Totally Disconnected L.C. Groups: Flat groups of automorphisms

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Lecture 1: The scale and minimising subgroups for an endomorphism

Lecture 2: Tidy subgroups and the scale

Lecture 3: Subgroups associated with an automorphism

Lecture 4: Flat groups of automorphisms

- Simultaneous minimisation
- Abelian groups of automorphisms are flat
- Examples and questions concerning flat groups

Flat groups of automorphisms

The preceding lectures concerned single automorphisms. This final lecture will describe groups of automorphisms that share a common minimising subgroup.

Definition

1. A subgroup $\mathcal{H} \leq \mathrm{Aut}(G)$ is *flat* if there is $V \in \mathcal{B}(G)$ that is minimising, or tidy, for every $\alpha \in \mathcal{H}$.
2. The *uniscalar* subgroup of $\mathcal{H}$ is

$$\mathcal{H}_1 = \left\{\alpha \in \mathcal{H} \mid s(\alpha) = 1 = s(\alpha^{-1})\right\}$$

$\mathcal{H}_1$ is a subgroup because $\alpha \in \mathcal{H}_1$ if and only if $\alpha(V) = V$ for any, and hence all, subgroups minimising for $\mathcal{H}$.

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The structure of simultaneously minimising subgroups

Theorem

Let $\mathcal{H}$ be a finitely generated flat group of automorphisms of G and suppose that V is tidy for $\mathcal{H}$. Then $\mathcal{H}_1 \lhd \mathcal{H}$ and there is $r \in \mathbb{N}$ such that

$$\mathcal{H}/\mathcal{H}_1 \cong \mathbb{Z}^r.$$

1. *There is $q \in \mathbb{N}$ such that*

$$V = V_0 V_1 \dots V_q, \tag{1}$$

where for every $\alpha \in \mathcal{H}$: $\alpha(V_0) = V_0$ and for every $j \in \{1, 2, \dots, q\}$ either $\alpha(V_j) \leq V_j$ or $\alpha(V_j) \geq V_j$.

2. For each $j \in \{1, 2, \ldots, q\}$ there is a homomorphism $\rho_j : \mathcal{H} \to \mathbb{Z}$ and a positive integer s_j such that

$$[\alpha(V_j) : V_j] = s_j^{\rho_j(\alpha)}.$$

3. $$\widetilde{V}_j := \bigcup_{\alpha \in \mathcal{H}} \alpha(V_j)$$

   is a closed subgroup of G for each $j \in \{1, 2, \ldots, q\}$.
4. The natural numbers r and q, the homomorphisms $\rho_j : \mathcal{H} \to \mathbb{Z}$ and positive integers s_j are independent of the subgroup V tidy for α.

Simultaneously minimising subgroups 2

- The numbers $s_j^{\rho_j(\alpha)}$ are analogues of absolute values of eigenvalues for α.
- The subgroups $\bigcup_{\alpha\in\mathcal{H}} \alpha(V_j)$ are the analogues of common eigenspaces for the automorphisms in $\mathcal{H}$.

Example

$G = SL(n, \mathbb{Q}_p)$, $H = \{\text{diagonal matrices in } GL(n, \mathbb{Q}_p)\}$ and $\alpha_h(x) = hxh^{-1}$. Then:

- $r = n - 1$;
- ρ_j are roots of H; and
- $\widetilde{V}_j$ are root subgroups of G.

Nilpotent groups of automorphisms are flat

Theorem

Every finitely generated nilpotent subgroup of Aut(G) *is flat. Every polycyclic subgroup of* Aut(G) *is virtually flat, i.e., has a finite index subgroup that is flat.*

The proof will be sketched for $\mathcal{H}$ finitely generated and *abelian*.

Lemma

Let α, β be commuting automorphisms of G. Then nub(α)nub(β) *is a compact $\langle \alpha, \beta \rangle$-stable subgroup of G.*

Abelian groups of automorphisms are flat

Lemma

Let $\mathcal{F} \subset \mathrm{Aut}(G)$ be finite and suppose that all elements of $\mathcal{F}$ commute. Then there is a subgroup $V \in \mathcal{B}(G)$ that is tidy, and therefore minimising, for all elements of $\mathcal{F}$.

A subgroup tidy for the elements of a generating set of an abelian group $\mathcal{H}$ need not be tidy for $\mathcal{H}$.

Proposition

Let $\mathcal{H} \leq \mathrm{Aut}(G)$ be finitely generated and abelian. Then there is a finite $\mathcal{F} \subset \mathcal{H}$ such that any $V \in \mathcal{B}(G)$ tidy for $\mathcal{F}$ is tidy for $\mathcal{H}$.

Flat groups of automorphisms

The converse direction, that a flat group is abelian modulo the uniscalar subgroup, depends on the next proposition.

Proposition

Let $\alpha, \beta \in \mathrm{Aut}(G)$ be such that $\langle \alpha, \beta \rangle$ is flat. Then $[\alpha, \beta]$ normalises any subgroup V that is tidy for $\langle \alpha, \beta \rangle$. Hence the derived subgroup of $\langle \alpha, \beta \rangle$ is contained in the uniscalar subgroup.

Remark

Unlike the case of linear transformations, there is no general method that produces a flat group containing a given automorphism α that is larger than $\langle \alpha \rangle$.

The flat-rank and uniscalar groups

Definition

The *rank* of the flat group, $\mathcal{H}$, of automorphisms of G is the rank of the free abelian group $\mathcal{H}/\mathcal{H}_1$.
The *flat-rank* of the t.d.l.c. group G is the maximum of the ranks of the flat subgroups $H \leq G$.
The t.d.l.c. group G is *uniscalar* if the scale function on G is identically equal to 1.

A group is uniscalar if and only if it has flat-rank 0. Every element of such a group normalises some compact open subgroup. It may happen however that a uniscalar group does not have a compact open normal subgroup.

The flat-rank in examples

- The flat-rank of a t.d.l.c. group agrees with other notions of rank in many classes of groups. For example, the flat-rank of a simple p-adic Lie group is equal to its usual rank. Automorphism groups of buildings have a rank that may be defined algebraically, in terms of abelian subgroups, or geometrically, in terms of the existence of geometric ‘flats’ in the building. These ranks agree with each other and with the flat-rank.
- There exist topologically simple groups with flat-rank 0 but they are not compactly generated. Are there compactly generated topologically simple groups with flat-rank 0?
- Neretin’s group of almost automorphisms of a regular tree is an example of a compactly generated, abstractly simple group having infinite flat-rank.

References

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