16. Functoriality of symmetric cube.

16.1 Weak Ramanujan property. The following is an important property of cuspidal representations of $GL_n(\mathbb{A})$: Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $GL_n(\mathbb{A})$. Let π_v be a spherical representation with a semi-simple conjugacy class $diag(\alpha_{1v}, ..., \alpha_{nv})$.

Definition 16.1. *We say π satisfies the weak Ramanujan property if given $\epsilon > 0$, there exists a density zero set T_ϵ such that $\max_i\{|\alpha_{iv}|, |\alpha_{iv}|^{-1}\} \leq q_v^\epsilon$ for $v \notin T_\epsilon$.*

Theorem 16.2. *Cuspidal representations of GL_2, GL_3, GL_4 satisfy the weak Ramanujan property.*

Proof. The cases of GL_2, GL_3 follow easily from the absolute convergence of the standard L-functions $L(s, \pi)$ for $Re(s) > 1$ (See [Ki-Sh2]). The case of GL_4 uses the exterior square lift of GL_4 (See [Ki5]). □

16.2 Functoriality of symmetric square. We first explain the method with the simplest example. Let $\phi = Ad : GL_2(\mathbb{C}) \longrightarrow SO_3(\mathbb{C}) \subset GL_3(\mathbb{C})$ be the adjoint representation. More explicitly, $Ad(diag(a, b)) = diag(ab^{-1}, 1, ba^{-1})$. Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $GL_2(\mathbb{C})$. By the local Langlands' correspondence, we have $Ad(\pi) = \otimes_v Ad(\pi_v)$, irreducible admissible representation of $GL_3(\mathbb{A})$. Note that $Sym^2(\pi) = Ad(\pi) \otimes \omega_\pi$, where ω_π is the central character of π. In this case, note that, for a grössencharacter χ,

$$L(s, \pi \times \tilde{\pi}) = L(s, Ad(\pi))L(s, 1), \quad L(s, (\pi \otimes \chi) \times \tilde{\pi}) = L(s, Ad(\pi) \otimes \chi)L(s, \chi).$$

Hence $L(s, Ad(\pi_v) \otimes \chi_v)$ and $\gamma(s, Ad(\pi_v) \otimes \chi_v, \psi_v)$ are Artin L and γ-factors. Here $L(s, Ad(\pi) \otimes \chi)$ appears in the Langlands-Shahidi method for $\mathbf{P} = \mathbf{MN} \subset Sp(4)$, where $\mathbf{M} \simeq GL_1 \times SL_2$: Take π_0 to be any irreducible constituent of $\pi|_{SL_2(\mathbb{A})}$ and consider $\Sigma = \chi \otimes \pi_0$. Then $(\mathbf{M}, \Sigma)$ gives rise to the fact that $m = 1$ and $L(s, Ad(\pi) \otimes \chi)$ is entire if $\chi^2 \neq 1$.

Apply the converse theorem to $Ad(\pi) = \otimes_v Ad(\pi_v)$, and $S = \{v_1\}$, where v_1 is any finite place. We obtain an automorphic representation Π of $GL_3(\mathbb{A})$ such that $\Pi_v \simeq Ad(\pi_v)$ for $v \neq v_1$. By the classfication of automorphic representations of GL_n, Π is equivalent to a subquotient of

$$Ind\, \sigma_1 |det|^{r_1} \otimes \cdots \otimes \sigma_k |det|^{r_k},$$

where $r_i \in \mathbb{R}$, and σ_i's are (unitary) cuspidal representations of GL_{n_i}, where $n_i = 1, 2, 3$. Since π satisfies the weak Ramanujan, so does Π. Hence $r_1 = \cdots = r_k = 0$, and $\Pi = Ind\, \sigma_1 \otimes \cdots \otimes \sigma_k$.

Now we apply the converse theorem again to $Ad(\pi) = \otimes_v Ad(\pi_v)$, and $S = \{v_2\}$, where $v_2 \neq v_1$ is any finite place, and obtain an automorphic representation Π' of $GL_3(\mathbb{A})$ such that $\Pi'_v \simeq Ad(\pi_v)$ for $v \neq v_2$. Also Π' is of the form $Ind\, \sigma'_1 \otimes \cdots \otimes \sigma'_k$. By the strong multiplicity one, $\Pi \simeq \Pi' \simeq Ad(\pi)$. Therefore $Ad(\pi)$ is an automorphic representation of $GL_3(\mathbb{A})$.

From the relation $L(s, (\pi \otimes \chi) \times \tilde{\pi}) = L(s, Ad(\pi) \otimes \chi)L(s, \chi)$, LHS has a pole at $s = 1$ if and only if $\pi \otimes \chi \simeq \pi$. Note that $L(s, Ad(\pi))$ does not have a pole at $s = 1$.

If $\chi \neq 1$ and $\pi \otimes \chi \simeq \pi$, π is a monomial representation, then $L(s, Ad(\pi) \otimes \chi)$ has a pole at $s = 1$. Hence $Ad(\pi)$ is not cuspidal and $Ad(\pi) \simeq \chi \boxplus \pi'$ (since $\chi^2 = 1$), where π' is an automorphic representation of GL_2. We have proved

Theorem 16.3 (Gelbart-Jacquet [Ge-J]). *Let π be a cuspidal representation of $GL_2(\mathbb{A})$. Then $Ad(\pi)$ is an automorphic representation of $GL_3(\mathbb{A})$. It is an isobaric sum of (unitary) cuspidal representations of GL, and it is cuspidal if and only if π is not monomial. Moreover, $Sym^2(\pi) = Ad(\pi) \otimes \omega_\pi$ is an automorphic representation of $GL_3(\mathbb{A})$.*

16.3 Functoriality of the tensor product of $GL_2 \times GL_3$. Consider $\phi : GL_2(\mathbb{C}) \times GL_3(\mathbb{C}) \longrightarrow GL_6(\mathbb{C})$ be the tensor product in Example 14.3. Let $\pi_1 = \otimes_v \pi_{1v}, \pi_2 = \otimes_v \pi_{2v}$ be cuspidal representations of $GL_2(\mathbb{A}_F)$ and $GL_3(\mathbb{A}_F)$, resp. By the local Langlands correspondence, we have $\pi_1 \boxtimes \pi_2 = \otimes_v (\pi_{1v} \boxtimes \pi_{2v})$, an irreducible admissible representation of $GL_6(\mathbb{A})$.

Let S be a finite set of finite places such that π_{1v}, π_{2v} are unramified for $v \notin S$, $v < \infty$. Let σ be a cuspidal representation of $GL_m(\mathbb{A})$, $m = 2, 3, 4$. The triple product L-functions $L(s, \pi_1 \times \pi_2 \times \sigma)$ are available from Langlands-Shahidi method. Consider three cases; $D_5 - 2$, $E_6 - 1$ and $E_7 - 1$, with all groups simply connected.

(1) $D_5 - 2$: in which case, $m = 2$; $\mathbf{G} = Spin(10)$, and the derived group of $\mathbf{M}$ is $SL_3 \times SL_2 \times SL_2$.
(2) $E_6 - 1$: in which case, $m = 3$; $\mathbf{G}$=simply connected E_6, and the derived group of $\mathbf{M}$ is $SL_3 \times SL_2 \times SL_3$.
(3) $E_7 - 1$: in which case, $m = 4$; $\mathbf{G}$=simply connected E_7, and the derived group of $\mathbf{M}$ is $SL_3 \times SL_2 \times SL_4$.

Note that given a finite place v, either one of π_{iv}, or σ_v is in the principal series for $\sigma \in \mathcal{T}^S(m)$. Hence by multiplicativity

$$L(s, (\pi_{1v} \boxtimes \pi_{2v}) \times \sigma_v) = L(s, \pi_{1v} \times \pi_{2v} \times \sigma_v),$$
$$\gamma(s, (\pi_{1v} \boxtimes \pi_{2v}) \times \sigma_v, \psi_v) = \gamma(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v) \tag{16.1}$$

Applying the converse theorem to $\pi_1 \boxtimes \pi_2 = \otimes_v (\pi_{1v} \boxtimes \pi_{2v})$ and S, we obtain

Proposition 16.4. *There exists an automorphic representation Π of $GL_6(\mathbb{A})$ such that $\Pi_v \simeq \pi_{1v} \boxtimes \pi_{2v}$ for all $v \notin S$.*

By the classification of automorphic representations, Π is equivalent to a subquotient of

$$Ind\, |det|^{r_1} \sigma_1 \otimes \cdots \otimes |det|^{r_k} \sigma_k$$

where σ_i is a (unitary) cuspidal representation of $GL_{n_i}(\mathbb{A}_F)$. Note that the central character of Π is $\omega_\Pi = \omega_{\pi_1}^3 \omega_{\pi_2}^2$. In particular, it is unitary. Hence $n_1 r_1 + \cdots + n_k r_k = 0$.

Here $n_i > 1$ since $L_S(s, \Pi \otimes \mu) = L_S(s, \pi_1 \times (\pi_2 \otimes \mu))$ is entire for every grössencharacter μ. Since cuspidal representations of $GL_2(\mathbb{A})$ and $GL_3(\mathbb{A})$ satisfy the weak Ramanujan property, so does Π. Hence $r_1 = \cdots = r_k = 0$. Therefore $\Pi = \sigma_1 \boxplus \cdots \boxplus \sigma_k$.

Proving $\Pi_v \simeq \pi_{1v} \boxtimes \pi_{2v}$ for $v \in S$ is difficult, especially when π_{1v}, π_{2v} are supercuspidal representations. The most difficult case is when π_{1v} is an extraordinary supercuspidal representation of $GL_2(F_v)$, i.e., $v|2$, and π_{2v} is a supercuspidal representation attached to non-normal cubic extension E_v/F_v. We briefly explain how it is done: Let T be the set of places where $v|2$, π_{1v} is an extraordinary supercuspidal representation of $GL_2(F_v)$, and π_{2v} is a supercuspidal representation of $GL_3(F_v)$ attached to a character of a non-normal cubic extension of F_v. Then, if $v \notin T$, we can show that, for every irreducible generic representation σ_v of $GL_m(F_v)$, $m = 1, 2, 3, 4$, the equalities (16.1) hold. We briefly explain how to prove it in the case when both π_{1v} and π_{2v} are supercuspidal representations. Suppose $v|2$ and $v \notin T$. Then π_{2v} corresponds to $Ind(W_{F_v}, W_K, \mu)$, where K/F_v is a cyclic cubic extension and μ is a character of K^*. We embed π_{1v}, σ_v as local components of cuspidal representations π_1, σ with unramified finite components everywhere else. We choose a cubic cyclic extension E/F such that $E_w = K$, $w|v$ (i.e., v is inert) and choose a grössencharacter η of E such that $\eta_w = \mu$. Let π_2 be a cuspidal representation corresponding to $Ind(W_F, W_E, \eta)$. Let π_1^E, σ^E be their base changes. By comparing functional equations of $L(s, \pi_1 \times \pi_2 \times \sigma)$ and $L(s, (\pi_1^E \otimes \eta) \times \sigma^E)$, and the adjointness property of base change (Proposition 2.3.1 of [Ra1], [A-C]), we obtain, up to a multiplication by an appropriate λ-function (see [A-C, p60] for the definition), the equality $\gamma(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v) = \gamma(s, (\pi_{1w}^K \otimes \eta_w) \times \sigma_w^K, \psi_w)$. The right hand side is now a Rankin-Selberg γ-factor and it is easy to see that it gives the correct Artin factors. The proof for the case $v \nmid 2$ is similar, except that we use a quadratic base change. If $v \in T$, by applying the converse theorem to the set $S = \{v\}$ and choosing appropriate cusp forms for which π_{1v} and π_{2v} are the only finite ramifications, one can get a unique local lift Π_v satisfying equalities (16.1) in place of $\pi_{1v} \boxtimes \pi_{2v}$. Using local-global arguments, one can show that our local lift differs from $\pi_{1v} \boxtimes \pi_{2v}$ by at most a quadratic character. We then combine this and equalities (16.1) for $m = 1, 2, 3$, which can be proved by non-normal base change, with a result in local Langlands correspondence (Bushnell-Henniart), to conclude that equalities (16.1) also hold for $m = 4$ for all v in T.

Now pick two finite places v_1 and v_2, where π_{jv_1} and $\pi_{jv_2}, j = 1, 2$, are unramified. Let $S_i = \{v_i\}$, $i = 1, 2$. We apply the converse theorem twice to $\pi_1 \boxtimes \pi_2 = \otimes_v \pi_{1v} \boxtimes \pi_{2v}$ with S_1 and S_2, and find two automorphic representations Π_1 and Π_2 of $GL_6(\mathbb{A})$, both induced from cuspidal representations, such that $\Pi_{1v} \simeq \pi_{1v} \boxtimes \pi_{2v}$ for $v \neq v_1$ and $\Pi_{2v} \simeq \pi_{1v} \boxtimes \pi_{2v}$ for $v \neq v_2$. By the strong multiplicity one, $\Pi_1 \simeq \Pi_2$. In particular, $\Pi_{1v_i} \simeq \Pi_{2v_i} \simeq \pi_{1v} \boxtimes \pi_{2v}$ for $i = 1, 2$.

In conclusion,

Theorem 16.5. *The representation $\pi_1 \boxtimes \pi_2$ of $GL_6(\mathbb{A}_F)$ is automorphic. Moreover, it is irreducibly induced from cuspidal representations, i.e., $\pi_1 \boxtimes \pi_2 = Ind\, \sigma_1 \otimes \cdots \otimes \sigma_k$, where σ_i's are cuspidal representations of $GL_{n_i}(\mathbb{A}_F)$, $n_i > 1$.*

16.4 Functoriality of symmetric cube. Let π be a cuspidal representation of $GL_2(\mathbb{A})$. Let $\phi = Sym^3 : GL_2(\mathbb{C}) \to GL_4(\mathbb{C})$, the four dimensional irreducible representation of $GL_2(\mathbb{C})$ on symmetric tensors of rank 3. Simply put, for each $g \in GL_2(\mathbb{C})$, $Sym^3(g) \in GL_4(\mathbb{C})$ can be taken to be the matrix that gives the change in coefficients of an arbitrary homogeneous cubic polynomial in two variables, under the change of variables by g. By the local Langlands' correspondence, we have the irreducible admissible representation $Sym^3(\pi) = \otimes_v \mathrm{Sym}^3(\pi_v)$, which we call the symmetric cube of π.

Applying Theorem 16.5 to $\pi_1 = \pi$ and $\pi_2 = Ad(\pi)$, we obtain that $\pi \boxtimes Ad(\pi)$ is an automorphic representation of $GL_6(\mathbb{A})$. However,

$$\pi \boxtimes Ad(\pi) = \pi \boxplus Sym^3(\pi) \otimes \omega_\pi^{-1}.$$

Theorem 16.6. *$Sym^3(\pi)$ is an automorphic representation of $GL_4(\mathbb{A})$. It is cuspidal, unless π is either of dihedral, or, of tetrahedral type, i.e., those attached to dihedral and tetrahedral representations of Galois group of $\overline{F}/F$. In particular, if $F = \mathbb{Q}$ and π is the automorphic representation attached to a non–dihedral holomorphic form of weight ≥ 2, then $Sym^3(\pi)$ is cuspidal.*

Proof. We only need to prove the cuspidality. For this, we can prove the following:

Lemma 16.7. *Let σ be a cuspidal representation of $GL_2(\mathbb{A}_F)$. Then the triple L-function $L_S(s, Ad(\pi) \times \pi \times \sigma)$ has a pole at $s = 1$ if and only if $\sigma \simeq \pi \otimes \chi$ and $Ad(\pi) \simeq Ad(\pi) \otimes (\omega_\pi \chi)$ for some grössencharacter χ. Here S is a finite set of places for which $v \notin S$ implies that both π_v and σ_v are unramified.*

If we use the global Langlands' correspondence (Langlands-Tunnell theorem), we can avoid the use of the appendix in [Ki-Sh2] by Bushnell-Henniart for the special case of the functoriality of $Sym^3(\pi)$. After applying the converse theorem, we obtain that there exists an automorphic representation Π of $GL_6(\mathbb{A})$ such that $\Pi_v \simeq \pi_v \boxtimes Ad(\pi_v)$ for $v \notin S$. Then $\Pi = \pi \boxplus \tau$ for an automorphic representation τ of $GL_4(\mathbb{A})$. We have $\tau_v \simeq Sym^3(\pi_v) \otimes \omega_{\pi_v}^{-1}$ for $v \notin S$. For simplicity, denote $Sym^3(\pi_v) \otimes \omega_{\pi_v}^{-1}$ by $A^3(\pi_v)$. If $v \nmid 2$, π_v is induced and hence $Ad(\pi_v)$ is not supercuspidal. Hence we can show easily that $\tau_v \simeq A^3(\pi_v)$ for $v \nmid 2$.

Since we need local-global argument, in order to avoid confusion, we use the following setup: Let k be a non-archimedean local field of characteristic zero. Let η_1, η_2 be supercuspidal representations of $GL_m(k), GL_2(k)$ with corresponding parametrizations $\phi_1 : W_k \longrightarrow GL_m(\mathbb{C}), \phi_2 : W_k \longrightarrow GL_2(\mathbb{C})$, resp. We can think of ϕ_i as a representation of $Gal(\bar{k}/k)$. By the local converse theorem, we need to show that

$$\gamma(s, \eta_1 \times A^3(\eta_2), \psi) = \gamma(s, \phi_1 \otimes A^3(\phi_2), \psi),$$

for $m = 1, 2$. It is equivalent to showing that

$$\gamma(s, \eta_1 \times \eta_2 \times Ad(\eta_2), \psi) = \gamma(s, \phi_1 \otimes \phi_2 \otimes Ad(\phi_2), \psi).$$

The case of $m = 1$ is the local Langlands' correspondence. Let $m = 2$. By appealing to [P-Ra, Lemma 3, section 4], we can find a number field F with $k = F_v$ and irreducible 2-dimensional representations σ_i of $Gal(\bar{F}/F)$ with solvable image such that $\sigma_{iv} = \phi_i$ and σ_{iu} are unramified for $u|2, u \neq v$. The global Langlands correspondence is available for representations with solvable image [La3, Tu], and hence we can find corresponding cuspidal representations π_i of $GL_2(\mathbb{A}_F)$ such that $\pi_{iv} = \eta_i$. We compare the functional equations for $L(s, \pi_1 \times \pi_2 \times Ad(\pi_2))$ and $L(s, \sigma_1 \otimes \sigma_2 \otimes Ad(\sigma_2))$. Even though we do not know the holomorphy of $L(s, \sigma_1 \otimes \sigma_2 \otimes Ad(\sigma_2))$, the functional equation is known and it suffices for our purpose. Since $L(s, \pi_{1w} \times \pi_{2w} \times Ad(\pi_{2w})) = L(s, \sigma_{1w} \otimes \sigma_{2w} \otimes Ad(\sigma_{2w}))$ for unramified places, we have an equality

$$\prod_{u \in S} \gamma(s, \pi_{1u} \times \pi_{2u} \times Ad(\pi_{2u}), \psi_u) = \prod_{u \in S} \gamma(s, \sigma_{1u} \otimes \sigma_{2u} \otimes Ad(\sigma_{2u}), \psi_u).$$

Note that π_{iu}'s are unramified if $u|2, u \neq v$. Also if $u \in S, u \nmid 2$, then $Ad(\pi_{2u})$ is not supercuspidal. Therefore, if $u \in S, u \neq v$, $Ad(\pi_{2u})$ is not supercuspidal. Hence

$$\gamma(s, \pi_{1u} \times \pi_{2u} \times Ad(\pi_{2u}), \psi_u) = \gamma(s, \sigma_{1u} \otimes \sigma_{2u} \otimes Ad(\sigma_{2u}), \psi_u),$$

for each $u \in S, u \neq v$. Therefore,

$$\gamma(s, \pi_{1v} \times \pi_{2v} \times Ad(\pi_{2v}), \psi_v) = \gamma(s, \sigma_{1v} \otimes \sigma_{2v} \otimes Ad(\sigma_{2v}), \psi_v).$$

17. Functoriality of symmetric fourth.

17.1 Functoriality of exterior square. Let $\wedge^2 : GL_4(\mathbb{C}) \longrightarrow GL_6(\mathbb{C})$, be the map given by exterior square in Example 14.5. Let $\pi = \otimes_v \pi_v$ be a cuspidal (automorphic) representation of $GL_4(\mathbb{A})$. By the local Langlands correspondence, we have $\wedge^2\pi = \otimes_v \wedge^2 \pi_v$, an irreducible admissible representation of $GL_6(\mathbb{A})$.

Consider $D_n - 3$ case with $\mathbf{G} = Spin(2n)$ in Example 1.36. We have an injection $f : \mathbf{M} \longrightarrow GL_{n-3} \times GL_4$. Let σ, π be cuspidal representations of $GL_{n-3}(\mathbb{A}), GL_4(\mathbb{A})$ with central characters ω_1, ω_2, resp. Let Σ be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and σ, π. Then $(\mathbf{M}, \Sigma)$ gives rise to, for almost all v,

$$\begin{aligned} L(s, \Sigma_v, r_1) &= L(s, \sigma_v \otimes \pi_v, \rho_{n-3} \otimes \wedge^2 \rho_4), \\ L(s, \Sigma_v, r_2) &= L(s, \sigma_v, \wedge^2 \otimes \omega_{2v}). \end{aligned}$$

We need the cases $n = 4, 5, 6, 7$. Let S be a finite set of finite places such that π_v's are unramified for $v \notin S$, $v < \infty$. We can show that if π_v is not supercuspidal,

$$\begin{gathered} \gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2 \rho_4, \psi_v) = \gamma(s, \sigma_v \times \wedge^2 \pi_v, \psi_v) \\ L(s, \sigma_v \otimes \pi_v, \rho_m \otimes \wedge^2 \rho_4) = L(s, \sigma_v \times \wedge^2 \pi_v). \end{gathered}$$

Note that if $v \in S$, σ_v is unramified. Hence by multiplicativity of γ and L-factors, it is enough to show the above equalities for σ_v, characters of $F_v \times$. However, if π_v is supercuspidal, we cannot show it yet. There are three ways to get around it:

First, use the stability of γ-factors. This is not known.

Second, use descent argument of Ramakrishnan. See [Ki5] for the details.

Third, use the global method in special cases such as the functoriality of the symmetric fourth: Let k be a non-archimedean local field of characteristic zero. Let η

be supercuspidal representations of $GL_2(k)$ with the corresponding parametrization $\phi : W_k \longrightarrow GL_2(\mathbb{C})$. We can think of ϕ as a representation of $Gal(\bar{k}/k)$. We need to show that

$$\gamma(s, A^3(\eta), \wedge^2\rho_4 \otimes \chi, \psi) = \gamma(s, A^3(\phi), \wedge^2\rho_4 \otimes \chi, \psi),$$

for any character χ of $k^\times$, which we identify it as a character of $Gal(\bar{k}/k)$. By appealing to [P-Ra, Lemma 3, section 4], we can find a number field F with $k = F_v$ and irreducible 2-dimensional representations σ of $Gal(\bar{F}/F)$ with solvable image such that $\sigma_v = \phi$ and σ_u is unramified for $u|2, u \neq v$. Let π be the cuspidal representation of $GL_2(\mathbb{A}_F)$ such that $\pi_v = \eta$, given by the global Langlands correspondence. Take a grössencharacter μ such that $\mu_v = \chi$. By comparing the functional equations of $L(s, A^3(\pi), \wedge^2\rho_4 \otimes \mu)$ and $L(s, A^3(\sigma), \wedge^2\rho_4 \otimes \mu)$, we obtain the equality, by noting that if $u|2, u \neq v$, π_u is unramified.

Apply the converse theorem to $\wedge^2\pi$ and S, and obtain an automorphic representation Π of $GL_6(\mathbb{A})$ such that $\Pi_v \simeq \wedge^2\pi_v$ for $v \notin S$. By the classification of the automorphic representation of GL_n, Π is equivalent to a subquotient of

$$Ind\, |det|^{r_1}\sigma_1 \otimes \cdots \otimes |det|^{r_k}\sigma_k$$

where σ_i is a (unitary) cuspidal representation of $GL_{n_i}(\mathbb{A}_F)$. However, proving that $r_1 = \cdots = r_k$ is not easy. It requires comparison of Hecke conjugacy classes of $\wedge^2\pi$ and Π. See [Ki5] for the details. However, if $\pi = Sym^3(\tau)$, where τ is a cuspidal representation of $GL_2(\mathbb{A})$ as we will do in order to obtain the functoriality of the symmetric fourth, then π satisifies the weak Ramanujan property, and so does $\wedge^2\pi$. It implies $r_1 = \cdots = r_k$. Proving that $\Pi_v \simeq \wedge^2\pi_v$ is not easy, especially when $v \mid 2$ and π_v is a supercuspidal representation. We have

Theorem 17.1. *Let T be the set of places where $v|2,3$, and π_v is a supercuspidal representation. Then there exists an automorphic representation Π of $GL_6(\mathbb{A})$ such that $\Pi_v \simeq \wedge^2\pi_v$ if $v \notin T$. Moreover, Π is of the form $Ind\,\tau_1 \otimes \cdots \otimes \tau_k$, where τ_i's are all cuspidal representations of $GL_{n_i}(\mathbb{A})$.*

17.2 Functoriality of symmetric fourth. Let $Sym^4 : GL_2(\mathbb{C}) \longrightarrow GL_5(\mathbb{C})$ be the 4^{th} symmetric power representation of $GL_2(\mathbb{C})$ on the space of symmetric tensors of rank 4. Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $GL_2(\mathbb{A})$ with central character ω_π. By the local Langlands' correspondence, $Sym^4(\pi) = \otimes_v Sym^m(\pi_v)$ is an irreducible admissible representation of $GL_5(\mathbb{A})$.

We apply Theorem 17.1 to $A^3(\pi) = Sym^3(\pi) \otimes \omega_\pi^{-1}$. Then $\wedge^2(A^3(\pi)) = Sym^4(\pi) \otimes \omega_\pi^{-1} \boxplus \omega_\pi$. Hence we have

Theorem 17.2. *$Sym^4(\pi)$ is an automorphic representation of $GL_5(\mathbb{A})$. Let $A^4(\pi) = Sym^4(\pi) \otimes \omega_\pi^{-1}$. It is a cuspidal representation of $GL_5(\mathbb{A})$ except for the following three cases*

(1) *π is monomial,*
(2) *π is not monomial and $A^3(\pi)$ is not cuspidal; this is the case when there exists a non-trivial grössencharacter μ such that $Ad(\pi) \simeq Ad(\pi) \otimes \mu$,*

(3) $A^3(\pi)$ *is cuspidal, but there exists a non-trivial quadratic character* η *such that* $A^3(\pi) \simeq A^3(\pi) \otimes \eta$*, or equivalently, there exists a non-trivial grössencharacter* χ *of* E *such that* $Ad(\pi_E) \simeq Ad(\pi_E) \otimes \chi$*, where* E/F *is the quadratic extension determined by* η *and* π_E *is the base change of* π*. In this case,* $A^4(\pi) = \sigma_1 \boxplus \sigma_2$*, where* $\sigma_1 = \pi(\chi^{-1}) \otimes \omega_\pi$ *and* $\sigma_2 = Ad(\pi) \otimes (\omega_\pi \eta)$.

Cases (1), (2), and (3) are equivalent to π *being of dihedral, tetrahedral, or octahedral type, respectively.*