# Amenability properties of Banach algebra valued continuous functions

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# Outline


1. Preliminaries
2. Amenability
3. generalized amenability
4. weak amenability

# $C(X, A)$

Let $X$ be a compact Hausdorff space and $A$ a Banach algebra. Denote

$$C(X.A) = \text{the space of } A\text{-valued continuous functions on } X.$$

With pointwise algebraic operations and the uniform norm

$$\|f\|_\infty = \sup\{\|f(x)\|_A : \ x \in X\}$$

$C(X, A)$ is a Banach algebra.

**Examples**

- $C(X, \ell_1) = \{(x_i(t)) : x_i \in C(X), \sum_{i=1}^{\infty} |x_i| \text{ converges uniformly on } X\}$
- Let $\mathfrak{M}$ be a W*-algebra and $E$ be its predual, Then $C(X, \mathfrak{M}) = \mathcal{K}(E, C(X))$, the space of compact operators from $E$ into $C(X)$.

Early investigation of $C(X, A)$ goes back to 1940's, when I. Kaplansky and A. Hausner studied the maximal ideal space of the algebra for commutative $A$.

We note

- $C(X, A)$ is a C*-algebra if and only if $A$ is a C*-algebra.
- $C(X, A)$ is commutative if and only if $A$ is commutative.
- $C(X, A)$ has a BAI if and only if $A$ has a BAI.

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We are concerned with the amenability properties of $C(X, A)$. We will show constructively, among other things, that

- $C(X, A)$ is amenable if and only if $A$ is amenable;
- if $A$ is commutative, then $C(X, A)$ is weakly amenable if and only if $A$ is weakly amenable.

# approximate diagonal

For Banach spaces $V$ and $W$, we denote by $V \otimes W$ the algebraic tensor product, and by $V\hat{\otimes}W$ the Banach space projective tensor product of $V$ and $W$. The norm of $V\hat{\otimes}W$ is denoted by $\|\cdot\|_p$.

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If $\mathcal{A}$ is a Banach algebra, then $\mathcal{A} \hat{\otimes} \mathcal{A}$ is a Banach $\mathcal{A}$-bimodule with the module actions determined by

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca$$

## Definition 1

*A net $(\alpha_\nu)) \subset \mathcal{A} \hat{\otimes} \mathcal{A}$ is called an approximate diagonal for $\mathcal{A}$ if*

$$\lim_\nu \|a \cdot \alpha_\nu - \alpha_\nu \cdot a\|_p = 0 \text{ and } \lim_\nu \pi(\alpha_\nu) a = a \quad (a \in \mathcal{A}),$$

*where $\pi \colon \mathcal{A} \hat{\otimes} \mathcal{A} \to \mathcal{A}$ is the product map defined by $\pi(a \otimes b) = ab$. If in addition there is constant $m > 0$ such that $\|\alpha_\nu\| \le m$ for all $\nu$, then $(\alpha_\nu)$ is called a bounded approximate diagonal.*

# amenability

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- For a locally compact group $G$, B.E. Johnson (1972) showed that $L^1(G)$ is amenable if and only if $G$ is an amenable group.
- Using Johnson's above result on $L^1(G)$ and the Stone-Weierstrass Theorem, M. V, Sheinberg (1977) showed that $C(X) = C(X, \mathbb{C})$ is amenable for any compact Hausdorff space $X$.
- A direct proof for the amenability of $C(X)$, by constructing a bounded approximate diagonal, was give by (Abtahi-Z. 2010).

We are concerned with general $C(X, A)$.

# Grothendieck inequality

The following inequality due to A. Grothendieck is important to us.

## Theorem 1 (Grothendieck)

*Let $K_1, K_2$ be compact Hausdorff spaces, and let $\Phi$ be a bounded scalar-valued bilinear form on $C(K_1) \times C(K_2)$. Then there are probability measures $\mu_1, \mu_2$ on $K_1, K_2$, respectively, and a constant $k > 0$ such that*

$$|\Phi(x, y)| \leq k\|\Phi\| \left( \int_{K_1} |x|^2 d\mu_1 \int_{K_2} |y|^2 d\mu_2 \right)^{\frac{1}{2}}$$

*for $x \in C(K_1)$ and $y \in C(K_2)$.*

The smallest constant $k$ in the above theorem is called the Grothendieck constant, denoted $K_G^{\mathbb{C}}$. We have known $4/\pi \leq K_G^{\mathbb{C}} < 1.405$. Therefore, the constant $k$ in the theorem may be chosen independent of the spaces $K_1$ and $K_2$.

As a consequence of the Grothendieck Theorem we have

## Corollary 2

*Let $K_1, K_2$ be compact Hausdorff spaces and $c = \frac{1}{2}K_G^{\mathbb{C}}$. Then for each $u = \sum_{i=1}^{n} x_i(t) \otimes y_i(t) \in C(K_1) \otimes C(K_2)$ we have*

$$\|u\|_p \le c \left( \| \sum_{i=1}^{n} |x_i(t)|^2 \|_\infty + \| \sum_{i=1}^{n} |y_i(t)|^2 \|_\infty \right).$$

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$$\|u\|_p \leq c\left(\|\sum_{i=1}^{n} |x_i(t)|^2\|_\infty + \|\sum_{i=1}^{n} |y_i(t)|^2\|_\infty\right).$$

## Proof.

We note that $(C(K_1)\hat{\otimes}C(K_2))^* = BL(C(K_1), C(K_2); \mathbb{C})$.

$$\|u\|_p = \sup_{\Phi\in[BL(C(K_1),C(K_2);\mathbb{C})]_1} |\Phi(u)| \leq K_G^{\mathbb{C}} \sum_{i=1}^{n} \left(\int |x_i|^2 d\mu_1 \int |y_i|^2 d\mu_2\right)^{1/2}$$

$$\leq c\left(\int \sum_{i=1}^{n} |x_i|^2 d\mu_1 + \int \sum_{i=1}^{n} |y_i|^2 d\mu_2\right) \leq c\left(\|\sum_{i=1}^{n} |x_i|^2\| + \|\sum_{i=1}^{n} |y_i|^2\|\right).$$

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Given a finite open covering $\{V_1, V_2, \dots V_n\}$ of the compact space $X$, from the partition of unity there are continuous functions $0 \le g_i \le 1$, $i = 1, 2, \dots, n$, such that $\mathrm{supp}(g_i) \subset V_i$ and

$$\sum_{i=1}^{n} g_i(t) = 1 \quad t \in X.$$

Now let $u_i = \sqrt{g_i}$ and $u = \sum_{i=1}^{n} u_i \otimes u_i$. Normally we can only estimate

$$\|u\|_p \le \sum_{i=1}^{n} \|u_i\|_\infty^2 \le n.$$

However, using the Grothendieck inequality we have

$$\|u\|_p \le c \left( \| \sum_{i=1}^{n} u_i^2 \|_\infty + \| \sum_{i=1}^{n} u_i^2 \|_\infty \right) = 2c \| \sum_{i=1}^{n} g_i \|_\infty = 2c.$$

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# Amenability of $C(X, A)$

## Theorem 3

*Let $X$ be a compact Hausdorff space and let $A$ be a Banach algebra. If $A$ has a bounded approximate diagonal, then so does $C(X, A)$.*

### Proof

It suffices to show that there is a constant $L$ so that, for any $\varepsilon > 0$ and any finite set $F \subset C(X, A)$, we can find $U = U_{(F,\varepsilon)} \in C(X, A)\hat{\otimes}C(X, A)$ such that

$$\|U\|_p \leq L, \quad \|f \cdot U - U \cdot f\|_p < \varepsilon \quad \text{and } \|\pi(U)f - f\|_\infty < \varepsilon$$

for all $f \in F$. Indeed, this will implies that the net $(U_{(F,\varepsilon)})$ is the desired bounded approximate diagonal for $C(X, A)$.

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for all $f \in F$. Indeed, this will implies that the net $(U_{(F,\varepsilon)})$ is the desired bounded approximate diagonal for $C(X, A)$.
To avoid complicated computation we only consider the case that $F$ contains only elements of the form $f(t) = h(t)a$ $(t \in X)$, where $h \in C(X)$ and $a \in A$.

For $u = \sum_i u_i \otimes v_i \in C(X) \otimes C(X)$ and $\alpha = \sum_j \alpha_j \otimes \beta_j \in A\hat{\otimes}A$, it is readily seen that

$$T(u, \alpha) = \sum_{i,j} u_i\alpha_j \otimes v_i\beta_j \in C(X, A)\hat{\otimes}C(X, A) \quad \text{and}$$

$$\|T(u, \alpha)\|_p \leq \|u\|_p\|\alpha\|_p.$$

Let $(\alpha_\nu) \subset A\hat{\otimes}A$ be a bounded approximate diagonal for $A$ such that $\|\alpha_\nu\|_p \leq M$ for all $\nu$.

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These are finite sets in, respectively, $A$ and $C(X)$.

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These are finite sets in, respectively, $A$ and $C(X)$. So there is $\alpha \in (\alpha_\nu)$ such that

$$\|a \cdot \alpha - \alpha \cdot a\|_p < \varepsilon, \quad \|\pi(\alpha)a - a\|_A < \varepsilon \quad (a \in F_A).$$

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Since $F_{\mathbb{C}}$ is finite, there are finite open sets $V_i \subset X$ ($i = 1, 2, \ldots, n$), such that $X = \cup_i V_i$ and

$$|h(t) - h(s)| < \varepsilon \quad (h \in F_{\mathbb{C}}, t, s \in V_i).$$

Apply partition of unity. We obtain continuous functions $g_1, g_2, \ldots, g_n \in C(X)$ such that $\mathrm{Supp}(g_i) \subset V_i$, $0 \leq g_i(x) \leq 1$ and $g_1 + g_2 + \cdots + g_n \equiv 1$ on $X$. Let $u_i = \sqrt{g_i}$ and set $u = \sum_{i=1}^{n} u_i \otimes u_i$. Then $u \in C(X) \otimes C(X)$ and $\pi(u) = 1$. By Grothendieck's inequality, $\|u\|_p \leq 2c$.

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$$\|h \cdot u - u \cdot h\|_p \le \|\sum_i (h - h(t_i)) u_i \otimes u_i\|_p + \|\sum_i u_i \otimes (h - h(t_i)) u_i\|_p < 4c\varepsilon$$

for all $h \in F_{\mathbb{C}}$.

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for all $h \in F_{\mathbb{C}}$. We now take $U = T(u, \alpha)$. Then $\|U\|_p \le \|u\|_p \|\alpha\| \le 2cM = L$ and for all $f(t) = h(t)a \in F$ we can have

$$\|f \cdot U - U \cdot f\|_p \le \text{const.} \cdot \varepsilon; \quad \|\pi(U)f - f\| \le \text{const.} \cdot \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we obtain the desired inequalities.

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Apply partition of unity. We obtain continuous functions $g_1, g_2, \ldots, g_n \in C(X)$ such that $\mathrm{Supp}(g_i) \subset V_i$, $0 \le g_i(x) \le 1$ and $g_1 + g_2 + \cdots + g_n \equiv 1$ on $X$. Let $u_i = \sqrt{g_i}$ and set $u = \sum_{i=1}^n u_i \otimes u_i$. Then $u \in C(X) \otimes C(X)$ and $\pi(u) = 1$. By Grothendieck's inequality, $\|u\|_p \le 2c$. Now from each $V_i$ we take a point $t_i$. We have

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Since $\varepsilon > 0$ is arbitrary we obtain the desired inequalities.

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**Corollary 4**

*$C(X, A)$ is amenable if and only if $A$ is amenable.*

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# unbounded approximate diagonal

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An approximate diagonal $(u_\alpha)$ for $A$ is called

- central if $a \cdot u_\alpha = u_\alpha \cdot a$ for all $a \in A$ and all $\alpha$;
- a compactly approximate diagonal if $\|a \cdot u_\alpha - u_\alpha \cdot a\|_p \to 0$ and $\pi(u_\alpha)a \to a$ uniformly on compact sets of $A$.

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The proof of Theorem 3 may be modified to get the following.

**Theorem 5**

*Let $X$ be a compact Hausdorff space and let $A$ be a Banach algebra. If $A$ has a central compactly approximate diagonal, then $C(X, A)$ has a compactly approximate diagonal.*

**Example**

*$C(X, \ell_1)$ has a compactly approximate diagonal and hence is pseudo amenable.*

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# weak amenability

Let $A$ be a Banach algebra and $Y$ a Banach $A$-bimodule. A linear map $D\colon A \to Y$ is a derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A).$$

In 1987 Bade, Curtis and Dales introduced weak amenability for commutative Banach algebras. Their definition is as follows.

**Definition 2**

*A commutative Banach algebra $A$ is weakly amenable if every continuous derivation from $A$ into a commutative Banach $A$-bimodule is necessarily zero.*

Here an $A$-bimodule $X$ is commutative if $a \cdot x = x \cdot a$ ($a \in A$, $x \in X$). They showed

**Theorem 6 (Bade-Curtis-Dales)**

*A commutative Banach algebra $A$ is weakly amenable if and only if every continuous derivation from $A$ into $A^*$ is necessarily zero.*

for $C(X, A)$, we have noticed that it is commutative if and only if $A$ is so.

### Lemma 1

*Let $A$ be a unital commutative Banach algebra. If $A$ is weakly amenable, then so is $C(X, A)$.*

for $C(X, A)$, we have noticed that it is commutative if and only if $A$ is so.

## Lemma 1

*Let $A$ be a unital commutative Banach algebra. If $A$ is weakly amenable, then so is $C(X, A)$.*

## Proof.

Let $Y$ be a commutative Banach $C(X, A)$-bimodule and $D$: $C(X, A) \to Y$ be a continuous derivation. Let $e$ be the unit of $A$. Identify $C(X)$ with the subalgebra $C(X)e$ of $C(X, A)$. Then $Y$ is naturally a commutative Banach $C(X)$- and also $A$-bimodule.

$$D_{C(X)} : h \mapsto D(he) \quad (h \in C(X)), \text{ and } D_A : a \mapsto D(a) \quad (a \in A)$$

are continuous derivations into $Y$. They must be zeros.Thus

$$D(ha) = hD_A(a) + D_{C(X)}(h)a = 0 \quad (h \in C(X), a \in A).$$

Then $D = 0$ on the dense subset $lin\{ha : h \in C(X), a \in A\}$ of $C(X, A)$. So $D = 0$ on the whole $C(X, A)$. □

For a commutative Banach algebra $A$ the following are well-known.

- $A$ is weakly amenable iff $A^\sharp$ is weakly amenable.;
- if $A$ is weakly amenable, then a closed ideal $I$ of $A$ is weakly amenable if and only if $I^2 = lin\{ab : a, b \in I\}$ is dense in $I$.
- If there is a Banach algebra epimorphism from $A$ onto $B$ and if $A$ is weakly amenable, then so is $B$.

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### Theorem 7

*let $X$ be a compact Hausdorff space and $A$ be a commutative Banach algebra. Then $C(X, A)$ is weakly amenable if and only if $A$ is weakly amenable.*

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### Theorem 7

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### Proof.

If $A$ is weakly amenable, then so is $A^\sharp$. So is $C(X, A^\sharp)$ due to previous lemma. Now $C(X, A)$ is a closed ideal of $C(X, A^\sharp)$, and $C(X, A)^2$ is dense in $C(X, A)$ iff and only if $A^2$ is dense in $A$ which is true since $A$ is weakly amenable. Thus $C(X, A)$ is weakly amenable. □

# non-commutative case

For non-commutative Banach algebras, B. E. Johnson suggested the following definition.

**Definition 3**

*A Banach algebra A is weakly amenable if every continuous derivation from A into $A^*$ is inner.*

In this definition all group algebras and all C*-algebras are weakly amenable. However many nice properties that a commutative weakly amenable Banach algebra has will no longer be available. For example, homomorphic image of a weakly amenable Banach algebra may not be weakly amenable. It is weakly amenable only when the kernel of the homomorphism has so called the trace extension property.

For $C(X, A)$ with $A$ being non-commutative, we take a $t_0 \in X$ and consider the the Banach algebra epimorphism

$$T_0 : C(X, A) \to A \text{ defined by } T_0(f) = f(t_0).$$

It can be verified that $\ker(T_0)$ has the trace extension property. So we have

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- If $C(X, A)$ is weakly amenable, then so is $A$.

We do not know whether the converse is true or not. The main difficulty is how to deal with a continuous derivation from $A$ into $C(X, A)^*$. We don't know this kind of derivation is inner or not, assuming $A$ is weakly amenable. However, it is not hard to see

- Suppose that $X$ is a finite set. If $A$ is weakly amenable, then so is $C(X, A)$.

# testing non-commutative case

Let $A$ be weakly amenable and have a bounded approximate identity. Then a continuous derivation $D\colon C(X,A) \to C(X,A)^*$ must satisfy

$$D(ha) = hD(a) \quad (h \in C(X), a \in A).$$

So to show $D$ is inner is equivalent to show $D|_A\colon A \to C(X,A)^*$ is inner.

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$$D(ha) = hD(a) \quad (h \in C(X), a \in A).$$

So to show $D$ is inner is equivalent to show $D|_A$: $A \to C(X, A)^*$ is inner. Consider the simplest compact space that has infinite elements: $X = \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, the one point compactification of $\mathbb{N}$. Then we have the $A$-module decomposition

$$C(\overline{\mathbb{N}}, A) = c_0(A) \oplus A, \quad \text{and } C(\overline{\mathbb{N}}, A)^* = \ell_1(A^*) \oplus A^*.$$

So $D|_A = (\oplus_{\ell_1} D_i) \oplus \tilde{D}$. Each term on the right side is a continuous derivation from $A$ to $A^*$ and hence is inner.

# testing non-commutative case

Let $A$ be weakly amenable and have a bounded approximate identity. Then a continuous derivation $D$: $C(X, A) \to C(X, A)^*$ must satisfy

$$D(ha) = hD(a) \quad (h \in C(X), a \in A).$$

So to show $D$ is inner is equivalent to show $D|_A$: $A \to C(X, A)^*$ is inner. Consider the simplest compact space that has infinite elements: $X = \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, the one point compactification of $\mathbb{N}$. Then we have the $A$-module decomposition

$$C(\overline{\mathbb{N}}, A) = c_0(A) \oplus A, \quad \text{and } C(\overline{\mathbb{N}}, A)^* = \ell_1(A^*) \oplus A^*.$$

So $D|_A = (\oplus_{\ell_1} D_i) \oplus \tilde{D}$. Each term on the right side is a continuous derivation from $A$ to $A^*$ and hence is inner. But there is a difficulty to control the norm of the elements of $A^*$ that implement these inner derivations. We can only conclude the following.

- If $A$ is weakly amenable, then $C(\overline{\mathbb{N}}, A)$ is approximately weakly amenable.

Thank You!