SELBERG'S CONJECTURES AND L-FUNCTIONS (JOINT WORK WITH RAM MURTY)

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Abstract. We will discuss the partial sums of coefficients of the Selberg class and apply this to study partial sums for the original conjecture by Birch and Swinnerton-Dyer.

This is a joint work with Ram Murty.

1. Selberg Class

Let $F(s)$ be a Dirichlet series, i.e., $F(s) = \sum_{n=1} a_n(F)/n^s$. In 1989, Selberg proposed a class $\mathcal{S}$ of Dirichlet series (see [17]), called the Selberg class, satisfying the following five conditions: for $F(s) \in \mathcal{S}$

1. (Dirichlet series)
$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$
a Dirichlet series absolutely convergent for $\Re s > 1$ and $a_1 = 1$;
2. (Analytic continuation) $F(s)$ extends to an entire function of finite order except possible pole at $s = 1$;
3. (Functional equation) it has a functional equation of the form:
$$\Phi(s) = \omega\overline{\Phi}(1-s),$$
where $\omega \in \mathbb{C},\ |\omega| = 1$ and
$$\Phi(s) = \Delta(s)F(s),\ \Delta(s) = Q^s \prod_{i=1}^{m} \Gamma(\alpha_i s + \gamma_i),$$
and $\overline{\Phi}(s) = \overline{\Phi(\overline{s})}$. Here $Q > 0$, $\alpha_i > 0$, $\Re(\gamma_i) \geq 0$, and m is a natural integer. The number Q is called the *conductor*.
4. (Euler Product) $F(s) = \prod_p F_p(s)$, where $F_p(s) = \exp(\sum_{k=1}^{\infty} b_{p^k} p^{-ks})$, $b_{p^k} = \mathbf{O}(p^{k\theta})$ for some $\theta < 1/2$, p are primes.
5. (Ramanujan hypothesis) $\forall \epsilon > 0$, $a_n = O(n^{\epsilon})$ and the constant only depending on F and ϵ.

Examples

1. Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$.

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2. Dirichlet L-functions $L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)/n^s$, where χ is a primitive character.
3. Ramanujan tau function $\tau(s)=\sum_{n=1}^{\infty}\tau_n/n^s$, where $\tau_n=\tau(n)/n^{11/2}$, and $\sum_{n=1}^{\infty}\tau(n)q^n = q\prod_{m=1}^{\infty}(1-q^m)^{24}$.
4. E: Elliptic curve, normalized L-functions $L_E(s)$ (due to the Taniyama-Shimura-Weil conjecture).
5. Assume the Ramanujan conjecture, all normalized L-functions attached to irreducible cuspidal representations.

The following are the basic properties of the Selberg class.

0. $\mathcal{S}$ is a multiplicative monoid.
1. Let $d=\sum 2\alpha_i$ be the degree of $f\in\mathcal{S}$, where α_i are in the functional equation. Then d and Q are independent of choices of functional equations.
2. $d=0\Longrightarrow F\equiv 1$. $d\le 1\Longrightarrow d=1$. (Selberg, Conrey & Ghosh)
3. The function f of degree 1 is either $\zeta(s)$ or $L(s+iA,\chi)$, for some real number A and primitive character χ. (Kaczorowski & Perelli)

We can introduce the idea of prime in $\mathcal{S}$.

Definition. F is called *primitive* is the equation $F=F_1\cdot F_2\Longrightarrow F_1\equiv 1$ or $F_2\equiv 1$

The following corollary is easily deduced from the work of Conrey and Ghosh.

Corollary. *Every element in $\mathcal{S}$ can be factorized into a finite product of primitive elements. (if $F=F_1F_2\Longrightarrow \deg F=\deg F_1+\deg F_2$)*

Selberg did not just put all L-functions at once. He also made conjectures on $\mathcal{S}$.

Conjecture (Selberg)**.** - *Conjecture A: for all $F\in\mathcal{S}$, there exists a positive integer n_F such that*

$$\sum_{p\le x}\frac{|a_p(F)|^2}{p}=n_F\log\log x+\mathbf{O}(1).$$

- *Conjecture B: For any two primitive functions F,G,*

$$\sum_{p\le x}\frac{a_p(F)\overline{a_P(G)}}{p}=\delta_{F,G}\log\log x+\mathbf{O}(1).$$

There are a lot of important consequences followed by Selberg's conjecture. For instance,

Theorem (R. Murty)**.** *Selberg's conjecture implies the Artin holomorphy conjecture and Langlands reciprocity conjecture for solvable group; i.e., the image of the Galois representation is solvable.*

Therefore, we can see the power of Selberg conjecture.

2. Summatory functions

Given a Dirichlet series $F(s)=\sum a_n n^{-s}$, the summatory function $S_F(x)$ is defined as follows

$$S_F(x)=\sum_{n\le x}a_n,\ x\ge 1.$$

Question. *What can we say about $S_F(x)$?*

1. $F \in \mathcal{S}$, by the Ramanujan hypothesis, the trivial estimate is $S_F(x) = \mathbf{O}(x^{1+\epsilon})$.
2. $F \in \mathcal{S}$, If $F(s)$ has only a simple pole at $s = 1$ on the line of $\Re(s) = 1$ and satisfies some conditions on coefficients, the celebrated Tauberian theorem gives us

$$S(x) = \kappa x + R(x),\ R(x) = o(x), \quad \text{for some constant } \kappa.$$

Definition. Given a Dirichlet series $F(s) = \sum a_n n^{-s}$. The exponent τ_F of F is defined as

$$\tau_F = \min\{\xi | \forall \epsilon > 0, S_F(x) = \mathbf{O}(x^{\xi+\epsilon}\}.$$

The reason why the exponent is important is by the following lemma:

Lemma. *Let $F(s)$ be a Dirichlet series. Assume that its summatory function $S(x)$ is $O(x^\delta)$. For $\Re(s) > \delta$, the series*

$$F(s) = \sum_{i=1}^{\infty} \frac{a_n}{n^s}$$

converges. Note that we do not assume that the $F(s)$ converges absolutely for $\Re(s) > \delta$.

To determine τ_F is not an easy job. If we can find some upper bound θ of τ_F less than 1, then the Dirichlet series $\sum_{n\geq 1} a_n n^{-s}$ will converge at $\Re(s) > \theta$. This Lemma gives the information inside the critical strip.

It is difficult to deal with general Dirichlet series. However, almost all interesting Dirichlet series admit analytic continuation or meromorphic continuation to the entire complex plane and satisfy a functional equation. More precisely, we assume that there is a function

$$\Delta(s) = Q^s \prod \Gamma(\alpha_i s + \gamma_i),\ Q > 0,\ \alpha_i > 0,$$

such that

$$F(s)\Delta(s) = \omega \overline{F}(1-s)\overline{\Delta}(1-s),$$

where ω is a complex number with $|\omega| = 1$, $\overline{F}(s) = \overline{F(\overline{s})}$, and $\overline{\Delta}(s) = \overline{\Delta(\overline{s})}$.

Now we assume that $F(s)$ is entire and satisfies Ramanujan hypothesis, i.e, for all $\epsilon > 0$, $|a_n| = O(n^\epsilon)$. The main term in the summatory function disappears.

The importance of the exponents is that we can estimate the size of $F(s)$ inside the the critical strip, which cannot be inferred from the functional equations.

We have the following theorems.

Theorem (Kuo-Murty)**.** *Given $F \in \mathcal{S}$ and F is entire, then*

$$S_F(x) = O(Q^{1-\theta} x^\theta),\ \theta = \frac{d_F}{d_F + 2} = 1 - \frac{2}{d_F + 2} < 1.$$

However, by employing the theorem of Chandrasekharan and Narasimhan, we can prove

Theorem (Kuo)**.** *Given $F(s) \in \mathcal{S}$ and assuming F is entire and $d_F \geq 2$. Then*

$$\forall \epsilon > 0, \forall \gamma > 1,\ S(x) = \sum_{n \leq x} a_n = O(Q^{1-\theta' - \epsilon} x^{\theta' + \gamma\epsilon}).$$

Here

$$\theta' = \frac{d_F - 1}{d_F + 1} < 1.$$

3. Birch-Swinnerton-Dyer Conjecture

Let $E/\mathbb{Q}$ be an elliptic curve, defined as

$$y^2 = x^3 + ax + b.$$

Question. *What can we say about $E(\mathbb{Q})$?*

Indeed we know that there is an addition law on $E(\mathbb{Q})$ and it is finitely generated under the addition law. Therefore we can write

$$E(\mathbb{Q}) = \mathbb{Z}^r \oplus E(\mathbb{Q})_{\mathrm{tor}}.$$

The torsion part is relatively "well-understood" due to the work of Nagell-Lutz and Mazur. Now the remaining part is the rank $\mathrm{rank}(E) = r$ of $E(\mathbb{Q})$.

For a prime p, $p \nmid \Delta = -16(4a^3 + 27b^2) \neq 0$, define

$$N_p = p + 1 - a_p = \#E(\mathbb{F}_p).$$

By the theorem of Hasse, we have

$$|a_p| \leq 2\sqrt{p}.$$

Now consider the L-function $L_E(s)$ attached to E defined as follows:

$$L_E(s) \colon = \prod_{p \nmid \Delta} (1 - a_p p^{-s} + p^{-2s})^{-1} \times \prod_{p \mid \Delta} l_p(E, s)^{-1} = \prod_{p \nmid \Delta} \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1} \times \prod_{p \mid \Delta} l_p(E, s)^{-1},$$

where $l_p(E, s)$ is a certain polynomial in p^{-s} with the property that $l_p(E, 1) \neq 0$ and

$$\alpha_p + \beta_p = a_p, \quad \alpha_p \cdot \beta_p = p \Longrightarrow |\alpha_p| = |\beta_p| = \sqrt{p}.$$

As a precursor to their celebrated conjecture, Birch and Swinnerton-Dyer formulated the following:

Conjecture ([1] & [2] Birch & Swinnerton-Dyer)**.** *For some constant c,*

$$P_E(x) = prod_{p \leq x, p \nmid \Delta} \frac{N_p}{p} \sim c (\log x)^r, \quad x \to \infty.$$

We use B-SD as the abbreviation of this Birch and Swinnerton-Dyer conjecture.

Later, they revised their conjecture.

Conjecture ([3] Birch & Swinnerton-Dyer)**.** *$L_E(s)$ extends to an entire function and*

$$\operatorname*{ord}_{s=1} L_E(s) = \mathrm{rank}(E)$$

By the work of Wiles [20], and Breuil, Conrad, Diamond and Taylor [4], $L_E(s)$ extends to an entire function and satisfies a functional equation relating $L_E(s)$ to $L_E(2-s)$.

In [7], Goldfeld examined the consequences of the original B-SD conjecture.

Theorem ([7] Goldfeld)**.** *Assume that first conjecture of B-SD, then $L_E(s)$ satisfies the Riemann Hypothesis and $r = \mathrm{ord}_{s=1} L_E(s)$. Note that the critical line here is $s = 1$.*

The sketch of the proof. It is easy to show the following identities

- $$P_E(x) = \prod_{p \le x, p \nmid \Delta} \left(1 - \frac{\alpha_p}{p}\right)\left(1 - \frac{\beta_p}{p}\right).$$

  Take log and we get

- $$-\log P_E(x) = {\sum_{p \le x}}' \sum_{k=1}^{\infty} \frac{\alpha_p^k + \beta_p^k}{kp^k}.$$

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$$(*) \qquad \log \tilde{L}_E(s+1) = {\sum_{p}}' \sum_{k \ge 1}^{\infty} \frac{\alpha_p^k + \beta_p^k}{kp^{(s+1)k}}. = \sum_{n \in \mathbb{N}} \frac{c_n}{n^s} = \int_1^{\infty} \frac{1}{x^s} dC(x) = s \int_1^{\infty} \frac{C(x)}{x^{s+1}} dx$$

and

$$(**) \qquad \log \tilde{L}_E(s) = {\sum_{p}}' \sum_{k \ge 1}^{\infty} \frac{\alpha_p^k + \beta_p^k}{kp^{sk}}. = \sum_{n \in \mathbb{N}} \frac{c_n'}{n^s} = \int_1^{\infty} \frac{1}{x^s} d\tilde{C}(x) = s \int_1^{\infty} \frac{\tilde{C}(x)}{x^{s+1}} dx.$$

where

$$C(x) := {\sum_{p^k \le x}}' \frac{\alpha_p^k + \beta_p^k}{kp^k} = \sum_{n \le x} c_n,$$

where

$$c_n = \begin{cases} \dfrac{\alpha_p^k + \beta_p^k}{kp^k} & n = p^k,\ p \text{ is a prime},\ k \text{ is a natural number},\ p \nmid \Delta, \\ 0 & \text{otherwise.} \end{cases}$$

Similar, we define

$$\tilde{C}(x) := {\sum_{p^k \le x}}' \frac{\alpha_p^k + \beta_p^k}{k} = \sum_{n \le x} \tilde{c}_n,$$

where

$$\tilde{c}_n = c_n \cdot n = \begin{cases} \dfrac{\alpha_p^k + \beta_p^k}{k} & n = p^k,\ p \text{ is a prime},\ k \text{ is a natural number},\ p \nmid \Delta, \\ 0 & \text{otherwise.} \end{cases}$$

For instance, $\tilde{c}_p = a_p$, $\tilde{c}_{p^2} = (a_p^2 - 2p)/2, \dots$ etc.

Lemma. *If we assume the B-SD conjecture, then*

$$C(x) = -r \log\log x + A + \mathbf{o}(1),$$

where A is a constant.

Now we use the lemma and $(*)$

$$\log \tilde{L}_E(s+1) = {\sum_{p}}' \sum_{k \ge 1}^{\infty} \frac{\alpha_p^k + \beta_p^k}{kp^{(s+1)k}} = s \int_{1^-}^{\infty} \frac{C(x)}{x^{s+1}} = s \int_{1^-}^{\infty} \frac{-r \log\log x + A + \mathbf{o}(1)}{x^{s+1}} dx.$$

Therefore, $\log \tilde{L}_E(s+1)$ is regular for $\Re(s) \geq 0$. It implies that $\tilde{L}_E(s+1)$ has no zero (or pole) for $\Re(s) \geq 1$. It is exactly the statement of Riemann Hypothesis for $\tilde{L}_E(s)$. The statement for r and the order of $L_E(s)$ can be proved, too. □

The summatory function $C(x)$ is not what we are familiar with. What we know more is $\tilde{C}(x)$. The function $\tilde{C}(x)$ is the summatory function of coefficients of $\log L_E(s)$. The dominant term is the sum over primes. Define

$$\overline{C}(x) := \sum_{p \leq x} \tilde{c}_p = \sum_{p \leq x} a_p.$$

Then

$$\overline{C}(x) \sim \tilde{C}(x).$$

Theorem. *Suppose that the B-SD conjecture is true, then*

$$\sum_{n \leq x} \tilde{c}_n = \mathbf{o}(x).$$

Proof. By partial summation,

$$\sum_{n \leq x} \tilde{c}_n = \sum_{n \leq x} c_n \cdot n = x \cdot C(x) - \int_3^x C(t)dt + \mathbf{o}(1),$$

where as usual $C(x) = \sum_{n \leq x} c_n$. By Lemma 3 in the previous section, we have

$$C(x) = -r \log \log x + A + \mathbf{o}(1).$$

Therefore,

$$\begin{aligned}
\sum_{n \leq x} \tilde{c}_n &= x \cdot C(x) - \int_3^x C(t)dt + \mathbf{o}(1) \\
&= x\{-r \log \log x + A + \mathbf{o}(1)\} - \int_3^x -r \log \log t + A + \mathbf{o}(1)dt + \mathbf{o}(1) \\
&= -rx \log \log x + cx - \left(\int_3^x -r \log \log t dt + cx + \mathbf{o}(x) \right) \\
&= -rx \log \log x + \int_3^x r \log \log t dt + \mathbf{o}(x).
\end{aligned}$$

Hence, after an easy integration,

$$\sum_{n \leq x} \tilde{c}_n = -rx \log \log x + r \left(x \log \log x + \mathbf{o}(x) + \mathbf{O}\left(\frac{x}{\log x} \right) \right) = \mathbf{o}(x).$$

□

Of course, we suppose to lose information to do such partial summation. The reason why we are more familiar with $\overline{C}(x)$ is because it behaves like the remainder term in Prime Number Theory. For precisely, let

$$\psi(x) = \sum_{p \leq x} \log p = x + R(x).$$

Take out the log weight and shift the critical line to $s = 1$, we have

$$\tilde{C}(x) \sim \overline{C}(x) \sim R(x)x^{1/2}/\log x.$$

Hypothesis	$R(x)$	$\tilde{C}(x) \sim R(x)x^{1/2}/\log x$
Riemann Hypothesis	$\mathbf{O}(x^{1/2}(\log x)^2$	$\mathbf{O}(x \log x$
Pair Correlation	$\mathbf{O}(x^{1/2}(\log x)^{3/2}$	$\mathbf{O}(x(\log x)^{1/2}$
Linear independence of zeros	$\mathbf{O}(x^{1/2}(\log\log\log x)^2)$	$\mathbf{O}(x(\log\log\log x)^2/\log x)$
	$\mathbf{\Omega}(x^{1/2}\log\log\log x)$	$\mathbf{\Omega}(x\log\log\log x/\log x)$

If we assume the Riemann hypothesis, we only can get $\tilde{C}(x) = \mathbf{O}(x \log x)$. Even if we assume the pair-correlation conjecture, we only have $\tilde{C}(x) = \mathbf{O}(x(\log x)^{1/2})$. On the other hand, one can show $\tilde{C}(x) = \mathbf{\Omega}(x \log\log\log x/\log x)$. However, in [15], Montgomery gave a heuristic argument treating the error term occurring in classical prime number theory. When applied to our context, this suggests that $\tilde{C}(x) = \mathbf{O}(x(\log\log\log x)^2/\log x)$. Thus, it is likely that the B-SD conjecture is true, from this perspective. If we would like to disprove it, what we have so far is

$$R(x) = \mathbf{\Omega}(x^{1/2}\log\log\log x)$$

by Ram Murty using the method of Littlewood. It is equivalent to

$$\tilde{C}(x) \sim \mathbf{\Omega}(x\log\log\log x/\log x).$$

It is still far away from $\mathbf{\Omega}(x)$.

Therefore, the condition

$$\sum_{n \le x} \tilde{c}_n = \tilde{C}(x) = \mathbf{o}(x).$$

is much deeper than what we know at present.

Amazingly, the converse of the previous theorem is true.

Theorem ([13] Kuo-Murty, [6] K. Conrad)**.** *If $\sum_{n\le x}\tilde{c}_n = \mathbf{o}(x)$, then the B-SD conjecture is true.*

4. Sato-Tate conjectures

Let $E/\mathbb{Q}$ be an elliptic curve over $\mathbb{Q}$. For each prime where E has good reduction, a_p satisfies Hasse's inequality

$$|a_p| \le 2p^{1/2}.$$

Thus, we can write

$$a_p = 2p^{1/2}\cos\theta_p,$$

for a uniquely defined angle θ_p satisfying $0 \le \theta_p < \pi$. The Sato-Tate conjecture is a statement how the angles θ_p are distributed in the interval $[0, \pi]$ as p varies.

For the non-CM case, the distribution is unknown at present. Sato and Tate (independently) predicted the law of distribution for the angles θ_p as follows:

$$\#\{p : p \le x, \theta_p \in (\alpha, \beta)\} \sim \left(\frac{2}{\pi}\int_\alpha^\beta \sin^2\theta d\theta\right)\pi(x),$$

as x tends to infinity, where $\pi(x)$ is the number of primes is less than x.

We can extend this conjecture to the cusp forms. Let

$$\mathcal{H} = \{z \in \mathbb{C}, \Im(z) > 0\}, \quad \mathcal{H}^* = \mathcal{H} \cup \text{cusps},$$

the upper half plane with cusps and Γ the finite index subgroup of the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

Definition. Given a primitive Dirichlet character ω. A *normalized Hecke eigenform of integer weight $k \geq 0$ for Γ of the Nebentypus ω* is a complex variable function f on $\mathcal{H}^*$ satisfying

1. for each $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have the modular transformation law $f(\gamma z) = \omega(d)(cz + d)^k f(z)$,
2. f is holomorphic on $\mathcal{H}$,
3. f extends holomorphically to every cusp of Γ and vanishes on cusps,
4. we can attach an L-function $L(s, f)$ to f, and then
$$L(s, f) = \prod_p \left(1 - a_p \cdot p^{-s} + \omega(p) p^{k-1} p^{-2s}\right)^{-1},$$
i.e., it admits an Euler product.

Let

$$H_k(\Gamma, \omega) = \{\text{normalized Hecke eigenforms of weight } k \text{ for } \Gamma \text{ of the Nebentypus } \omega\},$$
$$H(\Gamma, \omega) = \amalg_{k\geq 0} H_k(\Gamma, \omega).$$

The celebrated Ramanujan conjecture on $H(\Gamma, \omega)$ can be stated as follows:

Conjecture (Ramanujan). *For each $f \in H_k(\Gamma, \omega)$, we can rewrite the Euler product as*

$$L(s, f) = \prod_p \left(1 - \alpha_p \cdot p^{(k-1)/2} p^{-s}\right)^{-1} \left(1 - \beta_p \cdot p^{(k-1)/2} p^{-s}\right)^{-1},$$

where α_p and β_p are the roots of the quadratic equation $x^2 - a_p + \chi(p) = 0$. Then $|\alpha_p| = |\beta_p| = 1$.

Let's assume the Ramanujan conjecture. Therefore, for each $f \in H(\Gamma, \omega)$, we can associate it with $|\alpha_p| = |\beta_p| = 1$ for all prime p. Since $|\alpha_p| = |\beta_p| = 1$, we can write α_p and β_p as polar forms $\alpha_p = e^{i\theta_p}, \beta_p = e^{i\psi_p} = e^{-i\theta_p + t_p}, 0 \leq \theta_p, \psi_p < 2\pi$, where t_p is defined as $\omega(p) = e^{it_p}$. The question now is how θ_p, ψ_p distribute on $[0, 2\pi]$.

Definition. Fix a positive real number A. Given a sequence $S = \{x_i\}_{i\in I}$ of real numbers between 0 and A and its index set I which equips a map $N : I \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $N^{-1}(n)$ is a finite set, we say that S is *distributed (modulo A) with respect to a distribution dF on* $[0, A]$ if for any pair of real numbers $0 \leq \alpha < \beta \leq A$, we have

$$\#\{i \in I, N(i) \leq N, x_i \in (\alpha, \beta)\} \sim \left(\int_\alpha^\beta dF\right) \cdot \#\{i \in I, N(i) \leq N\}.$$

If S is distributed with respect to the normal measure dx, we say that S is *uniformly distributed.*

Theorem (Kuo-Murty)**.** *Let f be a non-CM cusp form of Neben type ω. Assume the Ramanujan conjecture holds. Then the sequence $S = \{\theta_p, \psi_p\}$ is not distributed with respect to the Sato-Tate measure $(2/\pi)\sin^2\theta d\theta$.*

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