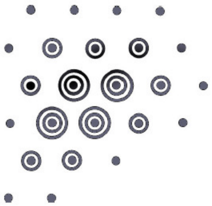

Hypergroups

and their amenability notions

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Contents

HYPERGROUPS

A locally compact space H is a hypergroup if
$\exists * : M(H) \times M(H) \to M(H)$ called convolution:

- $\forall x, y \in H$, $\delta_x * \delta_y$ is a positive measure with compact support and $\|\delta_x * \delta_y\|_{M(H)} = 1$.

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- $(x, y) \mapsto \delta_x * \delta_y$ is a continuous map from $H \times H$ into $M(H)$ equipped with the weak* topology.
- $(x, y) \to \operatorname{supp}(\delta_x * \delta_y)$ is a continuous mapping from $H \times H$ into $\mathcal{K}(H)$ equipped with the Michael topology.

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- $(x, y) \to \mathrm{supp}(\delta_x * \delta_y)$ is a continuous mapping from $H \times H$ into $\mathcal{K}(H)$ equipped with the Michael topology.

- $\exists e \in H$, δ_e is the identity of $M(H)$.
- $\exists$ a homeomorphism $x \to \check{x}$ of H called involution such that $(\delta_x * \delta_y)\check{} = \delta_{\check{y}} * \delta_{\check{x}}$.
- $e \in \mathrm{supp}(\delta_x * \delta_y)$ if and only if $y = \check{x}$.

Haar measure

Let $f \in C_c(H)$,

$$L_x f(y) = \delta_{\tilde{x}} * \delta_y(f) =: f(\delta_{\tilde{x}} * \delta_y).$$

A positive non-zero Borel measure h is called a Haar measure if

$$h(L_x f) = h(f), \quad \forall f \in C_c(H), x \in H.$$

For a commutative and/or compact and/or discrete hypergroup, the existence of a Haar measure can be proven.

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For a commutative and/or compact and/or discrete hypergroup, the existence of a Haar measure can be proven.

For $A, B \subseteq H$, $A * B \subseteq H$ where

$$A * B := \bigcup_{x \in A,\ y \in B} \operatorname{supp}(\delta_x * \delta_y).$$

Hypergroup algebra

For every $f, g \in L^1(H, h)$,

$$f * g = \int_H f(y)\, L_y g \, dh(y), \;\; f^*(y) = \overline{f(\tilde{y})}.$$

$L^1(H)(= L^1(H, h))$ forms a $*$-algebra called hypergroup algebra.

Fourier Space of Hypergroups

[Muruganandam, 07] defined Fourier Stieltjes space of hypergroups, similar to group case, and consequently Fourier space of H, $A(H)$.

$$A(H)^* = VN(H) = \lambda(L^1(H))'' \subseteq \mathcal{B}(L^2(H)).$$

Proposition. [A.'14]
For a hypergroup H,

$$A(H) := \{f * \bar{\tilde{g}} : f, g \in L^2(H)\}.$$

And $\|u\|_{A(H)} = \inf\{\|f\|_2 \|g\|_2\}$ for all $f, g \in L^2(H)$ s.t. $u = f * \tilde{g}$.

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The hypergroup H is called regular Fourier hypergroup if $A(H)$ is a Banach algebra with respect to pointwise multiplication.

Commutative hypergroups

Let H be a commutative hypergroup,

$$\widehat{H} := \{\alpha \in C_b(H) : \alpha(\delta_x * \delta_y) = \alpha(x)\alpha(y),\ \alpha(\check{x}) = \overline{\alpha(x)},\ \text{and } \alpha \neq 0\}.$$

$\widehat{H}$ is the Gelfand spectrum of $L^1(H)$. $\widehat{H}$ is called the dual of H.

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$\widehat{H}$ is not necessarily a hypergroup any more!

Fourier-Stieltjes transform and Fourier transform defined:

$$\mathcal{F} : M(H) \to C_b(H) \ \text{ where } \ \mathcal{F}(\mu)(\alpha) := \int_H \overline{\alpha}(x) d\mu(x).$$

$$\mathcal{F} : L^1(H) \to C_0(H) \ \text{ where } \ \mathcal{F}(f)(\alpha) := \int_H f(x)\overline{\alpha}(x) dh(x)$$

Plancherel measure

Theorem.
Let H be a commutative hypergroup. Then there exists a non-negative measure π on $\widehat{H}$, called Plancherel measure of $\widehat{H}$ such that

$$\int_H |f(x)|^2 dx = \int_{\widehat{H}} |\widehat{f}(\alpha)|^2 d\pi(\alpha)$$

for all $f \in L^1(H) \cap L^2(H)$.

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Note that for an arbitrary hypergroup H (unlike group case) the support of the Plancherel measure,

$$\mathrm{supp}(\pi) \neq \widehat{H}.$$

Example 0.

Locally compact groups

Every locally compact group G, it is a regular Fourier hypergroup.

EXAMPLE 1.

REPRESENTATIONS OF COMPACT GROUPS

Let G be a compact (quantum) group and $\widehat{G}$ the set of all irreducible unitary (co-)representations of G.

For each $\pi_1, \pi_2 \in \widehat{G}$, $\pi_1 \otimes \pi_2 \cong \sigma_1 \oplus \cdots \oplus \sigma_n$ for $\sigma_1, \cdots, \sigma_n \in \widehat{G}$.

Define a convolution on $\ell^1(\widehat{G})$ and make $\widehat{G}$ into a commutative discrete hypergroup which is called the fusion algebra of G.

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Define a convolution on $\ell^1(\widehat{G})$ and make $\widehat{G}$ into a commutative discrete hypergroup which is called the fusion algebra of G.

$\ell^1(\widehat{G})$ is isometrically isomorphic to
$ZA(G) = \{f \in A(G) : f(yxy^{-1}) = f(y) \text{ for all } x, y \in G\}$.

[A. '13]:
$\widehat{G}$ is a regular Fourier hypergroup and $A(\widehat{G}) \cong ZL^1(G)$.

EXAMPLE 2.

CONJUGACY CLASS OF $\overline{[\mathrm{FC}]}^{B}$ GROUPS

The space of all orbits in a locally compact group G for some relatively compact subgroup B of automorphisms of G including inner ones denoted by $\mathrm{Conj}_B(G)$.

$\mathrm{Conj}_B(G)$ forms a commutative hypergroup.
$L^1(\mathrm{Conj}_B(G))$ is isometrically isomorphic to
$Z_B L^1(G) = \{f \in L^1(G) : f \circ \beta = f \text{ for all } \beta \in B\}$.

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When B is the set of all inner automorphisms, we use $\mathrm{Conj}(G)$.
Then $A(\mathrm{Conj}(G)) \cong ZA(G)$.

- $\mathrm{Conj}(G)$ is a compact hypergroup if G is compact.
- $\mathrm{Conj}(G)$ is a discrete hypergroup if G is discrete.

EXAMPLE 3.

DOUBLE COSET HYPERGROUPS

Let G be a locally compact group and K be a compact subgroup of G.

$$G//K := \{KxK : x \in G\}.$$

forms a hypergroup.

$$L^1(G//K) \cong \{f \in L^1(G) : f \text{ is constant on double cosets of } K\}.$$

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[Muruganandam '08]:
$G//K$ is a regular Fourier hypergroup and

$$A(G//K) \cong \{f \in A(G) : f \text{ is constant on double cosets of } K\}.$$

Example 4.

Polynomial hypergroups

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $(a_n)_{n\in\mathbb{N}_0}$ and $(c_n)_{n\in\mathbb{N}_0}$ be sequences of non-zero real numbers and $(b_n)_{n\in\mathbb{N}_0}$ be a sequence of real numbers with the property

$$\begin{aligned} a_0 + b_0 &= 1 \\ a_n + b_n + c_n &= 1, \quad n \geq 1. \end{aligned}$$

If $(R_n)_{n\in\mathbb{N}_0}$ is a sequence of polynomials defined by

$$\begin{aligned} R_0(x) &= 1, \\ R_1(x) &= \frac{1}{a_0}(x - b_0), \\ R_1(x)R_n(x) &= a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x), \qquad n \geq 1, \end{aligned}$$

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Then,

$$R_n(x)R_m(x) = \sum_{k=|n-m|}^{n+m} g(n,m;k)R_k(x)$$

where $g(n,m;k) \in \mathbb{R}^+$ for all $|n-m| \leq k \leq n+m$.

EXAMPLE 4.

POLYNOMIAL HYPERGROUPS

Define $*$ on $\ell^1(\mathbb{N}_0)$ such that

$$\delta_n * \delta_m = \sum_{k=|n-m|}^{n+m} g(n,m;k)\delta_k$$

and $\check{n} = n$.

Then $(\mathbb{N}_0, *, \check{\ })$ is a discrete commutative hypergroup with the unit element 0 which is called the polynomial hypergroup on $\mathbb{N}_0$ induced by $(R_n)_{n\in\mathbb{N}_0}$.

Contents

Left invariant mean

[Skantharajah '92]:

A linear functional $m \in L^\infty(H)^*$ is called a mean if it has norm 1 and is non-negative, i.e. $f \geq 0$ a.e. implies $m(f) \geq 0$.

m is called left invariant mean if $m(L_x f) = m(f)$.

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Theorem.
Every commutative and/or compact hypergroup is amenable.

Reiter's conditions

[Skantharajah '92]:

H satisfies (P_r), $1 \leq r < \infty$, if whenever $\epsilon > 0$ and a compact set $E \subseteq H$ are given, then there exists $f \in L^r(H), f \geq 0, \|f\|_r = 1$ such that

$$\|L_x f - f\|_r < \epsilon \quad (x \in E).$$

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When H is commutative.

Note that $(P_1) \not\Rightarrow (P_2)$.

CONTENTS

LEPTIN CONDITION

[Singh '96]:

(L) H satisfies the Leptin condition if for every compact subset K of H and $\epsilon > 0$, $\exists V$ measurable in H such that $0 < h(V) < \infty$ and

$$\frac{h(K * V)}{h(V)} < 1 + \epsilon.$$

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Theorem. [Singh 96]
Let H be a hypergroup satisfying (L). Then it does (P_r) for $1 \leq r < \infty$.

Leptin hypergroups

Hypergroups satisfying (L) condition:

- Amenable locally compact groups. (Leptin '68)
- Some simple polynomial hypergroups. (Singh '96)
- Every compact hypergroup.
- $\widehat{SU(2)}$. (A. '13)

Modified Leptin Condition

[A. '14]:

(L_D) H satisfies the D-Leptin condition for some $D \geq 1$ if for every compact subset K of H and $\epsilon > 0$, $\exists V$ measurable in H such that $0 < h(V) < \infty$ and

$$\frac{h(K * V)}{h(V)} < D + \epsilon.$$

EXAMPLES OF (L_D) HYPERGROUPS.

[A.]:

- Let G be an FD group. Then Conj(G) satisfies the D-Leptin condition for $D = |G'|$.

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- By [Banica- Vergnioux '09]: dual of connected simply connected compact real Lie group satisfies some D-Leptin condition:

	classic computation	BV algorithm
$\widehat{SU(2)}$	1	15
$\widehat{SU(3)}$	$3^8 = 6561$	18240
$\widehat{SU(4)}$	?	$\geq 18 * 10^{14}$

An application of Leptin condition

Theorem. [Choi-Ghahramani '12]
Every proper Segal algebra of $\mathbb{T}^d$ is not approximately amenable.

AN APPLICATION OF LEPTIN CONDITION

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Theorem. [A. '14]
Let G be a compact group such that $\widehat{G}$ satisfies D-Leptin condition. Every proper Segal algebra of G is not approximately amenable.

D-Leptin and (P_2)?

Question.
Let H be a hypergroup satisfying (L_D). Does it satisfy (P_2)?

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D-Leptin and (P_2)?

Question.
Let H be a hypergroup satisfying (L_D). Does it satisfy (P_2)?

If H is a locally compact group: Yes!

1. (L_D) implies that $A(H)$ has a D-bounded approximate identity.
2. **Leptin's Theorem:** $A(H)$ has a bounded approximate identity if and only H is amenable
3. H is amenable if and only if (P_2).

$(L_D) \Rightarrow (P_2)$ FOR HYPERGROUPS

Proposition. [A. '14]
Let H be a regular Fourier hypergroup. If H satisfies (L_D). Then $A(H)$ has a D-bounded approximate identity.

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If H is a regular Fourier hypergroup. Then $A(H)$ has a bounded approximate identity if and only if H satisfies (P_2). Then there is a 1-bounded approximate identity for $A(H)$.

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$$(L_D) \Rightarrow (D - b.a.i \Leftrightarrow)(P_2).$$

Application of Leptin's Theorem.

Corollary. [A. '14]
Let G be a locally compact group. Then $G//K$ satisfies (P_2) for every compact subgroup K if and only if G is amenable .

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Proof.

- $G//K$ is a regular Fourier hypergroup and $A(G//K)$ is $f \in A(G)$ which are constant on double cosets of K. (Murugunandam '08)

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(i) If G is amenable if and only if $A(G)$ has a bounded approximate identity. (Leptin '68)

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(i) If G is amenable if and only if $A(G)$ has a bounded approximate identity. (Leptin '68)

(ii) $A(G)$ has a bounded approximate identity if and only if $A(G//K)$ has a bounded approximate identity.

(iii) By Leptin's Theorem, $G//K$ satisfies (P_2) if and only if $A(G//K)$ has a b.a.i.

Let H be a regular Fourier hypergroup.

(m) H is amenable.

(L_D) H satisfies the D-Leptin condition for some $D \geq 1$.

(B_D) $A(H)$ has a D-bounded approximate identity for some $D \geq 1$.

(P_2) H satisfies (P_2).

(P_1) H satisfies Reiter condition.

(AM) $L^1(H)$ is an amenable Banach algebra.

$$\begin{array}{ccccccccc} (L_1) & & \Longrightarrow & & (P_2) & \underset{\not\Leftarrow}{\Longrightarrow} & (P_1) & \underset{\not\Rightarrow}{\Longleftarrow} & (\mathrm{AM}) \\ \Downarrow & & & & \Updownarrow & & \Updownarrow & & \\ (L_D) & \Longrightarrow & (B_D) & \Longleftrightarrow & (B_1) & & (m) & & \end{array}$$

Contents

AMENABLE HYPERGROUP ALGEBRAS

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Amenable hypergroup algebras

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Theorem. [Skantharajah '92]
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Example. [Azimifard-Samei-Spronk '09]
$L^1(\mathrm{Conj}(SU(2)))(= ZL^1(SU(2)))$ is not amenable. But $\mathrm{Conj}(SU(2))$ is amenable.

Amenable polynomial hypergroup algebras

Chebyshev polynomial hypergroup on $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$:

$$\delta_m * \delta_n = \frac{1}{2}\delta_{|n-m|} + \frac{1}{2}\delta_{n+m}.$$

Amenable polynomial hypergroup algebras

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- $Z_{\pm 1}A(\mathbb{T})$ is a subalgebra of an amenable Banach algebra ($A(\mathbb{T})$) invariant under a finite subgroup of $Aut(A(\mathbb{T}))$. [Kepert '94]: $Z_{\pm 1}A(\mathbb{T})$ is amenable.

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Theorem. [Lasser '07]
Let $\mathbb{N}_0$ be a polynomial hypergroup and for each $N > 0$, $\{x \in \mathbb{N}_0 : h(x) \leq N\}$ is finite. Then $\ell^1(\mathbb{N}_0)$ is not amenable.

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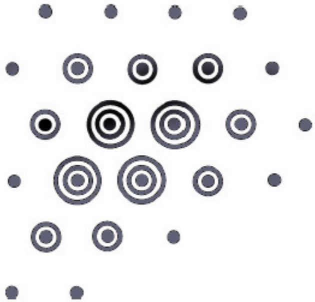
If G is a non-abelian connected compact group, then $L^1(\mathrm{Conj}(G))(= ZL^1(G))$ is not amenable.



Thank You