Taylor's functional calculus and derived categories

Alexei Yu. Pirkovskii

Department of Mathematics
National Research University Higher School of Economics
Moscow, Russia

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Locally Compact Quantum Groups and Amenability

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Outline

1. Taylor's joint spectrum and holomorphic functional calculus

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1. Taylor's joint spectrum and holomorphic functional calculus
2. Derived categories and derived functors
3. Quasi-coherent analytic Fréchet sheaves
4. A derived version of Taylor's functional calculus theorem

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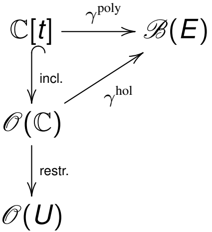

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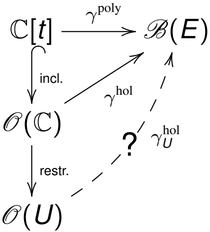

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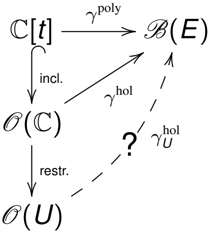

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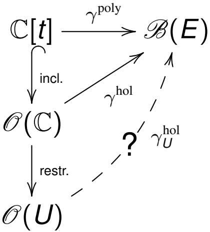

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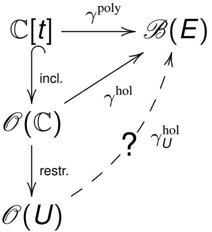

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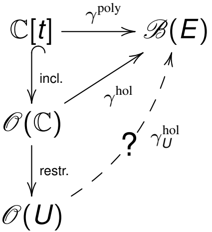

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- $\sigma(f(T)) = f(\sigma(T))$ *(the Spectral Mapping Theorem).*

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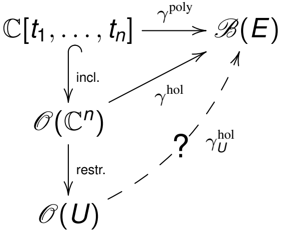

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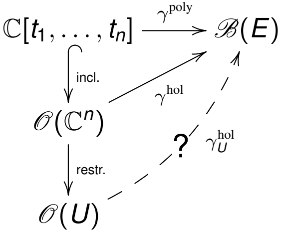

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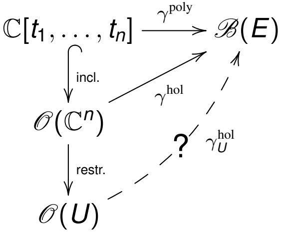

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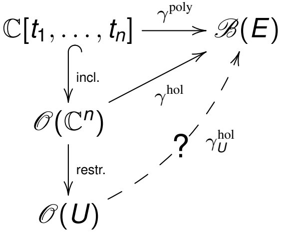

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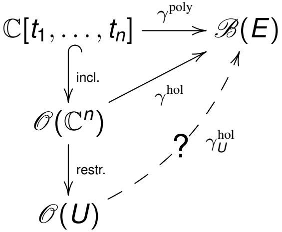

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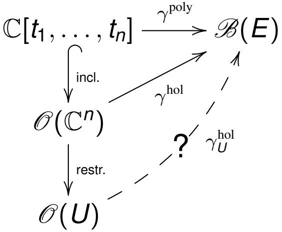

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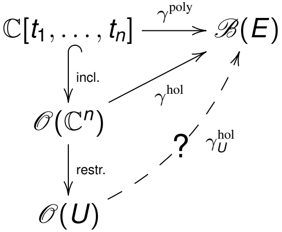

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Example

If $n = 1$, then $K(T, E) = (0 \leftarrow E \xleftarrow{T} E \leftarrow 0)$.

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$$0 \leftarrow E \xleftarrow{T-\lambda\mathbf{1}} E \leftarrow 0 \quad \text{is exact} \iff T - \lambda\mathbf{1} \quad \text{is invertible.}$$

Hence $\sigma(T)$ is the usual spectrum of T.

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Theorem (Taylor, 1970–72)

- *If U is an open subset of $\mathbb{C}^n$ and $\sigma(T) \subset U$, then there exists a unital continuous homomorphism*

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- Here $f(T) \stackrel{\mathrm{def}}{=} \gamma_U^{\mathrm{hol}}(f)$ if $m = 1$, and
- $f(T) \stackrel{\mathrm{def}}{=} (f_1(T), \ldots, f_m(T))$ if $f = (f_1, \ldots, f_m) \in \mathscr{O}(U, \mathbb{C}^m)$.

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- **Question.** Suppose $\sigma(M) \subset U$. By Taylor's theorem, N exists. Is it possible to construct N explicitly?

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- **Goal:**

$$\boxed{N = \mathsf{R}\Gamma(U, \mathscr{O}_{\mathbb{C}^n}) \mathbin{\widehat{\otimes}}^{\mathsf{L}}_{\mathscr{O}(\mathbb{C}^n)} M.}$$

- $\mathsf{R}\Gamma$ is the total right derived functor of Γ,
- $\widehat{\otimes}^{\mathsf{L}}_{\mathscr{O}(\mathbb{C}^n)}$ is the total left derived functor of $\widehat{\otimes}_{\mathscr{O}(\mathbb{C}^n)}$.

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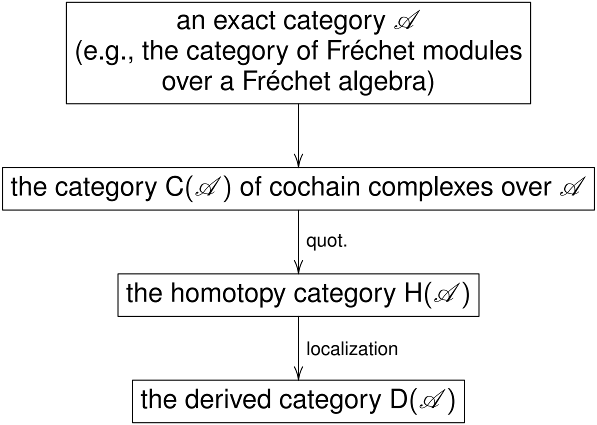
- **Solution:** add new morphisms to the category of complexes in such a way that $F(P)$ and $F(Q)$ become isomorphic in the new category.

The construction of the derived category



Exact categories

Quasi-definition (D. Quillen, 1973; A. Heller, 1958)

An exact category is $(\mathscr{A}, \mathscr{E})$, where $\mathscr{A}$ is an additive category and $\mathscr{E}$ is a class of diagrams in $\mathscr{A}$ of the form

$$X \xrightarrow{i} Y \xrightarrow{p} Z,$$

which are called exact pairs (or short admissible sequences) and which satisfy a number of axioms.

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- mod-A, $\underline{\mathrm{mod}}$-A, A-mod-B, A-$\underline{\mathrm{mod}}$-B ...

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- The shift functor $[1]\colon C(\mathscr{A}) \to C(\mathscr{A}), \quad H(\mathscr{A}) \to H(\mathscr{A})$:
  - If $X \in C(\mathscr{A})$, then $X[1]^n \overset{\text{def}}{=} X^{n+1}$ and $d^n_{X[1]} = -d^{n+1}_X$;
  - If $f\colon X \to Y$ is a morphism in $C(\mathscr{A})$, then $f[1]^n \overset{\text{def}}{=} f^{n+1}$.

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A cochain complex $X \in \mathrm{C}(\mathscr{A})$ is admissible if, for every $n \in \mathbb{Z}$,

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- Let $f\colon X \to Y$ be a morphism in $\mathsf{C}(\mathscr{A})$.
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$$M(f)^n = X^{n+1} \oplus Y^n, \quad d^n_{M(f)} = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_X^n \end{pmatrix}$$

Definition

A morphism $f\colon X \to Y$ in $\mathsf{C}(\mathscr{A})$ is a quasi-isomorphism (qis) if $M(f)$ is admissible.

Examples

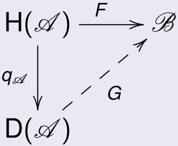
- If $\mathscr{A}$ is abelian, then $f\colon X \to Y$ is a qis $\iff$ $H^n(f)\colon H^n(X) \to H^n(Y)$ is an isomorphism for every n.
- If $\mathscr{A} = \mathscr{A}_{\mathrm{spl}}$, then $f\colon X \to Y$ is a qis $\iff$ f is a homotopy equivalence.
- If $\mathscr{A} = A$-mod, then $f\colon X \to Y$ is a qis $\iff$ f is a hmt. equiv. in $\mathsf{C}(\mathrm{Fr})$.
- If $\mathscr{A} = A$-$\underline{\mathrm{mod}}$, then $f\colon X \to Y$ is a qis $\iff$ $H^n(f)\colon H^n(X) \to H^n(Y)$ is an isomorphism for every n.

The derived category

Definition

The derived category of $\mathscr{A}$ is $(\mathrm{D}(\mathscr{A}), q_{\mathscr{A}})$, where

- $\mathrm{D}(\mathscr{A})$ is a category;
- $q_{\mathscr{A}} \colon \mathrm{H}(\mathscr{A}) \to \mathrm{D}(\mathscr{A})$ is a functor that takes quasi-isomorphisms to isomorphisms;
- For each category $\mathscr{B}$ and each functor $F \colon \mathrm{H}(\mathscr{A}) \to \mathscr{B}$ that takes quasi-isomorphisms to isomorphisms there exists a unique functor $G \colon \mathrm{D}(\mathscr{A}) \to \mathscr{B}$ making the following diagram commute:



Construction of $\mathrm{D}(\mathscr{A})$

- $\operatorname{Ob} \mathrm{D}(\mathscr{A}) = \operatorname{Ob} \mathrm{H}(\mathscr{A}) = \operatorname{Ob} \mathrm{C}(\mathscr{A})$.

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- Given $X, Y \in \operatorname{Ob}(\mathrm{D}(\mathscr{A}))$, let

$$\begin{aligned}\operatorname{Hom}_{\mathrm{D}(\mathscr{A})}(X, Y) &= \{\text{“right fractions” } fs^{-1} : s \text{ is a qis}\}\\ &= \{\text{eqv. classes of pairs } (f, s) : X \xleftarrow{s} Z \xrightarrow{f} Y,\\ &\qquad f \text{ is any morphism}, \ s \text{ is a qis}\}.\end{aligned}$$

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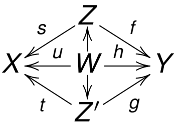
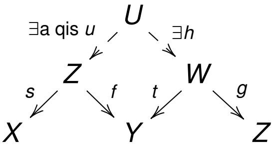
- $(f, s) \sim (g, t) \iff \exists (h, u)$ making the diagram



commute.

- Composition:



$$(gt^{-1}) \circ (fs^{-1}) \stackrel{\text{def}}{=} (gh)(su)^{-1}.$$

- $\mathrm{D}(\mathscr{A})$ is an additive category.

Subcategories of $\mathrm{D}(\mathscr{A})$

- $\mathrm{C}^+(\mathscr{A}) = \{X \in \mathrm{C}(\mathscr{A}) : \exists N \in \mathbb{Z} \text{ such that } X^n = 0 \ \forall n < N\}$.
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- $\mathsf{C}^\pm(\mathscr{A}) \xrightarrow{\text{quot.}} \mathsf{H}^\pm(\mathscr{A}) \xrightarrow{\text{loc.}} \mathsf{D}^\pm(\mathscr{A})$.
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- $\mathscr{A}$ is a full additive subcategory of $\mathrm{D}^b(\mathscr{A}) : \quad X \mapsto (0 \to X \to 0)$.

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- If $\mathscr{A}$ has enough projectives (i.e., $\mathscr{P}$ satisfies (FP1)), then $\mathscr{P}$ is F-projective for every F.

Derived functors

Fact

Let $\mathscr{P} \subset \mathscr{A}$ be a full additive subcategory satisfying (FP1). Then

- $\forall X \in \mathrm{C}^-(\mathscr{A}) \quad \exists P \in \mathrm{C}^-(\mathscr{P})$ and a qis $P \to X$.

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$$
\begin{array}{ccccc}
 & & \text{exact} & & \\
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\downarrow & & \downarrow & & \downarrow \\
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- **Definition.** The left derived functor of F is

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- Right derived functors are defined dually.

Classical derived functors

- Suppose that $\mathscr{B}$ is quasi-abelian, and let $n \in \mathbb{Z}$.

$$\mathsf{C}(\mathscr{B}) \xrightarrow{H^n} \mathscr{B}$$

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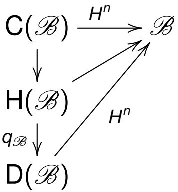

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$$\mathsf{H}(\mathscr{B}) \to \mathscr{B}$$

- Let $F \colon \mathscr{A} \to \mathscr{B}$ be an additive functor such that $\mathsf{L}F$ exists.

Definition

The nth classical left derived functor of F is

$$\mathsf{L}_n F = H^{-n} \circ \mathsf{L}F|_{\mathscr{A}} \colon \mathscr{A} \to \mathscr{B}.$$

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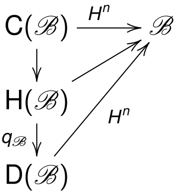

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- Classical right derived functors are defined similarly.

The derived projective tensor product

Example

- $A =$ a Fréchet algebra, $\mathscr{A} = A$-mod, $\mathscr{B} =$ LCS.
- $Y \in$ mod-A, $\quad F = Y \mathbin{\widehat{\otimes}}_A (-) \colon \mathscr{A} \to \mathscr{B}$.

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- A-mod has enough projectives $\Longrightarrow$ $\mathrm{L}F$ exists.
- $\mathrm{L}F(X) = Y \mathbin{\widehat{\otimes}}_A^{\mathrm{L}} X$ is the derived projective tensor product of Y and X.

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- $\mathrm{L}_n F(X) = \mathrm{Tor}^A_n(Y, X)$.

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- **Fact.** $(\mathrm{QCoh}(X), \text{all exact pairs})$ is an exact category (follows from M. Putinar, 1980).

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- Indeed, otherwise $\mathscr{M}(X) = 0$, but $\mathscr{M}(X) = M$.

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- $\mathrm{Tor}_k^A(\mathbb{C}_\lambda, M) \cong H_k(K(T - \lambda, M))$.
- Hence $\lambda \in \rho(M) \iff K(T - \lambda, M)$ is exact (Taylor's original definition (1970)).

Taylor's functional calculus

Theorem

Consider the following properties of $M \in \mathrm{D}^-(A\text{-}\underline{\mathrm{mod}})$*:*

(i) $\sigma(M) \subset U$;

(ii) $M = \mathscr{M}(X) \to \mathscr{M}(U)$ *is an isomorphism in* $\mathrm{D}^-(A\text{-}\underline{\mathrm{mod}})$;

(iii) $\sigma(M) \subset \overline{U}$.

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

If, in addition, X *is a manifold and* $M \in \mathrm{D}^-(A\text{-}\underline{\mathrm{Banmod}})$, *then* (i) $\iff$ (ii).

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Corollary

Let $M \in A\text{-}\underline{\mathrm{mod}}$*, and suppose that* $\sigma(M) \subset U$*. Then*

$$M \to H^0(\mathscr{M}(U))$$

is an isomorphism in A-mod.

As a consequence, the action of $\mathscr{O}(X)$ *on* M *extends to an action of* $\mathscr{O}(U)$.