8. Generic representations and their Whittaker models.

8.1 General case. Let $\mathbf{G}$ be a Chevalley group, and $\mathbf{B} = \mathbf{TU}$ be a Borel subgroup. Let Δ be the simple roots. Then

$$\mathbf{U}/[\mathbf{U},\mathbf{U}] \simeq \prod_{\alpha\in\Delta} U_\alpha.$$

First, let F be a local field. Let ψ be a character of $\mathbf{U}(F)$. Then $\psi = \prod_{\alpha\in\Delta}\psi_\alpha$, where ψ_α is a character of U_α.

Definition 8.1. *ψ is called generic or non-degenerate if each ψ_α is non-trivial.*

Note that ψ_α is a character of F. Any non-trivial character of F is of the form $x \longmapsto \eta(ax)$, $a \in F^*$, where η is a fixed non-trivial character of F. Hence a generic character ψ can be written as

$$\psi(u) = \eta(\prod_{\alpha\in\Delta} e_\alpha(a_i u_i)),$$

where $u \in U$, $a_i \in F^*$, and $u_i \in F$.

Example 8.2. Let $\mathbf{G} = GL_n$. Then

$$\prod_{\alpha\in\Delta} U_\alpha = (\begin{pmatrix} 1 & x_{12} & & & \\ & 1 & x_{23} & & \\ & & 1 & & \\ & & & \cdots & x_{n-1,n} \\ & & & & 1 \end{pmatrix}).$$

A generic character is of the form $\psi(u) = \eta(a_1x_{12} + \cdots + a_{n-1}x_{n-1,n})$, where $a_i \neq 0$ and η is any non-trivial character of F.

Suppose F is global.

Definition 8.3. *A character $\psi = \otimes_v\psi_v$ of $\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A})$ is called generic if each ψ_v is generic.*

From now on we fix a generic character ψ. Suppose $\pi = \otimes_v\pi_v$ is a cuspidal representation of $\mathbf{G}(\mathbb{A})$, and φ is a cuspidal function in the space of π. Let

$$W_\varphi(g) = \int_{\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A})} \varphi(ug)\overline{\psi(u)}\,du,$$

for $g \in \mathbf{G}(\mathbb{A})$. W_φ is called the Whittaker function attached to φ, and the space $\{W_\varphi\}$ is called the Whittaker model of π: W_φ satisfies the relation, $W_\varphi(ug) = \psi(u)W_\varphi(g)$.

Definition 8.4. *π is called globally ψ-generic if $W_\varphi \neq 0$ for some φ.*

Theorem 8.5 (Piatetski-Shapiro, Shalika). *Any cuspidal representation of GL_n is globally generic.*

Conjecture (generalization of Ramanujan conjecture). *Suppose $\pi = \otimes_v \pi_v$ is a globally generic cuspidal representation of $\mathbf{G}(\mathbb{A})$. Then each π_v is tempered.*

Let F be a local field, and ψ be a generic character of $U = \mathbf{U}(F)$.

Definition 8.6. *An irreducible representation (σ, V) is called ψ-generic if there exists a functional $\lambda : V \longrightarrow \mathbb{C}$ such that $\lambda(\sigma(u)\xi) = \psi(u)\lambda(\xi)$ for $u \in U$, $\xi \in V$. A non-zero such λ is called a Whittaker functional.*

If F is archimedean, we require λ to be continuous with respect to semi-norm topology defined by elements of universal enveloping algebra of the Lie algebra of $\mathbf{G}$, or more simply by right invariant differential operator on $\mathfrak{g}$. Note that by definition, a character $\chi : F^* \longrightarrow \mathbb{C}$ is not generic. More generally, if $\mathbf{G}$ is not quasi-split, any representation of $\mathbf{G}(F)$ is not generic.

Theorem 8.7 (Gelfand-Kazhdan, Rodier, Shalika). *The dimension of the space of Whittaker functionals for σ is at most 1.*

Given a Whittaker functional λ for (σ, V), we define a Whittaker function W_ξ for each $\xi \in V$ by

$$W_\xi(g) = \lambda(\sigma(g)\xi).$$

W_ξ satisfies the relation, $W_\xi(ug) = \psi(u)W_\xi(g)$ for $u \in U$.

Let $\mathcal{W} = \{W_\xi : \xi \in V\}$; we call it the Whittaker model for σ. Let $(\rho, \mathcal{W})$ be the right translation, namely, $\rho(g)W_\xi(h) = W_\xi(hg)$. Then $W_{\sigma(g)\xi} = \rho(g)W_\xi$, and we see that $(\rho, \mathcal{W}) \simeq (\sigma, V)$.

Conversely, if $\mathcal{W}$ is given, and we have a map $V \longrightarrow \mathcal{W}$; $\xi \longmapsto W_\xi$, we can define a Whittaker functional λ by $\lambda(\xi) = W_\xi(1)$. Thus specifying a Whittaker functional is equivalent to specifying a Whittaker model.

Theorem 8.8. *(1) Let $\pi = \otimes_v \pi_v$ be a globally ψ-generic cuspidal representation of $\mathbf{G}(\mathbb{A})$, where $\psi = \otimes_v \psi_v$ is a fixed generic character. Then each π_v is ψ_v-generic.*

(2) If $\varphi = \otimes_v \varphi_v$, where φ_v is a spherical vector for almost all v, then $W_\varphi = \otimes_v W_{\varphi_v}$, where $W_{\varphi_v}(k_v) = 1$ for almsot all v, $k_v \in \mathbf{G}(\mathcal{O}_v)$.

Note that the converse is not true. Namely, it can happen that each π_v is ψ_v-generic, and hence $\pi = \otimes_v \pi_v$ has a global Whittaker model, but π is not globally generic. There is an example of cuspidal representations of $\overline{SL_2(\mathbb{A})}$ due to Gelbart-Soudry. Kudla-Rallis-Soudry showed that in order that a cuspidal representation π of $Sp(4)$ be globally generic, one needs non-vanishing of the L-function $L(s, \pi)$ at $s = 1$.

Proof. Pick $\varphi^0 = \otimes_v \varphi_v^0$ such that $W_{\varphi^0}(e) = 1$. For each v, define $W_{\varphi_v}(g_v) = W_{i_v(\varphi_v)}(g_v)$ for $\varphi_v \in V_v$, where $i_v(\varphi_v) = \varphi_v \otimes \otimes_{w \neq v} \varphi_w^0$. Then W_{φ_v} is a Whittaker model for π_v, and $W_\varphi = \otimes_v W_{\varphi_v}$.

8.2 Whittaker models for induced representations. For the applications to L-functions, we need to consider Whittaker models for induced representations.

Suppose F is local. Let $\mathbf{P} = P_\theta = \mathbf{MN}$, $\theta \subset \Delta$. Let ψ_M be a generic character of $U_M = \mathbf{U}_M(F)$, where $\mathbf{U}_M = \mathbf{U} \cap \mathbf{M}$. Let ψ be a generic character of U such that $\psi|_{U_M} = \psi_M$.

Let (σ, V) be a ψ_M-generic representation of M, with λ_M being the ψ_M-Whittaker functional. Let $w_0 \in W$ be such that $w_0(\theta) \subset \Delta$, $w_0(\alpha) < 0$ for every α in $\mathbf{N}$. We choose w_0 so that ψ and w_0 are compatible. Let $\mathbf{P}' = \mathbf{M}'\mathbf{N}' = P_{w_0(\theta)}$. Let $I(\nu, \sigma)$ be the induced representation for $v \in \mathfrak{a}^*_{\mathbb{C}}$ with the representation space $V(\nu, \sigma)$. For every $f \in V(\nu, \sigma)$, define

$$\lambda_\psi(f) = \int_{N'} \overline{\psi(n')} \lambda_M(f(w_0^{-1} n'))\, dn'.$$

Proposition 8.9 (Casselman-Shalika, Shahidi). *λ is ψ-Whittaker functional for $I(\nu, \sigma)$, i.e., $I(\nu, \sigma)$ is ψ-generic. Moreover, as a function of ν, $\lambda_\psi(\nu, \sigma)$ is holomorphic for all ν.*

Proof. We need to prove

$$\lambda_\psi(\nu, \sigma)(I(\nu, \sigma)(u) f) = \psi(u) \lambda_\psi(\nu, \sigma)(f).$$

Since $(I(\nu, \sigma)(u)) f(h) = f(hu)$,

$$\lambda_\psi(\nu, \sigma)(I(\nu, \sigma)(u) f) = \int_{N'} \overline{\psi(n')} \lambda_M(f(w_0^{-1} n' u))\, dn'$$

Since $U = N' \cdot U_{M'}$, let $u = n'' \cdot u'$. By change of variables $n'n'' = n$, we see

$$\int_{N'} \overline{\psi(n')} \lambda_M(f(w_0^{-1} n' n'' u))\, dn' = \int_{N'} \psi(n'') \overline{\psi(n)} \lambda_M(f(w_0^{-1} n u))\, dn.$$

Note that $nu' = u' \cdot {u'}^{-1} n u'$. Again use the change of variables $n \longmapsto {u'}^{-1} n u'$. The right hand side is $\int_{N'} \psi(n'') \overline{\psi(n)} \lambda_M(f(w_0^{-1} u' n))\, dn$. Here $f(w_0^{-1} u' n) = f(w_0^{-1} n' w_0 \cdot w_0^{-1} n)$, and $w_0^{-1} u' w_0 \in U_M$. By the property of induced representation, $f(w_0^{-1} n' w_0 \cdot w_0^{-1} n) = \sigma(w_0^{-1} u' w_0) f(w_0^{-1} n)$. By the property of Whittaker functional, $\lambda_M(\sigma(w_0^{-1} u' w_0) f(w_0^{-1} n)) = \psi(w_0^{-1} u' w_0) f(w_0^{-1} n)$. Here by the compatibility of ψ and w_0, $\psi(w_0^{-1} u' w_0) = \psi(u')$. Hence $\lambda_M(\sigma(w_0^{-1} u' w_0) f(w_0^{-1} n)) = \psi(u') f(w_0^{-1} n)$. Therefore,

$$\lambda_\psi(\nu, \sigma)(I(\nu, \sigma)(u) f) = \overline{\psi(n'') \psi(u')} \int_{N'} \overline{\psi(n')} \lambda_M(f(w_0^{-1} n'))\, dn' = \psi(u) \lambda_\psi(\nu, \sigma) f.$$

□

The Whittaker function is

$$W_f(g) = \lambda_\psi(\nu, \sigma)(I(\nu, \sigma)(g) f) = \int_{N'} \overline{\psi(n')} \lambda_M(f(w_0^{-1} n' g))\, dn',$$

by noting that $(I(\nu,\sigma)(g))f(h)=f(hg)$. Especially, if $\mathbf{P}=\mathbf{B}$ is a Borel subgroup, then

$$W_f(g)=\int_U \overline{\psi(u)}f(w_l^{-1}ug)\,du,$$

where w_l is the longest Weyl group element in W.

Let us express $W_f(g)$ in terms of the Whittaker function of σ: Let $w_0^{-1}n'g = mnk$, where $m\in M, n\in N, k\in K$. Then $f(w_0^{-1}n'g)=\sigma(m)exp(\langle\nu{+}\rho_P, H_P(m)\rangle)f(k)$. For $\xi\in V$, let W_ξ be the Whittaker function of σ. Then $W_\xi(m)=\lambda_M(\sigma(m)\xi)$. Hence $\lambda_M(f(w_0^{-1}n'g)=\lambda_M(\sigma(m)exp(\langle\nu{+}\rho_P,H_P(m)\rangle)f(k))=W_{exp(\langle\nu+\rho_P,H_P(m)\rangle)f(k)}(m)$. By abuse of notation, we denote $W_{exp(\langle\nu+\rho_P,H_P(m)\rangle)f(k)}(m)$ by $W_\sigma(w_0^{-1}n'g)$. Therefore,

$$W_f(g)=\int_{N'}\overline{\psi(n')}W_\sigma(w_0^{-1}n'g))\,dn'. \tag{8.1}$$

Now we compute the Whittaker function in the case of SL_2. Let $G=SL_2(\mathbb{Q}_p)$ and χ is a character of $\mathbb{Q}_p^\times$. Let ψ be an unramified character of $\mathbb{Q}_p$, i.e., $\mathbb{Z}_p$ is the largest ideal of $\mathbb{Q}_p$ on which ψ is trivial. Let f_p^0 be the spherical function in $I(s,\chi)$ such that $f_p^0(e)=1$. It is the unique $SL_2(\mathbb{Z}_p)$-fixed function satisfying $f_p^0\left(\begin{pmatrix} a & x\\ 0 & a^{-1}\end{pmatrix}g\right)=\chi(a)|a|^{s+1}f_p^0(g)$. Let $N_p=\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\}$, and $w_0=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.

Theorem 8.10.

$$W_{f_p^0}(e)=\int_{N_p} f_p^0(w_0^{-1}n)\overline{\psi(n)}\,dn = L(s+1,\chi)^{-1}.$$

Proof. Note that for $n=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$,

$$w_0^{-1}n=\begin{pmatrix}0 & -1\\ 1 & x\end{pmatrix}=\begin{pmatrix}x^{-1} & 0\\ 0 & x\end{pmatrix}\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ x^{-1} & 1\end{pmatrix}.$$

If $x\in\mathbb{Z}_p$, $f_p^0(w_0^{-1}n)=1$. If $x=p^{-m}u$ with $m\geq 1$, then $f_p^0(w_0^{-1}n)=\chi(p^m)(p^{-m})^{s+1}$. Therefore,

$$W_{f_p^0}(e)=\int_{\mathbb{Z}_p} f_p^0(w_0^{-1}n)\overline{\psi(n)}\,dx+\sum_{m=1}^{\infty}\int_{p^{-m}\mathbb{Z}_p^\times} f_p^0(w_0^{-1}n)\overline{\psi(n)}\,dx.$$

Here

$$\int_{p^{-m}\mathbb{Z}_p^\times} f_p^0(w_0^{-1}n)\overline{\psi(n)}\,dx=\chi(p)^m(p^{-m})^{s+1}\int_{p^{-m}\mathbb{Z}_p^\times}\overline{\psi(n)}\,dx.$$

Since ψ is not trivial on $p^{-m}\mathbb{Z}_p$ for $m\geq 1$, and $p^{-m}\mathbb{Z}_p^\times=p^{-m}\mathbb{Z}_p-p^{-(m-1)}\mathbb{Z}_p$, we have

$$\int_{p^{-m}\mathbb{Z}_p^\times}\overline{\psi(n)}\,dx=\int_{p^{-m}\mathbb{Z}_p}\overline{\psi(n)}\,dx-\int_{p^{-(m-1)}\mathbb{Z}_p}\overline{\psi(n)}\,dx=\begin{cases}0, & \text{if } m>1\\ -1, & \text{if } m=1.\end{cases}$$

Therefore, $W_{f_p^0}(e) = 1 - \chi(p)p^{-s-1} = L(s+1,\chi)^{-1}$.

More generally over an arbitrary algebraic number field, we can show

$$W_{f_v^0}(e) = \int_{F_v} f_v^0(w_0^{-1}n)\overline{\psi(n)}\,dn = L(s+1,\chi_v)^{-1},$$

for $f_v^0 \in I(s,\chi_v)$.

Now let π_v be a spherical representation with the inducing character χ_v, and f_v^0 be a K_v-fixed vector in $I(s,\pi_v) \subset I(s\tilde{\alpha},\chi_v)$ such that $f_v^0(e) = 1$. Here it is important to remember that in (8.1), we need to choose the Whittaker function $W_{exp(\langle\nu+\rho_P,H_P(m)\rangle)f(k)}(m)$ so that $W_{f_v^0(e)}(e) = 1$ according to Theorem 8.8.[1] Hence $W_{\pi_v}(g) = c\int_{U_M} \overline{\psi(u)} f_v^0(w_{l,\theta}^{-1}ug)\,du$ for a constant c, where $w_{l,\theta}$ is the longest Weyl group element in W_θ such that $w_l = w_0 w_{l,\theta}$. Then (8.1) becomes

$$W_{f_v^0}(e) = c\int_U \overline{\psi(u)} f_v^0(w_l^{-1}ug)\,du.$$

We can verify it in the following way: Note that $U = N'\cdot U_{M'}$, and $w_{l,\theta} = w_{l,\theta}^{-1}$. Then

$$W_{f_v^0}(e) = c\int_{N'} \overline{\psi(n)} \int_{U_{M'}} \overline{\psi(u)} f_v^0(w_{l,\theta}^{-1}(w_0^{-1}nu))\,dudn.$$

Here $w_0^{-1}nu = (w_0^{-1}uw_0)w_0^{-1}u^{-1}nu$ and $w_0^{-1}uw_0 \in U_M$ for $u \in U_{M'}$. So by changing the variables $n \longmapsto unu^{-1}$,

$$W_{f_v^0}(e) = c\int_{N'} \overline{\psi(n)} W_{\pi_v}(w_0^{-1}n)\,dn.$$

Casselman-Shalika have shown that

$$\int_U \overline{\psi(u)} f_v^0(w_l^{-1}ug)\,du = \prod_{\beta>0} L(1+s\langle\tilde{\alpha},\beta\rangle,\chi_v\circ\beta^\vee)^{-1},$$

$$\int_{U_M} \overline{\psi(u)} f_v^0(w_{l,\theta}^{-1}ug)\,du = \prod_{\beta\in\Sigma_\theta^+} L(1+s\langle\tilde{\alpha},\beta\rangle,\chi_v\circ\beta^\vee)^{-1}.$$

Since $\langle\tilde{\alpha},\beta\rangle = 0$ for $\beta \in \Sigma_\theta^+$, the second integral is a constant. Now we can choose c such that $W_{\pi_v}(e) = 1$. Then

Theorem 8.11 (Casselman-Shalika).

$$W_{f_v^0}(e) = \prod_{\beta\in\Phi^+-\Sigma_\theta^+} L(1+s\langle\tilde{\alpha},\beta\rangle,\chi_v\circ\beta^\vee)^{-1} = \prod_{i=1}^{m} L(1+is,\pi_v,r_i)^{-1}.$$

[1] Thanks are due to Prof. F. Shahidi who pointed this out