4. Induced representations.

4.1 Harish-Chandra homomorphisms. Let $\mathbf{G}$ be a split reductive group and $\mathbf{P} = \mathbf{MN}$ be a parabolic subgroup defined over a field F. If F is a p-adic field with the ring of integer $\mathcal{O}$, then we have the Iwasawa decomposition: $\mathbf{G}(F) = \mathbf{P}(F)\mathbf{G}(\mathcal{O})$. In general, we have $\mathbf{G}(F) = \mathbf{P}(F)K$ for some maximal compact subgroup K.

Let $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$. Let $X^*(\mathbf{M}), X^*(\mathbf{A})$ be the group of characters of $\mathbf{M}, \mathbf{A}$, resp. Then by Proposition 1.9, $X^*(\mathbf{M})$ is of finite index in $X^*(\mathbf{A})$. Hence $X^*(\mathbf{M}) \otimes_{\mathbb{Z}} \mathbb{R} = X^*(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{R}$. Note that if $\chi \in X^*(\mathbf{M})$, χ is defined by a polynomial with coefficients in F, and hence defines a homomorphism of E-rational points $\mathbf{M}(E) \longrightarrow E^*$ for any extension field E/F. We denote it by the same χ.

Set $\mathfrak{a}^* = X^*(\mathbf{M}) \otimes_{\mathbb{Z}} \mathbb{R}$, and $\mathfrak{a}^*_{\mathbb{C}} = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$. Let $\mathfrak{a} = Hom(X^*(\mathbf{M}), \mathbb{R}) = Hom(X^*(\mathbf{A}), \mathbb{R})$ be the dual space. In fact, we define the pairing $\mathfrak{a} \times \mathfrak{a}^* \longrightarrow \mathbb{R}$ by: if $\chi \otimes r \in \mathfrak{a}^*$, $r \in \mathbb{R}$, and $\lambda \in \mathfrak{a}$, then $< \lambda, \chi \otimes r >= \lambda(\chi) r$.

Suppose first that F is a local field. Define the homomorphism $H_M : \mathbf{M}(F) \longrightarrow \mathfrak{a}$ by

$$exp(\langle \chi, H_M(m) \rangle) = |\chi(m)|,$$

where $\chi \in X^*(\mathbf{M})$ and $m \in \mathbf{M}(F)$ and $| \, |$ is the absolute value in F. If F is a p-adic field, then we replace exp by q_v. We extend H_M to $H_P : P \longrightarrow \mathfrak{a}$ by extending it trivially on $\mathbf{N}(F)$. We can extend H_P further on $\mathbf{G}(F)$ using the Iwasawa decomposition $\mathbf{G}(F) = \mathbf{P}(F)K$ by $exp(\langle \chi, H_P(mnk) \rangle) = |\chi(m)|$.

Suppose F is global. If $\chi \in X^*(\mathbf{M})$, for every place v of F, χ defines a homomorphism $\chi : \mathbf{M}(F_v) \longrightarrow F_v^*$. So we define the homomorphism $H_M : \mathbf{M}(\mathbb{A}) \longrightarrow \mathfrak{a}$ by

$$exp(\langle \chi, H_M(m) \rangle) = \prod_v |\chi(m_v)|_v,$$

where $\chi \in X^*(\mathbf{M})$ and $m = (m_v) \in \mathbf{M}(\mathbb{A})$. For almost all v, χ defined by a polynomial in $\mathcal{O}_v$, and $m_v \in \mathbf{M}(\mathcal{O}_v)$. Hence $\chi(m_v) \in \mathcal{O}_v^*$ for almost all v. So $|\chi(m_v)|_v = 1$ for almost all v, and the above product is a finite product. We extend H_M to H_P as in the local case. Observe that for each v, we can define H_{P_v} and, for $\chi \in X^*(\mathbf{M})$ and $m = (m_v) \in \mathbf{M}(\mathbb{A})$,

$$exp(\langle \chi, H_P(m) \rangle) = \prod_v exp(\langle \chi, H_{P_v}(m_v) \rangle),$$

We call H_P Harish-Chandra homomorphism. It is closely related to the modulus character δ_P of P: Note that P is not unimodular. The modulus character δ_P is the ratio of the right and the left invariant Haar measures on P (i.e., If $d_r p$ is the right Haar measure on P, then $\delta_P(m) d_r p$ is the left Haar measure $d_l p$). Let $Ad : \mathbf{M} \longrightarrow End(\mathfrak{n})$ be the adjoint representation, where $\mathfrak{n}$ is the Lie algebra of $\mathfrak{N}$. Then we can show that $\delta_P(m) = |det\, Ad(m)|$ for $m \in \mathbf{M}$.

Let $2\rho_P$ be the sum of positive roots in $\mathbf{N}$. More precisely, if $\mathbf{P} = P_\theta$, $\theta \subset \Delta$, then by Theorem 1.34, $2\rho_P$ is the sum of roots in $\Phi_+ - \Sigma_\theta^+$, where $\Sigma_\theta^+ = \{\theta\}_{\mathbb{Z}} \cap \Phi_+$.

Then we can show that $|(2\rho_P)(m)| = \delta_P(m)$. Hence

$$exp(\langle t\rho_P, H_M(m)\rangle) = exp(\frac{t}{2}\langle 2\rho_P, H_M(m)\rangle) = |(2\rho_P)(m)|^{\frac{t}{2}} = \delta_P(m)^{\frac{t}{2}}.$$

Example 4.1. Let $\mathbf{B} = \mathbf{TU}$. Suppose $t = \prod_{i=1}^{l} h_{\alpha_i}(t_i)$, where $t_i \in F^*$, where $\{\alpha_1, ..., \alpha_l\}$ are simple roots. Then

$$exp(\langle \rho_B, H_B(t)\rangle) = \prod_{i=1}^{l} |t_i|^{\langle \rho_B, \alpha_i\rangle}.$$

We show that $\langle \rho_B, \alpha_i\rangle = 1$ for all i: Note that $\langle \rho_B, \alpha_i\rangle = \frac{2(\rho_B,\alpha_i)}{(\alpha_i,\alpha_i)}$. First we show that $w_\alpha(\rho_B) = \rho_B - \alpha$ for any simple root α. Note that $w_\alpha(\alpha) = -\alpha$ and w_α leaves $\Phi_+ - \{\alpha\}$ invariant. Hence if ρ_α is the half-sum of elements of $\Phi_+ - \{\alpha\}$, $w_\alpha(\rho_\alpha) = \rho_\alpha$. Since $\rho_B = \rho_\alpha + \frac{\alpha}{2}$, $w_\alpha(\rho_B) = \rho_B - \alpha$.

Since $(\ ,\ )$ is the Weyl group invariant, $(w_\alpha(\rho_B), w_\alpha(\alpha)) = (\rho_B, \alpha)$. So $(\rho_B - \alpha, -\alpha) = (\rho_B, \alpha)$. This imples that $(\rho_B, \alpha) = \frac{1}{2}(\alpha, \alpha)$.

Now assume that $\mathbf{P}$ is a maximal parabolic and let $s \in \mathbb{C}$. Suppose $\mathbf{P} = P_\theta$, $\theta = \Delta - \{\alpha\}$ for $\alpha \in \Delta$. Set $\tilde{\alpha} = \langle \rho_P, \alpha\rangle^{-1}\rho_P \in \mathfrak{a}^*$, where $\langle \rho_P, \alpha\rangle = \frac{2(\rho_P,\alpha)}{(\alpha,\alpha)}$. Then $\tilde{\alpha}$ is the fundamental weight corresponding to α.

Example 4.2. (1) Let $\mathbf{P} = \mathbf{MN} \subset Sp(2n)$ such that $\mathbf{P} = P_\theta$, where $\theta = \Delta - \{2e_n\}$. So $\mathbf{M} \simeq GL_n$. Then we can show that $\rho_P = \frac{n+1}{2}(e_1 + \cdots + e_n)$, and $\tilde{\alpha} = e_1 + \cdots + e_n$. Hence $exp(\langle s\tilde{\alpha}, H_P(m)\rangle) = |det(m)|^s$. If we define $Sp(2n) = \{g \in SL(2n) \,|\, {}^tgJg = J\}$, where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, then $\mathbf{M} = \{m' = \begin{pmatrix} m & 0 \\ 0 & {}^tm^{-1} \end{pmatrix} \,|\, m \in GL_n\}$, and $\mathbf{N} = \{\begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix}\}$, where X is an $n \times n$ symmetric matrix. Hence $Ad : \mathbf{M} \longrightarrow End(\mathfrak{n})$ is given by $Ad(m') : X \longmapsto mX{}^tm$, where $m \in GL_n(F)$ and X is an $n \times n$ symmetric matrix. We leave as an exercise to show that $detAd(m') = det(m)^{n+1}$. Hence we again obtain $exp(\langle \tilde{\alpha}, H_P(m')\rangle) = |det(m)|$. More concretely, $H_P(m') = \log|det(m)|$.

(2) Let $\mathbf{P} = P_\theta = \mathbf{MN} \subset SO(2n+1)$, where $\theta = \Delta - \{e_n\}$ such that $\mathbf{M} \simeq GL_n$. In this case, $\rho_P = \frac{n}{2}(e_1 + \cdots + e_n)$ and $\tilde{\alpha} = \frac{1}{2}(e_1 + \cdots + e_n)$. Hence $exp(\langle s\tilde{\alpha}, H_P(m)\rangle) = |det(m)|^{\frac{s}{2}}$

(3) Let $\mathbf{P} = \mathbf{MN} \subset GL_{k+l}$, where $\mathbf{M} \simeq GL_k \times GL_l$. Then $\mathbf{M} = \{m = diag(m_1, m_2)\}$, where $m_1 \in GL_k, m_2 \in GL_l$, and $\mathfrak{n} = \{\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\}$, where X is an $k \times l$ matrix. Then $Ad : \mathbf{M} \longrightarrow End(\mathfrak{n})$ is given by $Ad(m) : X \longmapsto m_1Xm_2^{-1}$. Hence $detAd(m) = det(m_1)^l det(m_2)^{-k}$. In terms of roots, $2\rho_P = l(e_1 + \cdots + e_k) - k(e_{k+1} + \cdots + e_{k+l})$.

We continue to assume that $\mathbf{P}$ is maximal. If $\mathbf{G}$ is semi-simple, $\mathfrak{a} \simeq \mathbb{R}$. We write the kernel of $H_P : \mathbf{M}(\mathbb{A}) \longrightarrow \mathfrak{a}$ by $\mathbf{M}(\mathbb{A})^1$. Then we have a direct product decomposition

$$\mathbf{M}(\mathbb{A}) = \mathbf{M}(\mathbb{A})^1 \times \mathbf{A}(\mathbb{R})_+.$$

Since $\mathbf{A} \simeq GL(1)$, $\mathbf{A}(\mathbb{R})_+ \simeq \mathbb{R}_+$, and $\mathbf{A}(\mathbb{R})_+ = \{\tilde{\alpha}^\vee(t^{\frac{1}{n}}) : t \in \mathbb{R}_+\}$, where $n = \langle \tilde{\alpha}, \tilde{\alpha}^\vee \rangle$. Note that $exp(\langle s\tilde{\alpha}, H_P(\tilde{\alpha}^\vee(t^{\frac{1}{n}}))\rangle) = t^s$ for $t \in \mathbb{R}_+$. Clearly, $\mathbf{M}(F) \subset \mathbf{M}(\mathbb{A})^1$ and

$$\mathbf{M}(F)\backslash\mathbf{M}(\mathbb{A}) \simeq \mathbf{M}(F)\backslash\mathbf{M}(\mathbb{A})^1 \times \mathbf{A}(\mathbb{R})_+.$$

Here $\mathbf{M}(F)\backslash\mathbf{M}(\mathbb{A})^1$ is not compact in general, but it does have a finite volume.

Example 4.3. Let $I = \mathbb{A}^*$ be the ideles, i.e., $I = GL_1(\mathbb{A})$. Then we have $|\,| : I \longrightarrow \mathbb{R}_+$ defined by $|x| = \prod_v |x_v|_v$ if $x = (x_v)$. Denote the kernel by I^1. Then $F^* \subset I^1$, and we have a direct product decomposition $I \simeq I^1 \times \mathbb{R}_+$. Here $\mathbb{R}_+ \longrightarrow I$ is defined by $t \mapsto (t^{\frac{1}{n}}, ..., t^{\frac{1}{n}}, 1, ..., 1, ...)$, where $n = [F : \mathbb{Q}]$ and 1's are on the finite components. We denote $F_\infty^+ = \{(u, ..., u, 1, ..., 1, ...) \in I, u > 0\}$. Then $I/F^* \simeq I^1/F^* \times F_\infty^+$. Here I^1/F^* is compact.

Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$.

Normalization of cuspidal representations. *We normalize π by requiring that π is trivial on $\mathbf{A}(\mathbb{R})_+$.*

This is necessary later to put possible poles of Eisenstein series and L-functions on the real axis. This normalization gets rid of the twisting by the characters $|\,|_{\mathbb{A}}^t$, where $t \in i\mathbb{R}$, namely, we do not consider the cuspidal representations of the form $\pi \otimes |\,|_{\mathbb{A}}^t$. In the case of $\mathbf{M} = GL_n$, this means that we assume that the central character ω_π is trivial on F_∞^+. (If zI_n is in the center, we define $\omega_\pi(z) = \pi(zI_n)$.)

4.2 Induced representations: F local. Let (π, W) be an irreducible admissible representation of $\mathbf{M}(F)$. Let $I(s\tilde{\alpha}, \pi)$ be the induced representation of $\mathbf{G}(F)$;

$$I(s\tilde{\alpha}, \pi) = Ind_P^G\, \pi \otimes exp(\langle s\tilde{\alpha}, H_P(\,)\rangle) \otimes 1.$$

The representation space $V = V(s\tilde{\alpha}, \pi)$ is the vector space of all smooth functions $f : \mathbf{G}(F) \longrightarrow W$ such that

$$f(mng) = \pi(m) exp(\langle s\tilde{\alpha} + \rho_P, H_P(m)\rangle) f(g),$$

for all $m \in \mathbf{M}(F), n \in \mathbf{N}(F), g \in \mathbf{G}(F)$. The action is

$$(I(s\tilde{\alpha}, \pi)(g)f)(h) = f(hg),$$

for all $f \in V$ and $g, h \in \mathbf{G}(F)$. The reason we add ρ_P is to normalize $I(s\tilde{\alpha}, \pi)$ so that if π is unitary, then $I(s\tilde{\alpha}, \pi)$ is unitary for $s \in i\mathbb{R}$.

Examples 4.4. (1) Let $\mathbf{MN} \subset Sp(2n)$ with $\mathbf{M} \simeq GL_n$. Let π be a representation of $GL_n(F)$. Then $I(s\tilde{\alpha}, \pi) = Ind_P^G\, \pi \otimes |det(\,)|^s \otimes 1$. (Usually, we skip 1.)

(2) Let $\mathbf{MN} \subset GL_{m+n}$ with $\mathbf{M} \simeq GL_m \times GL_n$. Let π_1, π_2 be representations of $GL_m(F), GL_n(F)$, resp. In this case, because of the center, $dim\, \mathfrak{a} = 2$, and $\tilde{\alpha}$ and $e_1 + \cdots + e_{m+n}$ form a basis for $\mathfrak{a}$. So choosing the coordinates $s_1(e_1 + \cdots + e_m) + s_2(e_{m+1} + \cdots + e_{m+n}) \in \mathfrak{a}_{\mathbb{C}}^*$, we define the induced representation

$$I(s_1, s_2, \pi_1 \otimes \pi_2) = Ind_P^G \pi_1 \otimes |det|^{s_1} \otimes \pi_2 \otimes |det|^{s_2}.$$

Sometimes, we take $s_1 = \frac{s}{2}, s_2 = -\frac{s}{2}$, we denote $I(s, \pi_1 \otimes \pi_2) = I(\frac{s}{2}, -\frac{s}{2}, \pi_1 \otimes \pi_2)$.

We call this parabolic induction. Parabolic induction can be defined for arbitrary parabolic subgroups; $I(\nu, \sigma) = Ind_P^G \pi \otimes exp(\langle \nu, H_P(\,)\rangle)$ for $\nu \in (\mathfrak{a}_\theta)^*_{\mathbb{C}}$, where $\mathfrak{a}^*_\theta = X^*(M_\theta) \otimes \mathbb{R}$, where $\mathbf{P} = P_\theta = M_\theta N_\theta$, and σ is a representation of $\mathbf{M}(F)$.

Special case of principal series:. Let $\mathbf{B} = \mathbf{TU}$ be a Borel subgroup and let χ be a character of $\mathbf{T}(F)$. Then $I(\nu, \chi) = Ind_B^G \chi \otimes exp(\langle \nu, H_B(\,)\rangle)$ is called principal series, where $\nu \in \mathfrak{a}^*_{\mathbb{C}}$. Since $exp(\langle \nu, H_B(\,)\rangle)$ is a character of $\mathbf{T}(F)$, it is a convention just to write $I(\chi)$ by absorbing it into χ. The representation space is

$$V(\chi) = \{f : \mathbf{G}(F) \longrightarrow \mathbb{C} :\ f(tug) = \chi(t) exp(\langle \rho_B, H_B(t)\rangle) f(g)\},$$

where $t \in \mathbf{T}(F), u \in \mathbf{U}(F)$.

Example 4.5. Let $\mathbf{G} = GL_2$. Let $\chi = \mu_1 \otimes \mu_2$, where μ_1, μ_2 are characters of F^*, where F is a p-adic field with the ring of integer $\mathcal{O}$. In this case, $exp(\langle \rho_B, H_B(diag(a_1, a_2))\rangle) = |a_1 a_2^{-1}|^{\frac{1}{2}}$. So $I(\mu_1, \mu_2)$ is the induced representation on the space

$$V(\mu_1, \mu_2) = \{f : GL_2(F) \longrightarrow \mathbb{C} |\ f(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g) = \mu_1(a_1)\mu_2(a_2)|a_1 a_2^{-1}|^{\frac{1}{2}} f(g)\}.$$

We can show that $I(\mu_1, \mu_2)$ is irreducible if and only if $\mu_1 \mu_2^{-1} \neq |\,|^{\pm 1}$; $V = I(\mu|\,|^{\frac{1}{2}}, \mu|\,|^{-\frac{1}{2}})$ is reducible, where μ is a unitary character. We have a composition series: $0 \subset W \subset V$, where W is the unique invariant subspace. Then (σ, W) is square integrable, called Steinberg representation. $(\tau, V/W)$ is the one-dimensional representation of $GL_2(F)$, namely, $\tau = \mu \circ det$.

We can also show that if π is spherical, then $\pi \subset I(\mu_1, \mu_2)$ such that μ_1, μ_2 are unramified, i.e., $\mu_i|_{\mathcal{O}} = 1$. Let $\alpha = \mu_1(\varpi), \beta = \mu_2(\varpi)$, where ϖ is a uniformizer of F. Then π is uniquely determined by the semi-simple conjugacy class of $diag(\alpha, \beta)$.

4.3 Intertwining operators for $I(s, \pi)$. Suppose $\mathbf{P} = P_\theta$, where $\theta = \Delta - \{\alpha\}$. There exists a unique element $w_0 \in W$ such that $w_0(\theta) \subset \Delta$ and $w_0(\alpha) < 0$. Define, for $f \in V(s\tilde{\alpha}, \pi)$,

$$A(s, \pi, w_0) f(g) = \int_{\mathbf{N}'(F)} f(w_0^{-1} n g)\, dn,$$

where $\mathbf{N}'$ is the unipotent radical of the standard parabolic subgroup $\mathbf{P}' = P_{w_0(\theta)}$. $A(s, \pi, w_0)$ is called intertwining operator, since $A(s, \pi, w_0) : I(s, \pi) \longrightarrow I(-s, w_0(\pi))$.

Definition 4.6. *A maximal parabolic subgroup $\mathbf{P} = P_\theta$ is called self-conjugate if $w_0(\theta) = \theta$.*

Proposition 4.7. *The non self-conjugate maximal parabolic subgroups of split groups whose derived groups are almost simple, are the following:*

(1) *Type A_n: n even, all maximal parabolic subgroups, or n odd, all except $\theta = \Delta - \{e_{\frac{n-1}{2}} - e_{\frac{n+1}{2}}\}$. This is the case $GL_n \times GL_m \subset GL_{n+m}$, where $n \neq m$.*

(2) *Type D_n: n odd and $\theta = \Delta - \{\alpha_n\}$. This is the case $GL_n \subset SO_{2n}$.*
(3) *Type E_6: $\theta = \Delta - \{\alpha_3\}$. This is the case $\mathbf{P} = \mathbf{MN}$, where the derived group of $\mathbf{M}$ is $SL_2 \times SL_5$.*
(4) *Type E_6: $\theta = \Delta - \{\alpha_1\}$. This is the case $\mathbf{P} = \mathbf{MN}$, where the derived group of $\mathbf{M}$ is $Spin(10)$.*

Hence if $\mathbf{P}$ is self-conjugate, the intertwining operator is simpler, namely,

$$A(s, \pi, w_0)f(g) = \int_{\mathbf{N}(F)} f(w_0^{-1}ng)\, dn,$$

We can show that $A(s, \pi, w_0)$ is convergent for $Re(s) >> 0$.

Theorem 4.8. *$A(s, \pi, w_0)$ has a meromorphic continuation to all of $\mathbb{C}$.*

The intertwining operators can be defined for any parabolic subgroups $\mathbf{P} = P_\theta = \mathbf{MN}$. Let $w \in W$ such that $w(\theta) \subset \Delta$. Let $\overline{\mathbf{N}} = \mathbf{N}_{-\theta}$ be the unipotent subgroup opposed to $\mathbf{N}$. Let $\mathbf{N}_w = \mathbf{U} \cap w\overline{\mathbf{N}}w^{-1}$. Given $f \in I(\nu, \sigma)$, we define

$$A(\nu, \sigma, w)f(g) = \int_{\mathbf{N}_w(F)} f(w^{-1}ng)\, dn.$$

Then $A(\nu, \sigma, w) : I(\nu, \sigma) \longrightarrow I(w\nu, w\sigma)$.

4.4 Digression on admissible representations. Let $\mathbf{G}$ be a split reductive group defined over a p-adic field F. In this section, let $G = \mathbf{G}(F)$. Let (π, V) be a representation of G. If $K \subset G$, define $V^K = \{v \in V : \pi(k)v = v \text{ for all } k \in K\}$.

Definition 4.9. *(1) (π, V) is called smooth if every $v \in V$ lies in V^K for some open compact subgroup K. This is equivalent to the condition that π be continuous with respect to the discrete topology on V.*

(2) (π, V) is admissible if it is smooth and V^K has finite dimension for every open compact subgroup K.

(3) (π, V) is supercuspidal if for every proper parabolic subgroup P and any admissible representation σ of M, $Hom_G(V, I(\nu, \sigma)) = 0$. This is equivalent to the condition that V_N (Jacquet module)=0 for any unipotent radical N. This is also equivalent to the condition that the matrix coefficient $c_{v,\tilde{v}}(g) =< \pi(g)v, \tilde{v} >$ has compact support on G modulo Z (the center of G), where $v \in V, \tilde{v} \in \tilde{V}$ (the dual space).

(4) (π, V) is a discrete series (square integrable) if the central character of π is unitary and the matrix coefficient $c_{v,\tilde{v}}$ is square integrable modulo the center, i.e., $|c_{v,\tilde{v}}| \in L^2(G/Z)$.

(5) (π, V) is a tempered representation if $|c_{v,\tilde{v}}| \in L^{2+\epsilon}(G/Z)$ for every $\epsilon > 0$. This is equivalent to the condition that π is a direct summand of the induced representation $I(0, \sigma)$, where σ is a discrete series of M. (Determining direct summands and multiplicities are subjects of R-group)

Proposition 4.10 (Jacquet). *Let (π, V) be any irreducible admissible representation of G. Then there exists a parabolic subgroup $P = MN$, and a supercuspidal representation σ of M, such that $\pi \hookrightarrow I(\nu, \sigma)$.*

Proposition 4.11. *Let $(I(\nu, \sigma), V)$ be the induced representation, where $\mathbf{P} = P_\theta = \mathbf{MN}$.*

1. *There exists a composition series $0 = V_n \subset V_{n-1} \subset \cdots \subset V_1 \subset V_0 = V$. We call the resulting irreducible representations $(\pi_i, V_i/V_{i+1})$ subquotients of $I(\nu, \sigma)$.*
2. *If σ is supercuspidal, then $n \leq \#W(\theta, \theta)$, where $W(\theta, \theta) = \{w \in W | \, w\theta = \theta\}$. It is isomorphic to $N(\mathbf{M})/\mathbf{M}$, where $N(\mathbf{M})$ is the normalizer of $\mathbf{M}$ in $\mathbf{G}$.*
3. *If π is any subquotient, then there exists $w \in W(\theta, \theta)$ such that $\pi \hookrightarrow I(w\nu, w\sigma)$, where $w\sigma(m) = \sigma(w\sigma w^{-1})$.*

Suppose σ is tempered, and $\mathbf{P} = P_\theta$.

Definition 4.12. *We say that $I(\nu, \sigma)$ is in the Langlands' situation if $Re(\langle \nu, \alpha^\vee \rangle) > 0$ for all $\alpha \in \Delta - \theta$.*

Example 4.13. Let $\mathbf{G} = GL_n$. Let $\mathbf{P} = \mathbf{MN}$, $\mathbf{M} \simeq GL_{n_1} \times \cdots \times GL_{n_k}$, and π_i's are tempered representations of $GL_{n_i}(F)$. Then the induced representation

$$Ind_P^G \pi_1 |det|^{r_1} \otimes \cdots \otimes \pi_k |det|^{r_k},$$

is in the Langlands' situation if $r_1 > \cdots > r_k$. (Note that we can absorb the imaginary part of r_i's inside π_i, using the fact that $\pi_i \otimes \chi$ is tempered if χ is unitary.)

Theorem 4.14 (Langlands' classification theorem). *Let F be a local field. (1) Suppose $I(\nu, \sigma)$ is in the Langlands' situation. Then $I(\nu, \sigma)$ has a unique irreducible quotient, called Langlands' quotient, and we denote it by $J(\nu, \sigma)$.*

(2) $J(\nu, \sigma)$ is the image of the intertwining operator $A(\nu, \sigma, w_0)$, where w_0 is the longest Weyl group element of W/W_θ, where W_θ is the subgroup of W, generated by w_α, $\alpha \in \theta$.

(3) Any irreducible admissible representation π of $\mathbf{G}(F)$ occurs uniquely as some $J(\nu, \sigma)$. We call (P, ν, σ) Langlands' data for π.

Note: If $I(\nu, \sigma)$ is arbitrary, by the Weyl group action, we can make $I(w\nu, w\sigma)$ be in the Langlands' situation. The subquotient of $I(w\nu, w\sigma)$ is called Langlands' subquotient of $I(\nu, \sigma)$.

Definition 4.15 (Contragredient representation). *Let (π, V) be a representation of G. Let $\tilde{V}$ be the space of all continuous linear functionals on V. The contragredient representation of π is denoted by $(\tilde{\pi}, \tilde{V})$, and is defined by*

$$(\tilde{\pi}(g)f)(v) = f(\pi(g^{-1})v),$$

for $f \in \tilde{V}$ and $v \in V$.

Facts: (1) If $\pi = I(\nu, \sigma)$, then $\tilde{\pi} = I(-\nu, \tilde{\sigma})$. Hence in particular, if $\pi = I(\chi)$ (principal series), $\tilde{\pi} = I(\chi^{-1})$.

(2) If $G = GL_n(F)$, then $\tilde{\pi} \simeq \pi'$, where (π', V) is defined by $\pi'(g) = \pi({}^t g^{-1})$.

(3) If $G = GL_2(F)$, then $\tilde{\pi} \simeq \pi \otimes \omega_\pi^{-1}$, where ω_π is the central character of π.

(4) If π_v is a spherical representation of $\mathbf{G}(F_v)$, then $L(s, \pi_v, \tilde{r}) = L(s, \tilde{\pi}_v, r)$ for any finite dimensional representation $r : {}^L G \longrightarrow GL_N(\mathbb{C})$.

(5) Let $\mathbf{G} = Sp(2n)$, $\mathbf{P} = \mathbf{MN}$, $\mathbf{M} = \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix}$, where $g \in GL_n$, and $w_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Then for a representation of $\mathbf{M}(F)$, $w_0 \pi \simeq \pi$ if and only if $\pi \simeq \tilde{\pi}$, i.e., π is self-contragredient (or self-dual).

Let $\mathbf{G} = GL_{2n}$, $\mathbf{P} = \mathbf{MN}$, $\mathbf{M} = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$, where $g_1, g_2 \in GL_n$, and $w_0 = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. Then for representations π_1, π_2 of $GL_n(F)$, $w_0(\pi_1 \otimes \pi_2) \simeq \pi_1 \otimes \pi_2$ if and only if $\pi_1 \simeq \pi_2$.

Special cases of induced representations. (A) $\mathbf{P} = \mathbf{MN}$ maximal and σ is a supercuspidal representation of $\mathbf{M}(F)$, where F is a p-adic field: Let $I(s, \sigma)$ be the induced representation. We can assume that $s \in \mathbb{R}_+ \cup \{0\}$.

1. Unless $\mathbf{P}$ is self-conjugate or $w_0 \sigma \simeq \sigma$, then $I(s, \sigma)$ is irreducible for all s. If $I(s, \sigma)$ is reducible, then it has length 2.
2. Suppose $\mathbf{P}$ is self-conjugate and $w_0 \sigma \simeq \sigma$. Then there exists a unique $s_0 \in \mathbb{R}_0 \cup \{0\}$, such that $I(\pm s_0, \sigma)$ is reducible and $I(s, \sigma)$ is irreducible for all $s \neq s_0$.
3. If $s_0 > 0$, then $0 \subset V \subset I(s_0, \sigma)$ and (π, V) is an irreducible square integrable representation and $(\pi, I(s_0, \sigma)/V)$ is the Langlands' quotient.
4. If σ is generic (we will define later), then $s_0 \in \{0, \frac{1}{2}, 1\}$.

(B) $\mathbf{P} = \mathbf{B}$ and $I(\chi)$ is the principal series, where χ is a quasi-character of $\mathbf{T}(F)$, and F is a p-adic field. Let $K = \mathbf{G}(\mathcal{O})$ be the maximal compact subgroup. Let $G = \mathbf{G}(F)$.

Definition 4.16. *A representation π of G is called spherical if it has a non-zero K-fixed vector.*

Theorem 4.17 (Borel-Matsumoto). *If π is spherical, then there exists an unramified quasi-character χ of $\mathbf{T}(F)$ such that $\pi \hookrightarrow I(\chi)$.*

The quasi-character χ is also given by the Satake isomorphism: Suppose (π, V) is spherical. Then we have a representation of $\mathcal{H}(G, K)$ on the one-dimensional space $V^K = < v_0 >$, where $\mathcal{H}(G, K)$ is the space of $\mathbb{C}$-valued compactly supported functions on G which are left and right K-invariant. We denote it by λ_π. It is given by

$$\lambda_\pi : \mathcal{H}(G, K) \longrightarrow \mathbb{C}, \quad \lambda_\pi(\eta) v_0 = \pi(\eta) v_0 = \int_G \eta(g) \pi(g) v_0 \, dg.$$

The Satake isomorphism is given by

$$\mathcal{H}(G,K) \longrightarrow \mathcal{H}(T,T_0)^W, \quad f \longmapsto Sf,$$

where $T_0 = T \cap K$ and $Sf(t) = \delta(t)^{\frac{1}{2}} \int_U f(tu)\, du$. Since $T/T_0 \simeq \mathbb{Z}^n$, $\mathcal{H}(T,T_0) \simeq \mathbb{C}[x_1,...,x_n]$, where x_i is mapped to the characteristic function of t_iT_0, and t_i's are generators of T/T_0.

Hence λ_π gives rise to an unramified character $\sigma_\pi : T \longrightarrow T/T_0 \longrightarrow \mathbb{C}$. We call $\sigma_\pi(t_1),...,\sigma_\pi(t_n)$ Satake parameters. They determine π uniquely.

Theorem 4.18. $\sigma_\pi = \chi$.

Since $Hom(T/T_0,\mathbb{C}^\times) \simeq (\mathbb{C}^\times)^n \simeq {}^LT$, there exists an isomorphism $\chi \longmapsto t_\pi$ between the group X of unramified characters of $\mathbf{T}$ and ${}^LT$, such that $\chi(\alpha^\vee(\varpi)) = \alpha^\vee(t_\pi)$, where $\alpha^\vee$ on the right is regarded as a character of ${}^LT$, and ϖ is a uniformizer of F. The Satake parameters $\sigma_\pi(t_1),...,\sigma_\pi(t_n)$ map to t_π under the isomorphisms. We can show that there is a bijection between Weyl group orbits X/W and semi-simple conjugacy classes in ${}^LG$. So we call t_π a semi-simple conjugacy class in ${}^LT \subset {}^LG$.

Theorem 4.19. *Let χ be any unramified quasi-character of T. Then the Langlands' subquotient of $I(\chi)$ is spherical.*

Suppose I is the Iwahori compact subgroup. It is the inverse image of $\mathbf{B}(\mathbb{F}_q)$ of the canonical surjection $\mathbf{G}(\mathcal{O}) \longrightarrow \mathbf{G}(\mathcal{O}/\mathfrak{p})$. Here $\mathcal{O}/\mathfrak{p} = \mathbb{F}_q$.

Theorem 4.20 (Borel-Matsumoto, Casselman). *There is a category equivalence between irreducible admissible representations which have a non-zero fixed I-fixed vector and finite dimensional representations of $\mathcal{H}(G,I)$. Irreducible representations which have a non-zero fixed I-fixed vector are exactly subquotients of $I(\chi)$, where χ is unramified.*

4.5 Induced representations: F global. Suppose π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, and $\mathbf{P} = P_\theta = \mathbf{MN}$ is a maximal parabolic subgroup. Then we can define the global induced representation

$$I(s,\pi) = \otimes_v I(s,\pi_v).$$

It is the restricted tensor product, namely, let $V(s,\pi_v)$ be the representation space for $I(s,\pi_v)$ for all v. Then the representation space for $V(s,\pi)$ is defined as follows: Given $f \in V(s,\pi)$, there exist a finite set S of places, including archimedean places, such that $f \in \otimes_{v\in S} V(s,\pi_v) \otimes \otimes_{v\notin S} f_v^0$, where $f_v^0, v \notin S$, is the spherical vector with $f_v^0(k_v) = 1$ for $k_v \in \mathbf{G}(\mathcal{O}_v)$. Similarly, one can define the global induced representations for arbitrary parabolic subgroups.

For $f \in V(s,\pi)$, define

$$M(s,\pi)f(g) = \int_{\mathbf{N}'(\mathbb{A})} f(w_0^{-1}ng)\, dn,$$

where $\mathbf{N}'$ is the unipotent radical of $\mathbf{P}' = P_{w_0(\theta)}$. Then

$$M(s,\pi) = \otimes_v A(s,\pi_v,w_0).$$

It is called the global intertwining operator, and $M(s,\pi) : I(s,\pi) \longrightarrow I(-s,w_0(\pi))$.

Theorem 4.21 (Langlands). *σ is an automorphic representation of $\mathbf{G}(\mathbb{A})$ if and only if σ is a subquotient of $I(\nu, \pi)$ for some cuspidal representation π of $\mathbf{M}(\mathbb{A})$.*

Here $\sigma = \otimes_v \sigma_v$ and σ_v is a spherical representation for almost all v, and σ_v is a subquotient of $I(\nu, \pi_v)$ for all v. (We may take this as a definition of automorphic representations. For definitions of automorphic representations, see the lecture by J. Cogdell.)

4.6 Induced representations as holomorphic fiber bundles. In the trace formula and other applications, one needs to take derivatives of intertwining operators and scalar products. Our definition of $M(s, \pi)$ is that f has dependence on s. We want to separate the dependence on s from f. So we define $I(s, \pi)$ in a different way.

Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$. Recall that π occurs as a direct summand in the decomposition of $L_0^2(\omega) = L_0^2(\mathbf{M}(F)\backslash\mathbf{M}(\mathbb{A}), \omega)$, i.e., the representation space V of π is a subspace of $L_0^2(\omega)$. We have the Iwasawa decomposition

$$\mathbf{G}(\mathbb{A}) = \mathbf{N}(\mathbb{A})\mathbf{M}(\mathbb{A})K,$$

where K is the maximal compact subgroup, given as $K = \prod_v K_v$, where $K_v = \mathbf{G}(\mathcal{O}_v)$ for $v < \infty$. Then $\mathbf{N}(\mathbb{A})\mathbf{M}(F)\backslash\mathbf{G}(\mathbb{A}) = (\mathbf{M}(F)\backslash\mathbf{M}(\mathbb{A})) \cdot K$. Now we define $\mathcal{H}_P$ to be the set of functions

$$\phi : \mathbf{N}(\mathbb{A})\mathbf{M}(F)\backslash\mathbf{G}(\mathbb{A}) \longrightarrow \mathbb{C},$$

such that

(1) ϕ is right K-finite; this means that the space spanned by $\phi_k, k \in K$, is finite dimensional, where $\phi_k(g) = \phi(gk)$. This is equivalent to: Let $\psi = \phi|_K$. Then $\psi = \psi_1 \oplus \cdots \oplus \psi_r$, where ψ_i's are irreducible representations of K.
(2) for each $k \in K$, the function $m \longmapsto \phi(mk), m \in \mathbf{M}(\mathbb{A})$, belongs to V.

Then $I(s, \pi)$ is equivalent to $\{\phi exp(\langle s\tilde{\alpha} + \rho_P, H_P(\,)\rangle) |\, \phi \in \mathcal{H}\}$. Especially, $I(0, \pi)$ is equivalent to the space $\{\phi exp(\langle \rho_P, H_P(\,)\rangle) |\, \phi \in \mathcal{H}\}$. Then the intertwining operator $M(s, \pi)$ is an intertwining operator from $\mathcal{H}_P$ to $\mathcal{H}_{P'}$, given as

$$M(s,\pi)\phi(g)exp(\langle -s\tilde{\alpha}+\rho_{P'}, H_{P'}(g)\rangle) = \int_{\mathbf{N}'(\mathbb{A})} \phi(w_0^{-1}ng)exp(\langle s\tilde{\alpha}+\rho_P, H_P(w_0^{-1}ng)\rangle)\, dn.$$

Define a sesqui-linear form $(\,,\,)$ on $I(s,\pi) \times I(-\bar{s}, \pi)$ (or on $\mathcal{H} \times \mathcal{H}$) by

$$(\phi_1, \phi_2) = \int_{\mathbf{A}(\mathbb{R})_+\mathbf{N}(\mathbb{A})\mathbf{M}(F)\backslash\mathbf{G}(\mathbb{A})} \phi_1(g)\overline{\phi_2(g)}\, dg = \int_K \int_{\mathbf{M}(F)\backslash\mathbf{M}(\mathbb{A})^1} \phi_1(mk)\overline{\phi_2(mk)}\, dmdk.$$

Note that $\mathbf{M}(F)\backslash\mathbf{M}(\mathbb{A}) \simeq \mathbf{M}(F)\backslash\mathbf{M}(\mathbb{A})^1 \times \mathbb{A}(\mathbb{R})_+$. ("Sesqui" means "one and half-times". It is appropriate because it satisfies $(ax, y) = a(x, y)$, $(x, ay) = \bar{a}(x, y)$.)

Background from measure theory. We note the following two results from measure theory.

(1) Let G be a locally compact topological group and $H \subset G$ be a closed subgroup. Then there exist measures dg, dh, dx on $G, H, G/H$ such that

$$\int_G f(g)\, dg = \int_{G/H} (\int_H f(xh)\, dh)\, dx,$$

for $f \in C_c(G)$.

(2) Let G be a locally compact topological group, and A, B subgroups of G such that $A \cap B$ is compact, and $G = A \cdot B$. If G is unimodular, and da is a left invariant measure on A, and db is a right invariant measure on B, then we can choose an invariant measure dg on G such that

$$\int_G f(g)\, dg = \int_{A \times B} f(ab)\, da db,$$

for $f \in C_c(G)$.