In the case where the BPAI is contained in A_0, A is Tauberian. $\square$

We do not know if the last conclusion holds if we omit the condition that the BPAI is in A_0.

A theorem for Banach sequence algebras

We are wondering if each pointwise contractive Banach function algebra must be equivalent to a uniform algebra. This is true for Banach sequence algebras.

Theorem Let A be a natural, pointwise contractive Banach sequence algebra on S. Then A is equivalent to the uniform algebra $c_0(S)$.

Proof This uses a theorem of Bade and Curtis: Let A be a BFA on a compact K. Suppose that there exists $m > 0$ such that, for each disjoint pair $\{F, G\}$ of non-empty, closed subsets of K, there exists $f \in A$ with $\|f\| \leq m$, with $|1 - f(x)| < 1/2$ for $x \in F$, and with $|f(x)| < 1/2$ for $x \in G$. Then A is equivalent to the uniform algebra $C(K)$. $\square$

This does not work under the hypothesis that each maximal modular ideal has a BPAI: see the next example.

An example

Example 1 (Feinstein) For $\alpha = (\alpha_k) \in \mathbb{C}^{\mathbb{N}}$, set

$$p_n(\alpha) = \frac{1}{n} \sum_{k=1}^{n} k \left|\alpha_{k+1} - \alpha_k\right| \quad (n \in \mathbb{N})$$

and

$$\|\alpha\| = |\alpha|_{\mathbb{N}} + p(\alpha) \quad (\alpha \in A)\,;$$

define A to be $\{\alpha \in c_0 : \|\alpha\| < \infty\}$, so that A is a self-adjoint Banach sequence algebra on $\mathbb{N}$.

Then A is natural. Each maximal modular ideal of A has a BPAI of bound 4. Also

$$A^2 = A_0^2 = A_0\,,$$

A_0 is separable, and A is non-separable, and so A^2 is a closed subspace of infinite codimension in A. Thus A does not have an approximate identity. $\square$

The BSE norm

Definition Let A be a natural Banach function algebra on a locally compact space K. Then $L(A)$ is the linear span of $\{\varepsilon_x : x \in K\}$ as a subset of A', and

$$\|f\|_{\mathsf{BSE}} = \sup\{|\langle f, \lambda\rangle| : \lambda \in L(A)_{[1]}\} \quad (f \in A).$$

Clearly $K \subset L(A)_{[1]} \subset A'_{[1]}$; the space $\overline{L(A)}_{[1]}$ is weak-$*$ compact, and

$$|f|_K \leq \|f\|_{\mathsf{BSE}} \leq \|f\| \quad (f \in A).$$

In fact, $\|\cdot\|_{\mathsf{BSE}}$ is an algebra norm on A.

Definition A BFA A **has a BSE norm** if there is a constant $C > 0$ such that

$$\|f\| \leq C\,\|f\|_{\mathsf{BSE}} \quad (f \in A).$$

Clearly A has a BSE norm whenever it is equivalent to a uniform algebra.

BSE algebras

Let A be a BFA on a compact space K. An element $\lambda = \sum_{i=1}^{n} \alpha_i \varepsilon_{x_i} \in L(A)$ acts on $C(K)$ by setting

$$\langle f, \lambda \rangle = \sum_{i=1}^{n} \alpha_i f(x_i) \quad (f \in C(K)),$$

and now $\|f\|_{\mathsf{BSE}} = \sup\{|\langle f, \lambda\rangle| : \lambda \in L(A)_{[1]}\}$ for $f \in C(K)$. Set

$$C_{\mathsf{BSE}}(A) = \{f \in C(K) : \|f\|_{\mathsf{BSE}} < \infty\}.$$

This is also a BFA on K, with $A \subset C_{\mathsf{BSE}}(A)$. Then A is a **BSE algebra** if $A = C_{\mathsf{BSE}}(A)$.

Studied by Takahasi and Hatori, Kaniuth and Ülger, and others.

Theorem (Nearly Takahasi and Hatori) A BFA A on a compact space K is a BSE algebra iff $A_{[1]}$ is closed in the topology of pointwise convergence on K.

Examples on groups

Let Γ be a locally compact group.

Theorem (Kaniuth-Ülger) The Fourier algebra $A(\Gamma)$ is a BSE algebra iff Γ is amenable. $\square$

Theorem $A(\Gamma)$ always has a BSE norm, with $\|\cdot\|_{\mathrm{BSE}} = \|\cdot\|$. $\square$

Caution: It is stated in the paper of Kaniuth-Ülger that a BSE algebra has a BSE norm. This is false in general, but it is true whenever A has a BAI.

Some examples with a BSE norm

Example 1 Let A be the Banach sequence algebra ℓ^p, where $p > 1$. Then $L(A)$ is dense in $A' = \ell^q$, and so $\|\cdot\|_p = \|\cdot\|_{\mathrm{BSE}}$. $\square$

Example 2 For the Banach sequence algebras of Feinstein and of Blecher and Read, $\|\cdot\| = \|\cdot\|_{\mathrm{BSE}}$. Each Tauberian Banach sequence algebra with a BPAI has a BSE norm. $\square$

Example 3 A BFA with a BPAI such that A is an ideal in A'' has a BSE norm. $\square$

Example 4 The algebras $C^{(n)}[0,1]$ and $\mathrm{Lip}_\alpha K$ and $\mathrm{lip}_\alpha K$ have BSE norms; $C^{(n)}[0,1]$ and $\mathrm{Lip}_\alpha K$ are BSE algebras, but $\mathrm{lip}_\alpha K$ is not. $\square$

More examples with a BSE norm

Example 5 Let V be the Varopoulos algebra

$$(C[0,1]\widehat{\otimes}C[0,1], \|\cdot\|_\pi)\,,$$

which is a natural BFA on $[0,1]^2$. Then V has a BSE norm. (Is it a BSE algebra?)

So 'many' BFAs have BSE norms.

Example 6 A uniform algebra has a BSE norm, but it is not necessarily a BSE algebra: for a pointwise contractive uniform algebra A on a compact space K, we have $C_{\mathsf{BSE}}(A) = C(K)$, and there are examples with $A \neq C(K)$. **Caution**: this contradicts a result in Kaniuth–Ülger. $\square$

Classification theorem

Theorem Let A be a pointwise contractive BFA on a locally compact space K. Then the norms $|\cdot|_K$ and $\|\cdot\|_{\mathrm{BSE}}$ on A are equivalent.

Suppose, further, that A has a BSE norm. Then A is equivalent to a uniform algebra. $\square$

Theorem Let A be a natural Banach function algebra with a BSE norm.

(i) Suppose that A is contractive. Then A is equivalent to a Cole algebra.

(ii) Suppose that A is pointwise contractive. Then A is equivalent to a uniform algebra for which each singleton in Φ_A is a one-point Gleason part. $\square$

Thus, to find (pointwise) contractive BFAs that are not equivalent to uniform algebras, we must look for those that do not have a BSE norm.

Some examples without a BSE norm

Theorem Let $(B, \|\cdot\|_B)$ be an abstract Segal algebra with respect to a BFA $(A, \|\cdot\|_A)$. Suppose that A has a BSE norm and that B has a BPAI. Then $\|\cdot\|_{\mathrm{BSE},B}$ is equivalent to $\|\cdot\|_A$ on B. $\square$

Example Let G be a LCA group, and let $(S, \|\cdot\|_S)$ be a Segal algebra on G with a CPAI. Then the BSE norm on S is just the $\|\cdot\|_1$ from $L^1(G)$. Thus S has a BSE norm if and only if $S = L^1(G)$. For example, the earlier Segal algebra $S_p(\mathbb{R})$ does not have a BSE norm.

However $S_p(\mathbb{R})$ is not pointwise contractive. $\square$

BFAs on intervals

The first example, suggested by **Charles Read**, gives a BFA that is a pointwise contractive Ditkin algebra (so that $f \in \overline{fJ_x}$ for $f \in M_x$), but does not have a BAI and is not equivalent to a uniform algebra. Set $\mathbb{I} = [0, 1]$.

Example Consider the set A of functions $f \in C(\mathbb{I})$ with

$$I(f) := \int_0^1 \frac{|f(t) - f(0)|}{t} \, \mathrm{d}t < \infty \, .$$

Clearly A is a self-adjoint, linear subspace of $C(\mathbb{I})$ containing the polynomials, and so A is uniformly dense in $C(\mathbb{I})$. Indeed, A is 'large', in that it contains all the BFAs $(\mathrm{Lip}_\alpha(\mathbb{I}), \|\cdot\|_\alpha)$ (for $0 < \alpha \leq 1$).

Define $\|f\| = |f|_{\mathbb{I}} + I(f)$. Then $(A, \|\cdot\|)$ is a natural BFA on $\mathbb{I}$.

Example continued

Consider the maximal ideal

$$M = \{f \in A : f(0) = 0\}\,.$$

Set $f_0(t) = 1/\log(1/t)$ for $t \in (0,1]$, with $f_0(0) = 0$. Then $f_0 \in C(\mathbb{I})$, but $f_0 \notin M$.

We see that M is an abstract Segal algebra wrt $C_0((0,1])$. Thus the BSE norm on A is the uniform norm, so that A does not have a BSE norm.

All maximal ideals save for M have a CAI. But M^2 has infinite codimension in M, and so M does not have a BAI; it has a CPAI, and so A is pointwise contractive.

Take $B = M \oplus \mathbb{C}f_0$. Then B is pointwise contractive, but does not have any approximate identity. $\square$

Final example

Example We give a BFA A on the circle $\mathbb{T}$, but we identify $C(\mathbb{T})$ with a subalgebra of $C[-1,1]$. We fix α with $1 < \alpha < 2$.

Take $f \in C(\mathbb{T})$. For $t \in [-1,1]$, the **shift** of f by t is defined by

$$(S_t f)(s) = f(s-t) \quad (s \in [-1,1])\,.$$

Define

$$\Omega_f(t) = \|f - S_t f\|_1 = \int_{-1}^{1} |f(s) - f(s-t)| \,\mathrm{d}s$$

and

$$I(f) = \int_{-1}^{1} \frac{\Omega_f(t)}{|t|^\alpha} \,\mathrm{d}t\,.$$

Then $A = \{f \in C(\mathbb{T}) : I(f) < \infty\}$ and

$$\|f\| = |f|_{\mathbb{T}} + I(f) \quad (f \in A)\,.$$

We see that $(A, \|\cdot\|)$ is a natural, unital BFA on $\mathbb{T}$; it is homogeneous.

Final example continued

Let e_n be the trigonometric polynomial given by $e_n(s) = \exp(\mathrm{i}\pi ns)$ $(s \in [-1,1])$. Then $e_n \in A$, and so A is uniformly dense in $C(\mathbb{T})$. But $\|e_n\| \sim n^{\alpha-1}$, and so $(A, \|\cdot\|)$ is not equivalent to a uniform algebra.

We claim that A is contractive. Since A is homogeneous, it suffices to show that the maximal ideal $M := \{f \in A : f(0) = 0\}$ has a CAI.

For this, define

$$\Delta_n(s) = \max\{1 - n|s|, 0\} \quad (s \in [-1,1],\, n \in \mathbb{N}).$$

Then we can see that $I(\Delta_n) \sim 1/n^{2-\alpha}$, and so $\|1 - \Delta_n\| \leq 1 + O(1/n^{2-\alpha}) = 1 + o(1)$. Further, a calculation shows that $(1 - \Delta_n : n \in \mathbb{N})$ is an approximate identity for M.

We conclude that $((1-\Delta_n)/\|1-\Delta_n\| : n \in \mathbb{N})$ is a CAI in M, and so A is contractive. $\square$