Spectral Synthesis and Ideal Theory
Lecture 1

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A a commutative Banach algebra over $\mathbb{C}$

$\Delta(A) = \{\varphi : A \to \mathbb{C} \text{ surjective homomorphism}\} \subseteq A_1^*$

w^*-topology on $\Delta(A)$: weakest topology, for which all the functions $\widehat{a} : \Delta(A) \to \mathbb{C}, \varphi \to \widehat{a}(\varphi) = \varphi(a)$, $a \in A$, are continuous

$\Delta(A)$ is a locally compact Hausdorff space and $\overline{\Delta(A)} \subseteq \Delta(A) \cup \{0\}$

$\widehat{a}$ vanishes at infinity on $\Delta(A)$ (Riemann-Lebesgue), and $\Phi : a \to \widehat{a}$ is a norm decreasing homomorphism and

$$\sigma(a) \setminus \{0\} \subseteq \widehat{a}(\Delta(A)) \subseteq \sigma(a).$$

Φ is an isometry if and only if $\|a^2\| = \|a\|$ for every $a \in A$.

Φ is surjective if, in addition, $\Phi(A)$ is closed under complex conjugation.

Every commutative C^*-algebra A is isometrically isomorphic to $C_0(\Delta(A))$.

Definition

- $(\Delta(A), w^*)$ is called the *Gelfand spectrum* of A
- $A \to C_0(\Delta(A)), a \to \widehat{a}$ is called the *Gelfand homomorphism*
- A is *semisimple* if $a \to \widehat{a}$ is injective
- The w^*-topology is also called the *Gelfand topology*

Remark

(1) If A is unital, then $\Delta(A)$ is closed in A_1^*, hence compact

(2) When does $\Delta(A)$ compact imply that A is unital?

Ideals and Quotients

Let I be a closed ideal of A and $q : A \to A/I$ the quotient homomorphism

- $\varphi \to \varphi \circ q$ embeds $\Delta(A/I)$ topologically into $\Delta(A)$
- $\Delta(A/I)$ is closed in $\Delta(A)$
- $\Delta(A) \setminus \Delta(A/I) = \{\varphi \in \Delta(A) : \varphi|_I \neq 0\}$

Every $\psi \in \Delta(I)$ extends uniquely to some $\widetilde{\psi} \in \Delta(A)$ by

$$\widetilde{\psi}(a) = \frac{\psi(ab)}{\psi(b)}, \quad a \in A,$$

where $b \in I$ is such that $\psi(b) \neq 0$.
• $\psi \to \widetilde{\psi}$ is a homeomorphism from $\Delta(I)$ onto $\Delta(A) \setminus \Delta(A/I)$.

Maximal modular Ideals

Definition

Let A be a Banach algebra. An ideal I of A is called *modular* if the quotient algebra A/I has an idenity.

• Every modular ideal is contained in a maximal modular ideal

• Every maximal modular ideal is closed

Suppose that A is commutative.

• Then every maximal modular ideal has codimension one

• The map $\varphi \to \ker\varphi$ is a bijection between $\Delta(A)$ and $\mathrm{Max}(A)$, the set of all proper maximal modular ideals

The Hull-Kernel Topology

For $E \subseteq \Delta(A) = \mathrm{Max}(A)$ the *kernel* of E is defined by

$$k(E) = \{a \in A : \varphi(a) = 0 \text{ for all } \varphi \in E\} = \bigcap\{\ker(\varphi) : \varphi \in E\}$$

if $E \neq \emptyset$ and $k(\emptyset) = A$. If $E = \{\varphi\}$, write $k(\varphi)$ instead of $k(\{\varphi\})$ or $\ker(\varphi)$

For $B \subseteq A$, the *hull* of B is defined by

$$h(B) = \{\varphi \in \Delta(A) : \varphi(B) = \{0\}\} = \{M \in \mathrm{Max}(A) : B \subseteq M\}.$$

Remark

- $k(E)$ is a closed ideal of A
- $h(B)$ is a closed subset of $\Delta(A)$
- $E \subseteq h(k(E))$
- $h(k(E_1 \cup E_2)) = h(k(E_1)) \cup h(k(E_2))$

Definition

For $E \subseteq \Delta(A)$, let $\overline{E} = h(k(E))$. Then $E \to \overline{E}$ is a closure operation, i.e.

(1) $E \subseteq \overline{E}$ and $\overline{\overline{E}} = \overline{E}$

(2) $\overline{E_1 \cup E_2} = \overline{E_1} \cup \overline{E_2}$.

There exists a unique topology on $\Delta(A)$ such that $\overline{E}$ is the closure of E, the *hull-kernel topology*.

The *hk*-topology on $\Delta(A)$ is weaker than the Gelfand topology and in general not Hausdorff.

Problem: When do the two topologies on $\Delta(A)$ coincide?

Definition

A is called *regular* if for any closed subset E of $\Delta(A)$ which is closed in the Gelfand topology, and any $\varphi_0 \in \Delta(A) \setminus E$, there exists $a \in A$ such that

$$\varphi_0(a) \neq 0 \quad \text{and} \quad \varphi|_E = 0.$$

Theorem

For a commutative Banach algebra A, the following three conditions are equivalent.

1. *A is regular.*
2. *The hull-kernel topology and the Gelfand topology on $\Delta(A)$ coincide.*
3. *$\widehat{a}$ is continuous on $(\Delta(A), hk)$ for every $a \in A$.*

Proof of (1) $\Rightarrow$ (2)

Suppose that A is regular and let $E \subseteq \Delta(A)$ be closed in the Gelfand topology. To show that E is closed in the hk-topology, consider any $\varphi \in \Delta(A) \setminus E$:

- there exists $a_\varphi \in A$ such that $\widehat{a}_\varphi(\varphi) \neq 0$ and $\widehat{a}_\varphi = 0$ on E
- it follows that $k(E) \not\subseteq k(\varphi)$ for each $\varphi \in \Delta(A) \setminus E$
- thus $E = h(k(E))$, i.e. E is hk-closed

Since the Gelfand topology is the weak topology defined by the functions $\widehat{a}$, $a \in A$, the equivalence of (2) and (3) is clear. The proof of (3) $\Rightarrow$ (1) is somewhat more complicated.

Theorem

Let I be a closed ideal of the commutative Banach algebra A. Then the following conditions are equivalent.

- *A is regular*
- *I and A/I are both regular*

Theorem

A regular commutative Banach algebra A is even normal in the following sense.
Given a closed subset E of $\Delta(A)$ and a compact subset C of $\Delta(A)$ such that $C \cap E = \emptyset$, then there exists $a \in A$ such that

$$\widehat{a} = 1 \text{ on } C \quad \text{and} \quad \widehat{a} = 0 \text{ on } E.$$

Corollary

Let A be semisimple and regular. If $\Delta(A)$ is compact, then A has an identity.

Examples $C_0(X)$

X a locally compact Hausdorff space
$C_0(X) = \{f : X \to \mathbb{C} : f$ is continuous and vanishes at infinity$\}$

$C_0(X)$ is a commutative Banach algebra with pointwise operations and the sup-norm. For each closed subset E of X, let

$$I(E) = \{f \in C_0(X) : f = 0 \text{ on } E\}.$$

Theorem

The assignment $E \to I(E)$ is a bijection between the collection of all closed subsets E of X and the closed ideals of $C_0(X)$.

The proof is essentially an application of a variant of Urysohn's lemma: given a compact subset C of $X \setminus E$, there exists $f \in C_0(X)$ such that

$$f|_E = 0,, \ f|_C = 1 \text{ and } f(X) \subseteq [0,1].$$

Corollary

For $x \in X$, let

- $\varphi_x(f) = f(x)$ *for* $f \in C_0(X)$
- $M(x) = \{f \in C_0(X) : f(x) = 0\}$

Then $x \to \varphi_x$ *(resp.,* $x \to M(x) = \ker(\varphi_x)$*) is a homeomorphism between* X *and* $\Delta(C_0(X))$ *(resp.,* $\mathrm{Max}(C_0(X))$*). In particular,* $C_0(X)$ *is regular.*

Proof.

The map $x \to \varphi_x, X \to \Delta(C_0(X))$ is continuous since $x \to f(x)$ is continuous for each f.

Moreover, given $x \in X$ and an open neighbourhood V of x in X, by Urysohn's lemma there exists $f \in C_0(X)$ such that $f(x) \neq 0$ and $f = 0$ on $f = 0$ on $X \setminus V$. Thus

$$V \supseteq \{y \in X : |\varphi_y(f) - \varphi_x(f)| < |f(x)|,$$

which is a neighbourhood of x in the Gelfand topology. □

Example $C^n[a,b]$

Let $a, b \in \mathbb{R}$, $a < b$, $n \in \mathbb{N}$ and

$$C^n[a,b] = \{f : [a,b] \to \mathbb{C} : f \ \ n\text{-times continuously differentiable}\}.$$

With pointwise operations and the norm

$$\|f\| = \sum_{k=0}^{n} \frac{1}{n!} \|f^{(k)}\|_\infty,$$

$C^n[a,b]$ is a unital commutative Banach algebra. For $t \in [a,b]$, let

$$\varphi_t(f) = f(t), \quad f \in C^n[a,b].$$

Theorem

The map $t \to \varphi_t$ is a homeomorphism from $[a,b]$ onto $\Delta(C^n[a,b])$, and $C^n[a,b]$ is regular.

Outline of Proof

$t \to \varphi_t$ is an embedding of $[a, b]$ into $\Delta(C^n[a, b])$ because

- the mapping is injective and continuous
- $[a, b]$ is compact and $\Delta(C^n[a, b])$ is Hausdorff.

To show surjectivity, let $M \in \mathrm{Max}(C^n[a, b])$, and assume that $M \neq \ker(\varphi_t)$ for every $\in [a, b]$. Then, for each t, there exists $f_t \in M$ such that $f_t(t) \neq 0$. Then $f_t \neq 0$ in a neighbourhood V_t of t and hence

$$[a, b] = \bigcup_{j=1}^{r} V_{t_j}$$

for certain $t_1, \ldots, t_r$ and the function

$$f = \sum_{j=1}^{r} f_{t_j} \overline{f_{t_j}} \in M$$

has the property that $f(t) > 0$ for all $t \in [a, b]$. Then $\frac{1}{f} \in C^n[a, b]$, and hence $1 \in M$, which is a contradiction.

Regularity of $C^n[a, b]$:

Given $t_0 \in [a, b]$ and $\epsilon > 0$, construct $f \in C^n[a, b]$ such that $f(t_0) \neq 0$ and $f(t) = 0$ for $t \in [a, b]$ such that $|t - t_0| \geq \epsilon$.

To each $t \in [a, b]$ and $0 \leq k \leq n$, associate the closed ideal

$$I_k(t) = \{f \in C^n[a, b] : f^{(j)}(t) = 0 \text{ for } 0 \leq j \leq n\}.$$

It is clear that

$$I_n(t) \subseteq I_{n-1}(t) \subseteq \ldots \subseteq I_1(t) \subseteq I_0(t) = M(t),$$

and one can show that all the inclusions are proper.

Moreover, $h(I_k(t)) = \{t\}$ and there are now other closed ideals with hull $= \{t\}$.

$L^1(G)$, G abelian

G a locally compact abelian group, $\widehat{G}$ the dual group of G, equipped with the topology of uniform convergence on compact subsets of G.
For $\gamma \in \widehat{G}$, define $\varphi_\gamma : L^1(G) \to \mathbb{C}$ by

$$\varphi_\gamma(f) = \int_G f(x)\overline{\gamma(x)}\,dx, \quad f \in L^1(G).$$

• $\gamma \to \varphi_\gamma$ is a homeomorphism from $\widehat{G}$ onto $\Delta(L^1(G))$

• $L^1(G)$ is regular and semisimple

• $L^1(G)$ has an approximate identity with norm bound one, consisting of functions f such that $\widehat{f}$ has compact support

Examples

(1) $\widehat{\mathbb{R}^n} = \mathbb{R}^n$: $\gamma_y(x) = e^{i\langle x,y\rangle}$, $x, y \in \mathbb{R}^n$
(2) $\widehat{\mathbb{Z}} = \mathbb{T}$: $\gamma_z(n) = z^n$, $z \in \mathbb{T}$, $n \in \mathbb{Z}$

The Fourier Algebra $A(G)$

Let G be a locally compact group and $B(G)$ the Fourier-Stieltjes algebra of G.

The Fourier algebra $A(G)$ is the closure in $B(G)$ of the linear span of all functions of the form $f * \widetilde{g}$, $f, g \in C_c(G)$, where $\widetilde{g}(x) = \overline{g(x^{-1})}$. Then

- $A(G) = \{f * \widetilde{g} : f, g \in L^2(G)\}$
- $A(G) \subseteq C_0(G)$ and $A(G)$ is uniformly dense in $C_0(G)$.

Lemma

Let $x \in G$ and $u \in A(G)$ such that $u(x) = 0$. Then, given $\epsilon > 0$, there exists $v \in A(G)$ such that v vanishes in a neighbourhood of x and $\|u - v\| \leq \epsilon$.

Proof of the Lemma

• We can assume that $u \neq 0$, $u \in C_c(G)$, $\epsilon \leq \|u\|_\infty$ and $\epsilon < 1$. Let

$$W = \{y \in G : \|u - R_y u\|_{A(G)} \leq \epsilon\}.$$

• Choose $V \subseteq W$, V an open neighbourhood of e such that

$$\sup\{|u(xy)| : y \in V\} \leq \epsilon.$$

• Choose $U \subseteq V$, U a compact symmetric neighbourhood of e in G such that $|U| \geq |V|(1 - \epsilon)$.

• Let $f = |U|^{-1} 1_U$ and $g = 1_{xV} \cdot u$: $\quad f, g \in L^2(G)$

• Let $v = (u - g) * f \in A(G)$; then v has compact support and $v(y) = 0$ whenever $yU \subseteq xV$; so $v = 0$ in a neighbourhood of x

• $\|u - v\|_{A(G)} \leq \epsilon + \epsilon \left(\frac{1}{1-\epsilon}\right)^{1/2}$.

The Spectrum of $A(G)$

Lemma

Let C be a compact subset of G and U an open subset of G containing C. Then there exists a function u on G with the following properties:

(1) $0 \leq u \leq 1, u|_C = 1$ *and* $u|_{G \setminus U} = 0$.

(2) *u is a finite linear combination of functions in $P(G) \cap C_c(G)$.*

Proof.

There exists a compact neighbourhood V of e in G such that $V = V^{-1}$ and $CV^2 \subseteq U$. Then the function

$$u(x) = |V|^{-1} \left(1|_{CV} * 1|_V\right)(x) = |V|^{-1} \cdot |xV \cap CV|, \; x \in G,$$

satisfies (1). (2) follows from the polar identity for $f * g$. □

Theorem

Let G be an arbitrary locally compact group. For $x \in G$, let

$$\varphi_x : A(G) \to \mathbb{C}, \quad u \to u(x).$$

Then the map $x \to \varphi_x$ is a homeomorphism from G onto $\Delta(A(G))$. Moreover, $A(G)$ is regular.

Proof.

Clearly, $\varphi_x \in \Delta(A(G))$, and $x \to \varphi_x$ is injective. To show surjectivity, let $\varphi \in \Delta(A(G))$ be given and assume that $\varphi \neq \varphi_x$ for all $x \in G$. Then, for each $x \in G$, there exists $u_x \in A(G)$ such that

$$\varphi(u_x) = 1 \quad \text{and} \quad \varphi_x(u_x) = 0.$$

Then u_x is the limit of a sequence $(v_n)_n \subseteq A(G)$ such that $v_n = 0$ is a neighbourhood of x. Therefore, we can assume that $u_x = 0$ in a neighbourhood of x. □

Proof continued

Since $A(G) \cap C_c(G)$ is dense in $A(G)$, there exists $u_0 \in C_c(G) \cap A(G)$ with $\varphi(u_0) = 1$. Choose $x_1, \ldots, x_n \in \text{supp}(u_0)$ such that

$$\text{supp}(u_0) \subseteq \bigcup_{j=1}^{n} V_{x_j}$$

and let $u = u_0 \cdot \prod_{j=1}^{n} u_{x_j} \in A(G)$. Then $u(x) = 0$ for all $x \in G$, but

$$\varphi(u) = \varphi(u_0) \cdot \prod_{j=1}^{n} \varphi(u_{x_j}) = 1.$$

Thus the map $x \to \varphi_x, G \to \Delta(A(G))$ is surjective. It is a homeomorphism since, because $A(G)$ is uniformly dense in $C_0(G)$, the topology on G coincides with the weak topology defined by the set of functions

$$x \to u(x) = \varphi_x(u), \quad u \in A(G).$$