Amenability properties of the non-commutative Schwartz space

Krzysztof Piszczek

Adam Mickiewicz University
Poznań, Poland
e-mail: kpk@amu.edu.pl

Workshop on OS, LCQ Groups and Amenability,
Toronto, Canada, 26 – 30 May, 2014.

Plan

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2. Known results - short and informative

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3. Amenability

$$s = \left\{ \xi = (\xi_j)_{j\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}} \colon\ |\xi|_k^2 := \sum_{j=1}^{+\infty} |\xi_j|^2 j^{2k} < +\infty \text{ for all } k \in \mathbb{N}_0 \right\},$$

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$$L(s',s) := \{\text{linear and continuous maps } x\colon\ s' \to s\},$$

topology on $L(s',s)$ given by $\|x\|_k := \sup\{|x\xi|_k \colon\ |\xi|_k' \leqslant 1\}$.

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Definition

$$\mathcal{S} := (L(s', s), \cdot, *).$$

Non-commutative Schwartz space because

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Algebra of smooth operators because

$$\mathcal{S} \simeq C^{\infty}([a, b]).$$

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4. Locally convex operator spaces – work of Effros et al.

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$x = (\langle xe_j, e_i\rangle)_{i,j} = (x_{ij})_{i,j} \in \mathcal{S} \iff$

$$\|x\|_{k,\infty} := \sup_{i,j} |x_{ij}|(ij)^k < +\infty \text{ for all } k \in \mathbb{N}.$$

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Proposition (Bost, 1990; independently Domański, 2012)

If $\mathcal{S}_1 := \mathcal{S} \oplus \mathbb{C}$ *then* $\sigma_{\mathcal{S}_1}(x) = \sigma_{\mathcal{B}(\ell_2)}(x)$.

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2) Approximate identities

Definition

$(u_\alpha)_{\alpha\in\Lambda}$ is an a.i. if $u_\alpha x \to x$ and $x u_\alpha \to x$ for all x. It is bounded if the set of u_α's is bounded. It is sequential if $\Lambda = \mathbb{N}$.

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Proposition

If $u_n := \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$ *then* $(u_n)_n$ *is a sequential a.i. No b.a.i. in* $\mathcal{S}$.

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2. Characterization of closed, commutative $*$-subalgebras of $\mathcal{S}$.
3. Some consequences:
   (i) $\mathcal{S} = \operatorname{span} \mathcal{S}_+$.
   (ii) $x \geqslant 0 \Leftrightarrow x = y^* y$ for some $y \in \mathcal{S} \Leftrightarrow \langle x\xi, \xi \rangle \geqslant 0 \ \forall \xi \in s'$.

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Important in all 'automatic continuity' proofs: $\mathcal{S}$ is nuclear, i.e. $\mathcal{S}\tilde{\otimes}_\pi\mathcal{S} = \mathcal{S}\tilde{\otimes}_\varepsilon\mathcal{S}$, equivalently, unconditionally summable sequences are absolutely summable. Consequently, bounded subsets are relatively compact.

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Amenable are all:

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Theorem (B.E. Johnson, 1972)

TFAE for a locally compact group G:

1. *G is amenable,*
2. *for every $L^1(G)$-bimodule X and every continuous derivation $\delta\colon L^1(G) \to X'$ there is an $x \in X'$ with $\delta(a) = a \cdot x - x \cdot a$.*

Definition

A (Banach, Fréchet,...) algebra A is *amenable* if for every (Banach, Fréchet,...) A-bimodule X any continuous derivation $D\colon A \to X'$ is inner.

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A (Banach, Fréchet,...) algebra A is *super-amenable* if for every (Banach, Fréchet,...) A-bimodule X any continuous derivation $D\colon A \to X$ is inner.

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A (Banach, Fréchet,...) algebra A is *super-amenable* if for every (Banach, Fréchet,...) A-bimodule X any continuous derivation $D\colon A \to X$ is inner.

Fundamental Question

When a continuous derivation is inner?

A bunch of results.

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3. Helemskii: The only super-amenable and commutative Banach algebras are $\mathbb{C}^n$.
4. $\Rightarrow$ Connes, 1978; $\Leftarrow$ Haagerup, 1983: A C^*-algebra is amenable if and only if it is nuclear.

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Definition

A derivation $\delta\colon A \to X$ is uniformly approximately inner if

$$\delta(a) = \lim_{\alpha}(a \cdot x_\alpha - x_\alpha \cdot a) \ \ \forall\, a \in A$$

and $(x_\alpha)_\alpha$ is bounded in X.
A is uniformly approximately amenable if every continuous derivation into any dual A-bimodule is uniformly approximately inner.

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Question

Due to (weaker) notions of approximate amenability, find out how approximately amenable our algebra is?

Definition

A derivation $\delta\colon A \to X$ is boundedly approximately inner if

$$\delta(a) = \lim_{\alpha}(a \cdot x_\alpha - x_\alpha \cdot a) \quad \forall\, a \in A$$

and $(a \mapsto a \cdot x_\alpha - x_\alpha \cdot a)_\alpha$ is equicontinuous in $L(A, X)$.
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Due to (weaker) notions of approximate amenability, find out how approximately amenable our algebra is?

Definition

A derivation $\delta \colon A \to X$ is approximately inner if

$$\delta(a) = \lim_{\alpha}(a \cdot x_\alpha - x_\alpha \cdot a) \;\; \forall\, a \in A$$

and $(x_\alpha)_\alpha \subset X$.
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A – a Fréchet lmc algebra. An approximate identity $(u_\alpha)_\alpha \subset A$ is called a multiplier-bounded left approximate identity if the set $\{u_\alpha a \colon \alpha\}$ is bounded for every $a \in A$.

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Theorem (Choi, Ghahramani, Zhang, 2009)

If a Fréchet lmc algebra is boundedly approximately amenable and possesses both, multiplier bounded left and right approximate identity then it necessarily admits a bounded approximate identity.

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Theorem (Choi, Ghahramani, Zhang, 2009)

If a Fréchet lmc algebra is boundedly approximately amenable and possesses both, multiplier bounded left and right approximate identity then it necessarily admits a bounded approximate identity.

Corollary

The non-commutative Schwartz space is not boundedly approximately amenable. (Because $u_n := \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$ *is a m.b.a.i.)*

Theorem (Dales, Loy, Zhang, 2006)

A Fréchet lmc algebra $(A, (||\cdot||_n)_{n\in\mathbb{N}})$ is approximately amenable if and only if for each $\varepsilon > 0$, each finite subset S of A and every $k \in \mathbb{N}$ there exist $F \in A \otimes A$ and $u, v \in A$ such that $\pi(F) = u + v$ and for each $a \in S$:

(i) $||a \cdot F - F \cdot a + u \otimes a - a \otimes v||_k < \varepsilon$,

(ii) $||a - au||_k < \varepsilon$ *and* $||a - va||_k < \varepsilon$.

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Recall:

$\pi(a \otimes b) := ab,\ a \cdot (x \otimes y) := ax \otimes y,\ (x \otimes y) \cdot b := x \otimes yb.$

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Simplifications for $\mathcal{S}$: $\mathcal{S}\widehat{\otimes}\mathcal{S} = \mathcal{S}(\mathcal{S})$ is just matrices of matrices and it is enough to work with singletons.

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Proof.

For any $a \in \mathcal{S}$ and $u = v = u_n$ and $F = \mathrm{diag}(u_n, \ldots, u_n, 0, 0, \ldots)$:
$||a \cdot F - F \cdot a + u \otimes a - a \otimes v||_k < \varepsilon,\ ||a - au||_k < \varepsilon,\ ||a - va||_k < \varepsilon.$

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The problem is $\pi(F) = u_n \neq 2u_n = u + v$ and we need to slightly perturb the 'big matrix' F. The solution is

$$F = \begin{pmatrix} u_n + \frac{1}{n}e_{11} & \frac{1}{n}e_{21} & \ldots & \frac{1}{n}e_{n1} & 0 & \ldots \\ \frac{1}{n}e_{12} & u_n + \frac{1}{n}e_{22} & \ldots & \frac{1}{n}e_{n2} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n}e_{1n} & \frac{1}{n}e_{2n} & \ldots & u_n + \frac{1}{n}e_{nn} & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \qquad \square$$