13. Functoriality for the classical groups, II

[This are the notes that accompanied my talk at the Workshop on Automorphic L-functions. It has a slight overlap with Lecture 12.]

We let k be a number field, $\mathbb{A}$ its ring of adeles, and $\psi : k\backslash\mathbb{A} \to \mathbb{C}^\times$ a non-trivial additive character.

13.1. **Functoriality.** We will be interested in global functoriality from the split classical groups to GL_N. More precisely, let G_n be a split classical group of rank n defined over k as below:

(a) $G_n = SO_{2n+1}$ or SO_{2n} with respect to the split symmetric form

$$\Phi_m = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \text{ with } m = 2n+1, 2n.$$

(b) $G_n = Sp_{2n}$ with respect to the alternating form

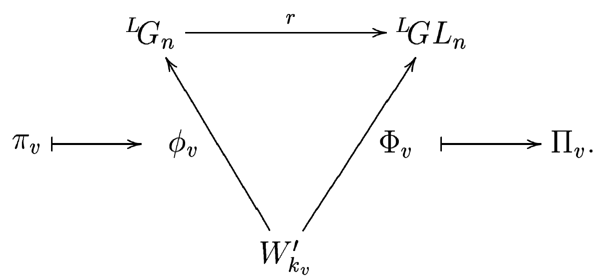
$$J_{2n} = \begin{pmatrix} & \Phi_n \\ -\Phi_n & \end{pmatrix}.$$

For each group there is a standard embedding $r : {}^LG \hookrightarrow GL_N(\mathbb{C}) = {}^LGL_N$:

G_n	$r : {}^LG_n \hookrightarrow {}^LGL_N$	GL_N
SO_{2n+1}	$Sp_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$	GL_{2n}
SO_{2n}	$SO_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$	GL_{2n}
Sp_{2n}	$SO_{2n+1}(\mathbb{C}) \hookrightarrow GL_{2n+1}(\mathbb{C})$	GL_{2n+1}

By Langlands principle of functoriality there should be an associated functorial lift or transfer of automorphic representations π of $G_n(\mathbb{A})$ to automorphic representations Π of $GL_N(\mathbb{A})$. Together with Kim, Piatetski-Shapiro and Shahidi, we have recently completed this functoriality for globally generic cuspidal representations π.

Theorem 13.1 (Functoriality). *Let $\pi = \otimes'\pi_v$ be a globally generic representation of $G_n(\mathbb{A})$. Then π has a functorial lift to an automorphic representation Π of $GL_N(\mathbb{A})$. More precisely, there is a finite set of* *(finite) places S such that for all $v \notin S$, Π_v is the local functorial lift of π_v in the sense of the local Langlands parameterization:*



Philosophically, through functoriality one hopes to pull back structural results from GL_N to the classical groups G_n. In this lecture I would like to outline some of what we know in these cases.

13.2. **Descent.** A bit earlier than our proof of functoriality, Ginzburg, Rallis, and Soudry were developing a theory of local and global descent from self dual automorphic representations Π of $GL_N(\mathbb{A})$ to cuspidal automorphic representations π of the classical groups $G_n(\mathbb{A})$. I would like to give some idea of the descent in the case of $GL_N = GL_{2n}$ to $G_n = SO_{2n+1}$.

Begin with Π a self dual cuspidal representation of $GL_N(\mathbb{A})$ having trivial central character. Let $H = SO_{4n}$. Then H has a maximal (Siegel) parabolic subgroup $P \simeq MN$ with Levi subgroup $M \simeq GL_{2n}$. Hence we can form the globally induced representation

$$\Xi(\Pi) = \mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})}(\Pi \otimes |\det|^{s-1/2}).$$

For any function $f \in V_{\Xi(\Pi)}$ we can then form an Eisenstein series $E(s, f, h)$ on $H(\mathbb{A})$ in the usual manner.

H also has a parabolic subgroup $P' = M'N'$ with Levi subgroup $M' \simeq (GL_1)^{n-1} \times SO_{2n+2}$. If one takes an appropriate additive character ψ' of N' then its stabilizer in M' is precisely $SO_{2n+1} = G_n$. Ginzburg, Rallis, and Soudry refer to the corresponding Fourier coefficient

$$E^{\psi'}(s, f, h) = \int_{N'(k)\backslash N'(\mathbb{A})} E(s, f, nh)\psi'(n)\, dn$$

a Gelfand-Graev coefficient. These naturally restrict to automorphic functions of $g \in G_n(\mathbb{A}) \hookrightarrow H(\mathbb{A})$. Hence if π is any cuspidal representation of $G_n(\mathbb{A})$ we can consider the Petersson inner products of $\varphi \in V_\pi$

with these coefficients

$$\langle \varphi, E^{\psi'}(s,f)\rangle = \int_{G_n(k)\backslash G_n(\mathbb{A})} \varphi(g) E^{\psi'}(s,f,g)\, dg.$$

This will vanish unless π is globally generic and in that case one finds that outside a finite set of places T we have

$$\langle \varphi, E^{\psi'}(s,f)\rangle \sim \frac{L^T(s,\pi\times\Pi)}{L^T(2s,\Pi,\wedge^2)}.$$

The condition for $L^T(s,\pi\times\Pi)$ to have a pole at $s=1$ is that Π be a functorial lift of π. On the other hand, if $L^T(s,\pi\times\Pi)$ is to have a pole at $s=1$ with a non-zero residue, then the above formula gives that V_π will have a non-zero G_n-invariant pairing with the space of residues

$$\pi_{\psi'}(\Pi) = \langle Res_{s=1}(E^{\psi'}(s,f,g)) \mid f \in V_{\Xi(\Pi)}\rangle.$$

On the other hand, if the Gelfand-Graev coefficients $E^{\psi'}(s,f)$ is to have a pole at $s=1$ then the full Eisenstein series $E(s,f)$ must as well and this happens iff (from the constant term calculation) $L^T(s,\Pi,\wedge^2)$ does.

If we run this analysis backwards, we obtain the descent theorem for self dual cuspidal representations Π of $GL_{2n}(\mathbb{A})$ such that $L^T(s,\Pi,\wedge^2)$ has a pole at $s=1$.

Theorem 13.2 (Descent). *Let Π be a self dual cuspidal representation of $GL_{2n}(\mathbb{A})$ with trivial central character and such that $L^T(s,\Pi,\wedge^2)$ has a pole at $s=1$. Let*

$$\pi_{\psi'}(\Pi) = \langle Res_{s=1}(E^{\psi'}(s,f,g)) \mid f \in V_{\Xi(\Pi)}\rangle.$$

Then

(i) $\pi_{\psi'}(\Pi) \neq 0$,
(ii) $\pi_{\psi'}(\Pi)$ *is cuspidal,*
(iii) *each summand of* $\pi_{\psi'}(\Pi)$ *is globally generic,*
(iv) *each summand of* $\pi_{\psi'}(\Pi)$ *functorially lifts to* Π,
(v) $\pi_{\psi'}(\Pi)$ *is multiplicity one,*
(vi) *if π is a globally generic cuspidal representation which functorially lifts to Π then π has a non-zero invariant pairing with* $\pi_{\psi'}(\Pi)$.

Of course the conjecture is the following.

Conjecture 13.1. *$\pi_{\psi'}(\Pi)$ is irreducible (and hence cuspidal and the only globally generic representation of $G_n(\mathbb{A})$ which functorially lifts to Π).*

One has precisely the same result for the other classical groups with modifications as indicated in the following table.

GL_N	H	G_n	$L^T(s,\Pi,\beta_{G_n})$ with a pole at $s=1$
GL_{2n}	SO_{4n}	SO_{2n+1}	$L^T(s,\Pi,\wedge^2)$
GL_{2n}	SO_{4n+1}	SO_{2n}	$L^T(s,\Pi,Sym^2)$
GL_{2n+1}	$\widetilde{Sp}_{4n+2}$	Sp_{2n}	$L^T(s,\Pi,Sym^2)$

As a consequence of this descent theorem, they obtain the following characterization of the image of functoriality.

Theorem 13.3 (Characterization of the image)**.** *Let π be a globally generic representation of $G_n(\mathbb{A})$. Then any functorial lift of π to an automorphic representation Π of $GL_N(\mathbb{A})$ has trivial central character and is of the form*

$$\Pi = \mathrm{Ind}(\Pi_1 \otimes \cdots \otimes \Pi_{\mathrm{d}}) = \Pi_1 \boxplus \cdots \boxplus \Pi_{\mathrm{d}}$$

with $\Pi_i \not\simeq \Pi_j$ for $i \neq j$ and such that each Π_i is a unitary self dual cuspidal representation of $GL_{N_i}(\mathbb{A})$ such that $L^T(s,\Pi_i,\beta_{G_n})$ has a pole at $s=1$. Moreover, any such Π is the functorial lift of some π.

Note that since these functorial lifts are isobaric, then by the Strong Multiplicity One theorem for GL_N, the functorial image Π is completely determined by the π_v for $v \notin S$, i.e, those places where we know the local functorial lifts.

13.3. **Bounds towards Ramanujan.** As a first consequence of functoriality, we obtain bounds towards Ramanujan for globally generic cuspidal representations of G_n by pulling back the known bounds for GL_N.

Theorem 13.4. *Let $\pi \simeq \otimes' \pi_v$ be a globally generic cuspidal representation of $G_n(\mathbb{A})$. Then at the places v where π_v is unramified the Satake parameters for π_v satisfy*

$$q_v^{-(\frac{1}{2}-\frac{1}{N^2+1})} < |\alpha_v| < q_v^{\frac{1}{2}-\frac{1}{N^2+1}}.$$

As far as we know, these are the first Ramanujan type bounds for classical groups.

13.4. **The local converse theorem.** The results in this section are due to Jiang and Soudry. We now restrict ourselves to the case of $G_n = SO_{2n+1}$. Also, let v be a non-archimedean place of k, so k_v is a p-adic field. One of the most powerful local results for $SO_{2n+1}(k_v)$ to be pulled back from GL_{2n} is the "local converse theorem" for GL_{2n}, first formulated by Jacquet, Piatetski-Shapiro, and Shalika but finally proved by Henniart.

Theorem 13.5 (Local Converse Theorem)**.** *Let $\pi_{1,v}$ and $\pi_{2,v}$ be two irreducible admissible generic representations of $SO_{2n+1}(k_v)$. Suppose that*

$$\gamma(s, \pi_{1,v} \times \pi'_v, \psi_v) = \gamma(s, \pi_{2,v} \times \pi'_v, \psi_v)$$

for all irreducible super-cuspidal π'_v of $GL_d(k_v)$ for $1 \leq d \leq 2n-1$. Then $\pi_{1,v} \simeq \pi_{2,v}$.

To obtain this result, Jiang and Soudry needed to combine global functoriality from SO_{2n+1} to GL_{2n} with local descent from GL_{2n} to $\widetilde{Sp}_{2n}$ and then the local theta correspondence from $\widetilde{Sp}_{2n}$ to SO_{2n+1} to be able to pull Henniart's result back from GL_{2n} to SO_{2n+1}.

One immediate consequence of this is that Conjecture 13.1 is true for $G_n = SO_{2n+1}$ and hence the global functoriality from SO_{2n+1} to GL_{2n} for globally generic cuspidal representations is injective. This allows them to pull the Strong Multiplicity One Theorem for $GL_{2n}(\mathbb{A})$ back to globally generic cuspidal representations of $SO_{2n+1}(\mathbb{A})$ to obtain a rigidity theorem.

Theorem 13.6 (Rigidity)**.** *Let π_1 and π_2 be two globally generic cuspidal representations of $SO_{2n+1}(\mathbb{A})$. Suppose there is a finite set of places S such that $\pi_{1,v} \simeq \pi_{2,v}$ for all $v \notin S$. Then $\pi_{1,v} \simeq \pi_{2,v}$ for all v, that is. $\pi_1 \simeq \pi_2$.*

More importantly, having this Local Converse Theorem allow one to pull back local results from GL_{2n} to SO_{2n+1} through global functoriality. As a first result, Jiang and Soudry are able to complete the local functoriality from $SO_{2n+1}(k_v)$ to $GL_{2n}(k_v)$ for generic representations at those places where it was not previously known. The first step is the following.

Let $\mathcal{A}_0^g(SO_{2n+1})$ denote the set of irreducible generic super-cuspidal representations of $SO_{2n+1}(k_v)$ up to equivalence and let $\mathcal{A}_\ell(GL_{2n})$ denote the set of all Π_v of $GL_{2n}(k_v)$ of the form

$$\Pi_v \simeq \mathrm{Ind}(\Pi_{1,v} \otimes \cdots \otimes \Pi_{d,v})$$

where each $\Pi_{i,v}$ is an irreducible super-cuspidal self dual representation of some $GL_{2n_i}(k_v)$ such that $L(s, \Pi_{i,v}, \wedge^2)$ has a pole at $s = 0$ and $\Pi_{i,v} \not\simeq \Pi_{j,v}$ for $i \neq j$.

Theorem 13.7 (Local functoriality)**.** *There exists a unique bijection* $\mathcal{A}_0^g(SO_{2n+1}) \to \mathcal{A}_\ell(GL_{2n})$, *denoted* $\pi_v \mapsto \Pi_v = \Pi(\pi_v)$ *such that*

$$L(s, \pi_v \times \pi_v') = L(s, \Pi_v \times \pi_v')$$
$$\varepsilon(s, \pi_v \times \pi_v', \psi_v) = \varepsilon(s, \Pi_v \times \pi_v', \psi_v)$$

for all irreducible super-cuspidal representations of $GL_d(k_v)$ *for all* d.

Using the way in which one builds a general generic representation from generic super-cuspidal ones, they later extended this theorem to all of $\mathcal{A}^g(SO_{2n+1})$, the set of all irreducible admissible generic representations of $SO_{2n+1}(k_v)$.

Through local functoriality they were then able to pull back the local Langlands conjecture (or arithmetic Langlands parameterization) for $\mathcal{A}_0^g(SO_{2n+1})$. Let $\Phi_0^g(SO_{2n+1})$ denote the set of admissible, completely reducible, multiplicity free representations $\rho : W_{k_v} \to Sp_{2n}(\mathbb{C}) = {}^L SO_{2n+1}$ which are symplectic (so each irreducible constituent must be symplectic).

Theorem 13.8 (Local Langlands conjecture)**.** *There exists a unique bijection* $\Phi_0^g(SO_{2n+1}) \to \mathcal{A}_0^g(SO_{2n+1})$ *denoted* $\rho_v \mapsto \pi_v = \pi(\rho_v)$ *such that*

$$L(s, \rho_v \otimes \phi_v) = L(s, \pi(\rho_v) \times \pi'(\phi_v))$$
$$\varepsilon(s, \rho_v \otimes \phi_v, \psi_v) = L(s, \pi(\rho_v) \times \pi'(\phi_v), \psi_v)$$

for all irreducible admissible representations $\phi_v : W_{k_v} \to GL_d(k_v)$, *that is* $\phi_v \in \Phi_0(GL_d)$.

In the same subsequent paper, they extended this result to all of $\mathcal{A}^g(SO_{2n+1})$ as well.

There are other local consequences to be found in their papers, such as the existence of a unique generic member of each tempered L-packet for SO_{2n+1}, but we shall end here.

References

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