Spectral Synthesis and Ideal Theory
Lecture 3

Eberhard Kaniuth

University of Paderborn, Germany

Fields Institute, Toronto, April 2, 2014

The Restriction Map $A(G) \to A(H)$

Theorem

Let H be a closed subgroup of G. For every $u \in A(H)$, there exists $v \in A(G)$ such that

$$v|_H = u \quad \text{and} \quad \|v\|_{A(G)} = \|u\|_{A(H)}.$$

This important result was independently shown by McMullen and Herz:

C. Herz, *Harmonic synthesis for subgroups*, Ann. Inst. Fourier **23** (1973), 91-123.
J.R. McMullen, *Extension of positive definite functions*, Mem. Amer. Math. Soc. **117**, 1972.

Remark

If H is open in G, then v can be defined to be zero on $G \setminus H$. In the general case, the proof is fairly technical. We give a brief indication for second countable groups.

Suppose that G is second countable. Then there exists a Borel subset S of G with the following properties:

- $S \cap H = \{e\}$
- S intersects each right coset of H in exactly one point
- for each compact subset C of G, $HC \cap S$ is relatively compact
- there exists a closed neighbourhood V of e in G such that $HV = V$ and $V \cap S$ is relatively compact.

For $x \in G$, let $\beta(x)$ denote the unique element of H such that $x = \beta(x)s$ for some $s \in S$. For any function f on G, define f_V on G by

$$f_V(x) = f(\beta(x))1_V(x), \quad x \in G.$$

Lemma

Let $G, H, S, V, \ldots$ be as above. There exists a constant $c > 0$ such that $f \to c\, f_V$ is a linear isometry of $L^2(H)$ into $L^2(G)$. Moreover, for all $f, g \in L^2(H)$ and $h \in H$,

$$c^2\,(f_V *_G \widetilde{g}_V)\,(h) = (f *_H \widetilde{g})(h).$$

Remark

What is c?
If $f \in C_c(H)$, then f_V is bounded and measurable and has compact support. Thus we can define a linear functional I on $C_c(H)$ by

$$I(f) = \int_G f_V(x)dx.$$

Check that I is left invariant and if $f \geq 0$ and $f \neq 0$, then $I(f) > 0$. Thus I is a left Haar integral on H. By uniquenes, there exists $c > 0$ such that

$$c \int_G f_V(x)dx = \int_H f(h)dh.$$

Amenable Groups

Definition

A locally compact group G is called *amenable* if there exists a left invariant mean, i.e. a linear functional m on $L^\infty(G)$ such that $m(\bar{f}) = \overline{m(f)}$ for all $f \in L^\infty(G)$, $m(f) \geq 0$ if $f \geq 0$ and $m(1) = 1$.

Amenability of G can also be characterized through the existence of left invariant means on various other function spaces on G.

Examples

(1) Compact groups and abelian locally compact groups

(2) If N is a closed normal subgroup of G and N and G/N are both amenable, then G is amenable

(3) Closed subgroup of amenable groups are amenable

Further Examples

(4) If there exists an increasing sequence

$$\{e\} = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_r = G$$

of closed subgroups of G such that H_{j-1} is normal in H_j and every quotient group H_j/H_{j-1} is amenable, $1 \leq j \leq r$, then G is amenable

(5) Free groups and $SL(n, \mathbb{Z})$ are not amenable

(6) Noncompact semisimple Lie groups is not amenable

(7) If $G = \bigcup_\alpha H_\alpha$, where $(H_\alpha)_\alpha$ is an upwards directed system of closed amenable subgroups of G, then G is amenable.

For a locally compact group G with left Haar measure, let λ_G denote the left regular representation, i.e. the representation on $L^2(G)$ defined by

$$\lambda_G(x)f(y) = f(x^{-1}y), \quad f \in L^2(G),\ x \in G.$$

The coordinate functions of λ_G are the functions of the form

$$u_{f,g}(x) = \langle \lambda_G(x)f, g \rangle, \quad f, g \in L^2(G).$$

Theorem

For a locally compact group G, the following are equivalent:

1. *G is amenable*
2. *1_G is weakly contained in λ_G: the function 1 can be approximated uniformly on compact subsets of G by functions $u_{f,g}$*
3. *For every $f \in L^1(G)$, $f \geq 0$, $\|\lambda_G(f)\| = \|f\|_1$.*

Existence of a Bounded Approximate Identity in $A(G)$

Theorem

For a locally compact G, the following three conditions are equivalent:

1. *G is amenable*
2. *$A(G)$ has an approximate identity $(u_\alpha)_\alpha$ such that, for every α, $\|u_\alpha\| \leq 1$ and u_α is a positive definite function with compact support*
3. *$A(G)$ has a bounded approximate identity.*

H. Leptin, *Sur l'algèbre de Fourier d'une groupe localement compact*, C.R. Math. Acad. Sci. Paris Ser. A **266** (1968), 1180-1182.

The proof outlined below is taken from an unpublished thesis of Nielson and appears in

J. de Canniere and U. Haagerup, *Multipliers of the Fourier algebra of some simple Lie groups and their discrete subgroups*, Amer. J. Math. **107** (1985), 455-500.

Outline of Proof

Have to show (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (1)

(1) $\Longrightarrow$ (2): Amenability of G is equivalent to that 1_G is weakly contained in $\lambda_G \Longrightarrow$ given $K \subseteq G$ compact and $\epsilon > 0$, there exists $u_{K,\epsilon} \in P(G)$ such that

- $|u_{K,\epsilon} - 1| \leq \epsilon$ for all $x \in K$
- $u_{K,\epsilon}$ is a coordinate function of λ_G.

Since $C_c(G)$ is dense in $L^2(G)$, we can assume that $u_{K,\epsilon}$ has compact support. (2) follows now from the following lemma, applied to $u = 1_G$.

Lemma

Let $(u_\alpha)_\alpha$ be a net in $P(G)$ and $u \in P(G)$ such that $u_\alpha \to u$ uniformly on compact subsetes of G. Then

$$\|(u_\alpha - u)v\|_{A(G)} \to 0$$

for every $v \in A(G)$.

For (3) $\Longrightarrow$ (1) one shows that $\|\lambda_G(f)\| = \|f\|_1$ for every $f \in C_c(G)$, $f \geq 0$.

This implies amenability of G.

Let $(u_\alpha)_\alpha$ be an approximate identity for $A(G)$ bounded by $c > 0$. Let $K = \text{supp}(f)$ and choose a compact symmetric neighbourhood V of e in G. Set

$$u = |V|^{-1}(1_V * 1_{VK}) \in A(G).$$

Then $u = 1$ on K and hence, since $\|u_\alpha u - u\|_{A(G)} \to 0$, $u_\alpha \to 1$ uniformly on K. This implies, since $f \geq 0$,

$$\begin{aligned} \|f\|_1 &= \lim_\alpha |\langle u_\alpha, f\rangle| = \lim_\alpha |\langle u_\alpha, \lambda_G(f)\rangle| \\ &\leq c\,\|\lambda_G(f)\|. \end{aligned}$$

Replacing f with the n-fold convolution product f^n, it follows that

$$\|f\|_1^n = \|f^n\|_1 \leq c\,\|\lambda_G(f^n)\| \leq c\,\|\lambda_G(f)\|^n$$

and therefore

$$\|f\|_1 \leq \|\lambda_G(f)\| \cdot \lim_{n\to\infty} c^{1/n} = \|\lambda_G(f)\| \leq \|f\|_1.$$

This completes the proof of (3) $\Longrightarrow$ (1).

When does Spectral Synthesis hold for $A(G)$?

Necessary Condition: $u \in \overline{uA(G)}$ for every $u \in A(G)$.

Sufficient Condition: $G = \Delta(A(G))$ is discrete and $u \in \overline{uA(G)}$ for every $u \in A(G)$.

Remark

The hypothesis that $u \in \overline{uA(G)}$ for every $u \in A(G)$ is satisfied in the following cases:

- G is amenable: then $A(G)$ has a bounded approximate identity
- $G = \mathbb{F}_2$, $G = SL(2, \mathbb{R})$ or $G = SL(2, \mathbb{R})$: then $A(G)$ has an approximate identity, which is bounded in the multiplier norm (Haagerup).

Question: Do we always have $u \in \overline{uA(G)}$ for every $u \in A(G)$?

Theorem

Let G be an arbitrary locally compact group. Then spectral synthesis holds for $A(G)$ (if and) only if G is discrete and $u \in \overline{uA(G)}$ for each $u \in A(G)$.

E. Kaniuth and A.T. Lau, *Spectral synthesis for $A(G)$ and subspaces of $VN(G)$*, Proc. Amer. Math. Soc. **129** (2001), 3253-3263.

Independently, this result was also shown in

K. Parthasarathy and R. Prakash, *Malliavin's theorem for weak synthesis on nonabelian groups*, Bull. Sci. Math. **134** (2010), 561-576.

Lemma

Let H be a closed subgroup of G, and let

$$I(H) = \{u \in A(G) : u|_H = 0\}.$$

Then the restriction map $A(G) \to A(H)$ induces an isometric isomorphism

$$A(G)/I(H) \to A(H), \quad u + I(H) \to u|_H.$$

Proof.

The map $u + I(H) \to u|_H$ is an algebra isomorphism from $A(G)/I(H)$ into $A(H)$. By the restriction theorem, it is surjective, and it is an isometry, since

$$\begin{aligned} \|u|_H\|_{A(H)} &= \inf\{\|v\|_{A(G)} : v \in A(G), v|_H = u|_H\} \\ &= \inf\{\|v\|_{A(G)} : v - u \in I(H)\} \\ &= \|u + I(H)\| \end{aligned}$$

for every $u \in A(G)$. □

Lemma

Let K be a compact normal subgroup of G, $q : G \to G/K$ the quotient homomorphism and E a closed subset of G/K. If $q^{-1}(E)$ is a set of synthesis for $A(G)$, then E is a set of synthesis for $A(G/K)$.

Proof.

Given $u \in k(E)$ and $\epsilon > 0$, consider $u_1 = u \circ q$. Then $u_1 \in k(q^{-1}(E))$ and hence there exists $v_1 \in j(q^{-1}(E))$ such that $\|u_1 - v_1\| \leq \epsilon$. Define v on G/K by

$$v(xK) = \int_K v_1(xk)\, dk = \int_K (R_k v_1)(x)\, dk.$$

Then $v \in A(G/K)$ and

$$\|u - v\|_{A(G/K)} \left\| \int_K R_k(u_1 - v_1) dk \right\|_{A(G/K)} \leq \|u - v\|_{A(G)} \leq \epsilon.$$

□

Proof continued

Moreover, $v \in j(E)$ since:

- $C = \text{supp}(v_1)$ is compact and $C \cap q^{-1}(E) = \emptyset$
- hence there exists a symmteric neighbourhood V of e in G such that $C \cap Vq^{-1}(E) = \emptyset$
- v vanishes on the neighbourhood $q(Vq^{-1}(E))$ of E since $v_1 = 0$ von $Vq^{-1}(E)$
- supp $V \subseteq q(C)$

Lemma

Let G be a connected locally compact group. If spectral synthesis holds for $A(G)$, then G is trivial.

Proof.

Assume that $G \neq \{e\}$.

- G connected $\Longrightarrow$ G contains a compact normal subgroup K such that G/K is a Lie group
- spectral synthesis holds for $A(G/K)$
- the nontrivial connected Lie group G/K contains a closed nondiscrete abelian subgroup H (a one-parameter subgroup)
- spectral synthesis holds for $A(H)$ since $A(H)$ is a quotient of $A(G)$
- this contradicts Malliavin's theorem for abelian groups □

Proof of the Theorem

Suppose that synthesis holds for $A(G)$

• then synthesis holds for G_0, the connected component of the identity

• $G_0 = \{e\}$ by the preceding lemma, i.e. G is totally disconnected

Fix a compact open subgroup K of G, and assume that K is infinite.

• by a theorem of Zelmanov, every infinite compact group contains an infinite abelian subgroup, say H

• then spectral synthesis holds for $A(H)$, which contradicts Malliavin's theorem

Fourier Algebras of Coset Spaces

G a locally compact group, K a compact subgroup of G with normalized Haar measure

G/K the space of left cosets of K, equipped with the quotient topology, $q : G \to G/K$ the quotient map

Definition

$A(G/K) = \{u : G/K \to \mathbb{C} : u \circ q \in A(G)\}$ is called the *Fourier algebra of* G/K.

Let $p_K : A(G) \to A(G/K)$ be defined by

$$p_K(u)(xK) = \int_K u(xk)dk, \quad u \in A(G), x \in G.$$

Then p_K maps the subalgebra

$$\{u \in A(G) : u(xk) = u(x) \text{ for all } k \in K \text{ and all } x \in G\}$$

of $A(G)$ isometrically onto $A(G/K)$.

The spaces $A(G/K)$ are precisely the norm closed left translation invariant subspaces of $A(G)$ (Takesaki/Tatsuuma).

Theorem

1. *$A(G/K)$ is regular and semisimple*
2. *$\Delta(A(G/K)) = G/K$: the map $xK \to \varphi_{xK}$, where $\varphi_{xK}(u) = u(xK)$, is a homeomorphism*
3. *$A(G/K)$ has a bounded approximate identity if and only if G is amenable*

B.E. Forrest, *Fourier analysis on coset spaces*, Rocky Mountain J. Math. **28** (1998), 173-190.

When does Spectral Synthesis hold for $A(G/K)$?

Yes, if K is open in G and $u \in \overline{uA(G/K)}$ for every $u \in A(G/K)$.

Conjecture: The converse is true.

Theorem

Let G contain a nilpotent open subgroup. If K is a compact subgroup of G and spectral synthesis holds for $A(G/K)$, then K is open in G.

Corollary

Suppose that G_0, the connected component of the identity, is nilpotent. If K is a compact subgroup of G and spectral synthesis holds for $A(G/K)$, then $G_0 \subseteq K$.

E. Kaniuth, Weak spectral synthesis in Fourier algebras of coset spaces, *Studia Math.* **197** *(2010), 229-246.*

Lemma

Let H be a closed subgroup and K a compact subgroup of G. Then the restriction map

$$A(G/K) \to A(H/H \cap K), \quad u \to u|_H$$

is surjective in any of the two cases:

- *H is contained in the normalizer of K*
- *H is open in G.*

Lemma

Let $i : H/H \cap K \to G/K$, $x(H \cap K) \to xK$, $x \in H$, and suppose that

$$u \to u|_H,\ A(G/K) \to A(H/H \cap K)$$

is surjective. Let E be a closed subset of $H/H \cap K = \Delta(A(H/H \cap K))$. If $i(E)$ is a set of synthesis (Ditkin set) for $A(G/K)$, then E is a set of synthesis (a Ditkin set) for $A(H/H \cap K)$.

Corollary

1. *Singletons $\{xK\}$ are sets of synthesis for $A(G/K)$*
2. *If G is amenable, then finite subsets of G/K are Ditkin sets for $A(G/K)$.*

Proof.

Take $H = K$ and recall that xK is a set of synthesis for $A(G)$ and that xK is a Ditkin set if G is amenable. □

(1) and (2) for sets of synthesis were already proved by Forrest l.c..