9. Local coefficients and non-constant terms.

9.1 Non-constant terms of Eisenstein series. Let $\mathbf{P} = \mathbf{M}\mathbf{N}$ be a maximal parabolic subgroup and ψ_M be a generic character of $\mathbf{U}_M(F)\backslash \mathbf{U}_M(\mathbb{A})$, where $\mathbf{U}_M = \mathbf{U}\cap \mathbf{M}$. Let ψ be a generic character of $\mathbf{U}(F)\backslash \mathbf{U}(\mathbb{A})$ such that $\psi|_{\mathbf{U}_M(\mathbb{A})} = \psi_M$. Let π be globally ψ_M-generic. Recall the Eisenstein series

$$E(s,\pi,f_s,g) = \sum_{\gamma\in \mathbf{P}(F)\backslash \mathbf{G}(F)} f_s(\gamma g),$$

for $f_s \in I(s,\pi)$.

Definition 9.1.

$$E_\psi(s,\pi,f_s,g) = \int_{\mathbf{U}(F)\backslash \mathbf{U}(\mathbb{A})} E(s,\pi,f_s,ug)\overline{\psi(u)}\,du,$$

is called ψ-th Fourier coefficient (or non-constant term) of $E(s,\pi,f_s,g)$.

Proposition 9.2. *Suppose $f_s = \otimes_v f_{s,v}$, where $f_{s,v} = f_v^0$ for $v\notin S$, where S is a finite set of places, including archimedean places, and π_v, ψ_v are unramified for $v\notin S$. Then, for $g = \otimes g_v$,*

$$E_\psi(s,\pi,f_s,g) = \prod_v W_{f_{s,v}}(g_v), \quad E_\psi(s,\pi,f_s,e) = \prod_{v\in S} W_{f_{s,v}}(e)\prod_{i=1}^m L_S(1+is,\pi,r_i)^{-1}.$$

Proof. We use Bruhat decomposition in the proof of Theorem 5.3:

$$\mathbf{G}(F) = \bigcup_{w\in W_\theta\backslash W/W_{w_0(\theta)}} \mathbf{P}(F)w^{-1}\mathbf{P}'(F).$$

where $\mathbf{P}' = P_{w_0(\theta)} = \mathbf{M}'\mathbf{N}'$. Here if $w = w_0$, then $\mathbf{P}(F)w^{-1}\mathbf{P}'(F) = \mathbf{P}(F)w_0^{-1}\mathbf{N}'(F)$. Let $\mathbf{N}_1 = w\mathbf{N}w^{-1}\cap \mathbf{N}'$. So

$$E_\psi(s,\pi,f_s,g) = \sum_{w\in W(\theta,w_0(\theta))}\sum_{n'\in \mathbf{N}_1(F)\backslash \mathbf{N}'(F)}\int_{\mathbf{U}(F)\backslash \mathbf{U}(\mathbb{A})} f_s(w^{-1}n'ug))\overline{\psi(u)}\,du.$$

Note that $\mathbf{U} = \mathbf{N}'\cdot \mathbf{U}_{M'}$. So

$$\begin{aligned} &E_\psi(s,\pi,f_s,g)\\ &= \sum_{w\in W(\theta,w_0(\theta))}\sum_{n'\in \mathbf{N}_1(F)\backslash \mathbf{N}'(F)}\int_{\mathbf{U}_{M'}(F)\backslash \mathbf{U}_{M'}(\mathbb{A})}\int_{\mathbf{N}'(F)\backslash \mathbf{N}'(\mathbb{A})}\overline{\psi(n''u)}f_s(w^{-1}n'n''ug)\,dn''du\\ &\quad= \sum_{w\in W(\theta,w_0(\theta))}\int_{\mathbf{U}_{M'}(F)\backslash \mathbf{U}_{M'}(\mathbb{A})}\overline{\psi(u)}\int_{\mathbf{N}_1(F)\backslash \mathbf{N}'(\mathbb{A})}\overline{\psi(n)}f_s(w^{-1}nug)\,dndu.\end{aligned}$$

Here we use the fact that $\psi|_{U(F)} = 1$. If $w^{-1}\mathbf{N}'w \cap \mathbf{M} \neq 1$, we can see that $w\mathbf{M}w^{-1} \cap \mathbf{N}'$ is a product of unipotent groups attached to non-simple positive roots. (They are generated by positive roots $\alpha \in \Phi_+ - \Sigma^+_{w_0(\theta)}$ such that $\alpha = w(\beta)$ for $\beta \in \Sigma^+_\theta$. Since $w^{-1}(w_0(\theta)) > 0$, α cannot be simple. If α were simple, then $w^{-1}(\Delta) > 0$. contradiction.) Hence if $n \in (w\mathbf{M}w^{-1} \cap \mathbf{N}')(\mathbb{A})$, $\psi(n) = 1$. By the cuspidalidty of π, we see that

$$\int_{\mathbf{N}_1(F)\backslash\mathbf{N}'(\mathbb{A})} \overline{\psi(n)} f_s(w^{-1}nug)\, dn = 0.$$

If $w^{-1}\mathbf{N}'w \cap \mathbf{M} = 1$, then only $w = 1$ and w_0 can contribute. If $w = 1$ (it can occur only when $\mathbf{P}$ is self-conjugate), we need to calculate

$$\int_{\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A})} f_s(ug))\overline{\psi(u)}\, du.$$

Let $\mathbf{U} = \mathbf{N} \cdot \mathbf{U}_M$. Then the above integral is

$$\int_{\mathbf{U}_M(F)\backslash\mathbf{U}_M(\mathbb{A})} \int_{\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})} \overline{\psi(nu)} f_s(nug)\, dndu.$$

Here f_s is a function on $\mathbf{N}(\mathbb{A})\mathbf{M}(F)\backslash\mathbf{G}(\mathbb{A})$, and hence $f_s(nug) = f_s(ug)$. Therefore, the above integral is

$$\int_{\mathbf{U}_M(F)\backslash\mathbf{U}_M(\mathbb{A})} \overline{\psi(u)} f_s(ug) \int_{\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})} \overline{\psi(n)}\, dndu.$$

Since ψ is non-trivial on $\mathbf{N}(\mathbb{A})$, $\int_{\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})} \overline{\psi(n)}\, dn = 0$.

Hence only $w = w_0$ contributes. So

$$E_\psi(s, \pi, f_s, g) = \int_{\mathbf{N}'(\mathbb{A})} \overline{\psi(n)} \left(\int_{\mathbf{U}_{M'}(F)\backslash\mathbf{U}_{M'}(\mathbb{A})} f_s(w_0^{-1}nug)\overline{\psi(u)}\, du \right) dn.$$

Here $w_0^{-1}nu = w_0^{-1}uw_0w_0^{-1}u^{-1}nu$ and $w_0^{-1}uw_0 \in \mathbf{U}_M$ if $u \in \mathbf{U}'_M$. By the change of variables, $n \longmapsto unu^{-1}$ and noting that $\psi(w_0^{-1}uw_0) = \psi(u)$ by the compatibility of ψ and w_0,

$$E_\psi(s, \pi, f_s, g) = \int_{\mathbf{N}'(\mathbb{A})} \overline{\psi(n)} \left(\int_{\mathbf{U}_M(F)\backslash\mathbf{U}_M(\mathbb{A})} f_s(uw_0^{-1}ng)\overline{\psi(u)}\, du \right) dn.$$

Let $w_0^{-1}ng = n_1mk$, where $n_1 \in \mathbf{N}(\mathbb{A}), m \in \mathbf{M}(\mathbb{A}), k \in K$. Then $f_s(uw_0^{-1}ng) = f_s(umk)$. Here for $k \in K$, the function $f_{s,k} : m \longmapsto f_s(mk)$ belongs to the space $\pi \otimes exp(\langle s\tilde{\alpha} + \rho_P, H_P(\,)\rangle)$. Hence

$$\int_{\mathbf{U}_M(F)\backslash\mathbf{U}_M(\mathbb{A})} f_s(uw_0^{-1}ng)\overline{\psi(u)}\, du = W_{f_{s,k}}(m),$$

the Whittaker function of $f_{s,k}$. By abuse of notation, we denote $W_{f_{s,k}}(m)$ by $W_\pi(w_0^{-1}ng)$. Then

$$E_\psi(s,\pi,f_s,g)=\int_{\mathbf{N}'(\mathbb{A})}\overline{\psi(n)}W_\pi(w_0^{-1}ng)\,dn.$$

We write $W_\pi=\otimes_v W_{\pi_v}$, and

$$E_\psi(s,\pi,f_s,g)=\prod_v\int_{\mathbf{N}'(F_v)}\overline{\psi(n)}W_{\pi_v}(w_0^{-1}ng)\,dn.$$

Here $W_{f_{s,v}}(g)=\int_{\mathbf{N}'(F_v)}\overline{\psi(n)}W_{\pi_v}(w_0^{-1}ng)\,dn$. Hence

$$E_\psi(s,\pi,f_s,g)=\prod_v W_{f_{s,v}}(g_v).$$

By Theorem 8.11, for $v\notin S$, $W_{f_{s,v}}(e)=\prod_{i=1}^m L(1+is,\pi_v,r_i)^{-1}$. Hence

$$E_\psi(s,\pi,f_s,e)=\prod_{v\in S}W_{f_{s,v}}(e)\prod_{i=1}^m L_S(1+is,\pi,r_i)^{-1}.$$

Corollary 9.3. *$\prod_{i=1}^m L_S(1+is,\pi,r_i)$ has no zeros for $Re(s)=0$. Especially, if π_1,π_2 are cuspidal representations of GL_k, GL_l, resp, then $L_S(s,\pi_1\times\pi_2)$ has no zeros for $Re(s)=1$.*

Proof. Just note that $E(s,\pi,f_s,e)$ is holomorphic for $Re(s)=0$, and given s, we can find $f_{s,v}$ such that $W_{f_{s,v}}(e)\neq 0$ (See Lemma 9.4). For the second assertion, consider the case $GL_k\times GL_l\subset GL_{k+l}$.

More generally, by Corollary 5.5, if $\mathbf{P}$ is not self-conjugate or $w_0(\pi)\not\simeq\pi$, $M(s,\pi)$ is holomorphic for $Re(s)\geq 0$, and hence $E(s,\pi,f_s,e)$ is holomorphic for $Re(s)\geq 0$. First, we need

Lemma 9.4. *(1) If $v<\infty$, there exists $f_{s,v}$ such that $W_{f_{s,v}}(e)$ is a non-zero constant (independent of s).*

(2) If $v|\infty$, $W_{f_{s,v}}(e)$ is entire and of finite order for $Re(s)\geq 0$. Given s, we can choose $f_{s,v}$ such that $W_{f_{s,v}}(e)\neq 0$. But given $f_{s,v}$, $W_{f_{s,v}}(e)$ can have zeros for $Re(s)\geq 0$.

Therefore, we have

Corollary 9.5. *If $\mathbf{P}$ is not self-conjugate or $w_0(\pi)\not\simeq\pi$, $\prod_{i=1}^m L_S(1+is,\pi,r_i)$ has no zeros for $Re(s)\geq 0$.*

In the next two examples, let $F=\mathbb{Q}$. There exists a standard character ψ_0 on $\mathbb{Q}\backslash\mathbb{A}$, namely, $\psi_0=\otimes\psi_{0p}$, where $\psi_{0\infty}(x)=e^{-2\pi ix}$ for $x\in\mathbb{R}$, and $\psi_{0p}(x)=e^{2\pi ir}$, where $r\in\mathbb{Q}$ and $x-r\in\mathbb{Z}_p$. Such r is uniquely determined up to $\mathbb{Z}$. This ψ_0 has the property that the conductor of ψ_{0p} is $\mathbb{Z}_p$ for each p. Then all other characters of $\mathbb{Q}\backslash\mathbb{A}$ is of the form ψ_a for $a \in \mathbb{Q}$, where $\psi_a(x) = \psi_0(ax)$. Hence for $r \in \mathbb{Q}$,

$$\prod_{p<\infty} \int_{\mathbb{Z}_p} \psi_{0p}(rx_p)\, dx_p = \begin{cases} 1, & r \in \mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}$$

We know that the fundamental domain for $\mathbb{Q}\backslash\mathbb{A}$ is $\mathbb{R}/\mathbb{Z} + \hat{\mathbb{Z}}$, where $\hat{\mathbb{Z}} = \prod_p' \mathbb{Z}_p$. If f is a function on $\mathbb{R}/\mathbb{Z}$ with a Fourier series expansion $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi inx}$, then we can think of f as a function on $\mathbb{Q}\backslash\mathbb{A}$ which is right $\hat{\mathbb{Z}}$-invariant. Then, for $r \in \mathbb{Q}$,

$$\int_{\mathbb{Q}\backslash\mathbb{A}} f(x)\psi_0(-rx)\, dx = \int_{\mathbb{R}/\mathbb{Z}} f(x_\infty)e^{2\pi irx_\infty}\, dx_\infty \prod_{p<\infty} \int_{\mathbb{Z}_p} \psi_{0p}(-rx_p)\, dx_p.$$

Hence if $r \notin \mathbb{Q}$, it is zero. If $r \in \mathbb{Z}$, then $\int_{\mathbb{Q}\backslash\mathbb{A}} f(x)\psi_0(-rx)\, dx = a_r$.

Example 9.6. This example was done by J. Cogdell in his lecture. Consider the automorphic form ϕ_f attached to a holomorphic modular form $f(z) = \sum_{n=1}^\infty a_n e^{2\pi inz}$ with respect to $SL_2(\mathbb{Z})$. Then ϕ_f is right $K = \prod K_p$-invariant, where $K_p = GL_2(\mathbb{Z}_p)$ for all p. Since $u \longrightarrow \phi_f(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g)$ is periodic, we have a Fourier expansion

$$\phi_f(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g) = \sum_{r\in\mathbb{Q}} W_{\phi_f,r}(g)\psi_0(ru),$$

where

$$W_{\phi_f,r}(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} \phi_f(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g)\psi_0(-ru)\, du.$$

Then $\phi_f(g) = \sum_{r\in\mathbb{Q}} W_{\phi_f,r}(g)$. Let us compute $W_{\phi_f,r}(g)$ for $g = \begin{pmatrix} y^{\frac{1}{2}} & y^{-\frac{1}{2}}x \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}$. In this case, $g \cdot i = x + yi = z$, and $\phi_f(g) = f(z)$. Then $u \longrightarrow \phi_f(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g)$ is a function on $\mathbb{Q}\backslash\mathbb{A}$ which is invariant under $\hat{\mathbb{Z}}$. By the above computation, $W_{\phi_f,r}(g) = 0$ unless $r \in \mathbb{Z}$. If $r \in \mathbb{Z}$, then

$$W_{\phi_f,r}(g) = \int_{\mathbb{R}/\mathbb{Z}} f(x + u_\infty + yi)e^{2\pi iru_\infty}\, du_\infty = a_r e^{2\pi iz}.$$

So we retrieve the original Fourier expansion.

Example 9.7. Consider $E(s,1,g)$ in Example 5.1. Since $x \longrightarrow E(s,1,\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g)$ is periodic, we have a Fourier expansion

$$E(s,1,\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g) = \sum_{r\in\mathbb{Q}} c_r(g)\psi_0(ru),$$

where

$$c_r(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} E(s, 1, \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g)\psi_0(-ru)\, du.$$

By Bruhat decomposition, we can see that if $r = 0$, we obtain the constant term. Suppose $r \neq 0$. Then we get

$$c_r(g) = \int_{\mathbb{A}} f(w_0^{-1} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g)\psi_0(-ru)\, du,$$

where $f(g) = exp(\langle s\tilde{\alpha} + \rho_P, H_P(g)\rangle)$. This is a non-constant term. First recall how to compute $H_P(g)$ for $g \in SL_2(\mathbb{R})$, $exp(\langle \rho_P, H_P(g)\rangle) = ||(0,1)g||^{-1}$, where $||(a,b)|| = \sqrt{a^2+b^2}$. This can be seen by letting $g = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} k$, where $k \in K$. So

$$f(w_0^{-1} \begin{pmatrix} 1 & u_\infty \\ 0 & 1 \end{pmatrix} g) = (y + y^{-1}(x+u_\infty)^2)^{-\frac{s+1}{2}} = y^{\frac{s+1}{2}}((x+u_\infty)^2 + y^2)^{-\frac{s+1}{2}}.$$

Also

$$f(w_0^{-1} \begin{pmatrix} 1 & u_p \\ 0 & 1 \end{pmatrix}) = \begin{cases} (p^{-m})^{s+1}, & \text{if } x = p^{-m}u \text{ with } m \geq 1, \\ 1, & \text{if } x \in \mathbb{Z}_p \end{cases}$$

We can calculate in two ways: First, we write

$$c_r(g) = \sum_{\alpha\in\mathbb{Q}} \int_{\mathbb{Q}\backslash\mathbb{A}} f(w_0^{-1} \begin{pmatrix} 1 & u+\alpha \\ 0 & 1 \end{pmatrix} g)\psi_0(-ru)\, du.$$

Then since $u \longrightarrow f(w_0^{-1} \begin{pmatrix} 1 & u+\alpha \\ 0 & 1 \end{pmatrix} g)$ is invariant under $\hat{\mathbb{Z}}$, we have

$$\int_{\mathbb{Q}\backslash\mathbb{A}} f(w_0^{-1} \begin{pmatrix} 1 & u+\alpha \\ 0 & 1 \end{pmatrix} g)\psi_0(-ru)\, du = \int_{\mathbb{R}/\mathbb{Z}} y^{\frac{s+1}{2}}((x+u_\infty+\alpha)^2+y^2)^{-\frac{s+1}{2}} e^{2\pi i r u_\infty}\, du_\infty,$$

where $r \in \mathbb{Z}$. If we write $\alpha = \frac{n}{m}$, where $m > 0$, $gcd(m,n) = 1$,

$$c_r(g) = e^{2\pi i r x} y^{\frac{s+1}{2}} \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} m^{s+1} \int_0^1 ((mu_\infty + n)^2 + m^2y^2)^{-\frac{s+1}{2}} e^{-2\pi i r u_\infty}\, du_\infty.$$

This is exactly the same sum in Ram Murty's lecture.

Second, we write $c_r(g) = \prod_p W_{p,r}(g_p)$, where

$$W_{\infty,r}(g_\infty) = \int_{\mathbb{R}} y^{\frac{s+1}{2}}((x+u_\infty)^2 + y^2)^{-\frac{s+1}{2}} e^{2\pi i r u_\infty}\, du_\infty,$$

$$W_{p,r}(g_p) = \int_{\mathbb{Q}_p} f(w_0^{-1} \begin{pmatrix} 1 & u_p \\ 0 & 1 \end{pmatrix})\psi_{0p}(-ru_p)\, du_p.$$

We can see that $W_{\infty,r}(g_\infty) = e^{-2\pi i r u_\infty} \int_{\mathbb{R}} y^{\frac{s+1}{2}}((u_\infty)^2 + y^2)^{-\frac{s+1}{2}} e^{2\pi i r u_\infty}\, du_\infty$. We can show that $W_{p,r}(g_p) = \int_{\mathbb{Q}_p} f(w_0^{-1}\begin{pmatrix} 1 & u_p \\ 0 & 1 \end{pmatrix})\psi_{0p}(-ru_p) = 0$ if $r \notin \mathbb{Z}_p$ and, if $r = p^l u$, $l \in \mathbb{Z}_+ \cup \{0\}$ and $u \in \mathbb{Z}_p^\times$,

$$W_{p,r}(g_p) = \int_{\mathbb{Q}_p} f(w_0^{-1}\begin{pmatrix} 1 & u_p \\ 0 & 1 \end{pmatrix})\psi_{0p}(-ru_p) = \frac{(1-p^{-s-1})(1-(p^{-s})^{l+1})}{1-p^{-s}}$$
$$= (1-p^{-s-1})(1+p^{-s}+\cdots+(p^{-s})^l).$$

Hence $c_r(g) = 0$ unless $r \in \mathbb{Z}$. For $r \in \mathbb{Z}$, let $r = p_1^{k_1}\cdots p_j^{k_j}$.

$$\prod_{p<\infty} W_{p,r}(g_p) = \frac{1}{\zeta(s+1)}\prod_{i=1}^{j}(1+p_i^{-s}+\cdots+(p_i^{-s})^{k_i}) = \frac{1}{\zeta(s+1)}(\sum_{d|r} d^{-s}).$$

This coincides with the Fourier expansion in Ram Murty's lecture.

9.2 Local coefficients and crude functional equation. Let $I(\nu,\sigma)$ be the induced representation and $A(\nu,\sigma,w)$ be the intertwining operator from $I(\nu,\sigma)$ to $I(w\nu,w\sigma)$. Denote by $\lambda_\psi(w\nu,w\sigma)$ be the Whittaker functional for $I(w\nu,w\sigma)$. Then $\lambda_\psi(w\nu,w\sigma)A(\nu,\sigma,w)$ is another non-zero Whittaker functional for $I(\nu,\sigma)$. Since the Whittaker functionals are unique up to a scalar, there exists a scalar $C_\psi(\nu,\sigma,w)$, called local coefficient, such that

$$\lambda_\psi(\nu,\sigma) = C_\psi(\nu,\sigma,w)\lambda_\psi(w\nu,w\sigma)A(\nu,\sigma,w).$$

Special case of maximal parabolic subgroups: Suppose $I(s,\pi)$ be the induced representation, where π be an admissible representation of $\mathbf{M}(F)$, F local field, where $\mathbf{P} = \mathbf{MN}$ is a maximal parabolic subgroup. Then

$$\lambda_\psi(s,\pi) = C_\psi(s,\pi,w_0)\lambda_\psi(-s,w_0(\pi))A(s,\pi,w_0).$$

So $C_\psi(s,\pi,w_0)$ is a complex function. In terms of Whittaker functions, for $f \in I(s,\pi)$,

$$C_\psi(s,\pi,w_0) = \frac{W_f(e)}{W_{A(s,\pi,w_0)f}(e)}.$$

Theorem 9.8 (Crude functional equation).

$$\prod_{i=1}^{m} L_S(is,\pi,r_i) = \prod_{v\in S} C_{\psi_v}(s,\pi_v,w_0)\prod_{i=1}^{m} L_S(1-is,\pi,\tilde{r}_i).$$

Proof. Consider the functional equation of the Eisenstein series

$$E(s,\pi,f_s,e) = E(-s,w_0(\pi),M(s,\pi)f_s,e).$$

Then

$$E_\psi(s,\pi,f_s,e)=E_\psi(-s,w_0(\pi),M(s,\pi)f_s,e).$$

Note that $M(s,\pi)f_s=\otimes_v A(s,\pi_v,w_0)f_{s,v}$. So the right hand side is

$$\prod_{v\in S} W_{A(s,\pi_v,w_0)f_{s,v}}(e)\times\prod_{v\notin S} W_{A(s,\pi_v,w_0)f_v^0}(e).$$

Here $A(s,\pi_v,w_0)f_v^0=\prod_{i=1}^m\frac{L(is,\pi_v,r_i)}{L(1+is,\pi_v,r_i)}\tilde f_v^0$, where $\tilde f_v^0$ is the unique $\mathbf{G}(\mathcal{O}_v)$-fixed vector in $I(-s,w_0(\pi_v))$ such that $\tilde f_v^0(e)=1$. Hence

$$W_{\tilde f_v^0}(e)=\prod_{i=1}^m L(1-is,w_0(\pi_v),r_i)=\prod_{i=1}^m L(1-is,\pi_v,\tilde r_i).$$

Therefore,

$$\prod_{v\notin S} W_{A(s,\pi_v,w_0)f_v^0}(e)=\prod_{i=1}^m\frac{L_S(is,\pi,r_i)}{L_S(1+is,\pi,r_i)}\frac{1}{L_S(1-is,\pi,\tilde r_i)}.$$

So

$$\prod_{i=1}^m L_S(is,\pi,r_i)=\prod_{v\in S}\frac{W_{f_{s,v}}(e)}{W_{A(s,\pi_v,w_0)f_{s,v}}(e)}\prod_{i=1}^m L_S(1-is,\pi,\tilde r_i).$$

By the definition of local coefficients, $\frac{W_{f_{s,v}}(e)}{W_{A(s,\pi_v,w_0)f_{s,v}}(e)}=C_{\psi_v}(s,\pi_v,w_0)$. Hence our assertion follows. $\square$

By induction on i as in Theorem 7.4, we obtain the functional equation $L_S(s,\pi,r_i)=\prod_{v\in S}\gamma_i(s,\pi_v,r_i,\psi_v)L_S(1-s,\pi,\tilde r_i)$. In order to extract more precise functional equations, we need to define the local L and ϵ-factors at bad places. We will do it by studying the local coefficients $C_{\psi_v}(s,\pi_v,w_0)$. They are basically a product of γ-functions, namely, γ_i is the usual γ-factor. They should be canonical, in the sense that they are Artin L and ϵ-factors. So in the next section, we review the local Langlands correspondence and Artin L and ϵ-factors.

10. Local Langlands' correspodence.

Let $\mathbf{G}$ be a split reductive group defined over a local field F. Let ${}^LG$ be the dual group and W_F be the Weil group. It is defined as follows:

If $F=\mathbb{C}$, $W_F=\mathbb{C}^\times$.

If $F=\mathbb{R}$, $W_F=\mathbb{C}^\times\rtimes\{\tau\}$, where $\tau^2=-1$, $\tau z\tau^{-1}=\bar z$ for $z\in\mathbb{C}^\times$. Here $W_{\mathbb{R}}^{ab}=\mathbb{R}^\times$.

Suppose F is a p-adic field. Let $Gal(\bar F/F)$ be the absolute Galois group, and $k=\mathcal{O}/\mathfrak{p}$ be the residue field with $|k|=q$. Let $Gal(\bar k/k)$ the Galois group. Let $Fr: x\longmapsto x^q$ be the Frobenius automorphism in $Gal(\bar k/k)$. We can show that $<Fr>=\{Fr^n : n\in\mathbb{Z}\}\neq Gal(\bar k/k)$. Let $h: Gal(\bar F/F)\longrightarrow Gal(\bar k/k); \sigma\longmapsto\sigma|_{\bar k}$ be the canonical surjection. We define $W_F=h^{-1}<Fr>$. Denote $Ker(h)=I_F$, the inertia group. Then we have an exact sequence

$$1\longrightarrow I_F\longrightarrow W_F\longrightarrow\mathbb{Z}\longrightarrow 1.$$

Theorem 10.1 (reciprocity isomorphism of local class field theory). *There exists an isomophism $r_F : F^\times \longrightarrow W_F^{ab}$.*

Naive version of local Langlands' conjecture. *Let $\phi : W_F \times SL_2(\mathbb{C}) \longrightarrow {}^LG$ be an admissible homomorphism, namely, $\phi|_{W_F}$ is semi-simple, and $\phi|_{SL_2(\mathbb{C})}$ is a complex representation. Then*

(1) *ϕ "parametrizes" a finite set Π_ϕ, called L-packet, of isomorphism classes of irreducible representations of $G = \mathbf{G}(F)$.*
(2) *Every admissible representations of G belongs to Π_ϕ for a unique ϕ.*
(3) *The representations in the L-packet Π_ϕ are parametrized by the component group $C_\phi = S_\phi / Z_{{}^LG} S_\phi^0$, where S_ϕ is the centralizer of $Im(\phi)$ in ${}^LG$, and S_ϕ^0 is the connected component of the identity in S_ϕ, and $Z_{{}^LG}$ is the center of ${}^LG$.*

Properties of L-packets.

(1) The elements of Π_ϕ are square integrable if and only if $Im(\phi)$ is not contained in any proper parabolic subgroups of ${}^LG$.
(2) The elements of Π_ϕ are tempered if and only if $\phi(W_F)$ is bounded.
(3) Given ϕ, and $r : {}^LG \longrightarrow GL_N(\mathbb{C})$, there is a definition of Artin L and ϵ-factors $L(s, r \circ \phi), \epsilon(s, r \circ \phi, \psi)$. Artin γ-factor is defined by

$$\gamma(s, r \circ \phi, \psi) = \epsilon(s, r \circ \phi, \psi) \frac{L(1-s, \tilde{r} \circ \phi)}{L(s, r \circ \phi)}.$$

If $F = \mathbb{R}$ or $\mathbb{C}$, Langlands proved the local Langlands correspondence (We only need W_F, not $W_F \times SL_2(\mathbb{C})$). If F is p-adic, the local Langlands correspondence is available only for GL_n and a few other low rank groups. (The most difficult problem is constructing and parametrizing supercuspidal representations.)

Special case of $\mathbf{G} = GL_n$. In this case, for any subset S in ${}^LG$, the centralizer $Z_{{}^LG}(S)$ is connected. Hence $C_\phi = 1$ for any ϕ. Therefore, Π_ϕ is a singleton for any ϕ. Let $\mathcal{G}_F(n)$ be the set of equivalent classes of admissible representations of $W_F \times SL_2(\mathbb{C})$; two representations $\phi_1, \phi_2 : W_F \times SL_2(\mathbb{C}) \longrightarrow GL_n(\mathbb{C})$ are said to be equivalent if they are conjugate by an element of $GL_n(\mathbb{C})$.

Let $\mathcal{A}_F(n)$ be the set of equivalent classes of admissible representations of $GL_n(F)$. We identify $\mathcal{G}_F(1)$, the characters of W_F, with $\mathcal{A}_F(1)$, the characters of $F^\times$, via the reciprocity isomorphism $r_F : F^\times \longrightarrow W_F^{ab}$.

Theorem 10.2 (Harris-Taylor, Henniart). *For each $n \geq 1$, there exists a canonical bijection*

$$\pi_F : \mathcal{G}_F(n) \longrightarrow \mathcal{A}_F(n), \quad \rho \longmapsto \pi_F(\rho),$$

such that

(1) *$\pi_F(\rho(\chi)) = \pi_F(\rho)(\chi)$ for any character χ of $F^\times$*
(2) *$det\rho$ corresponds to $\omega_{\pi_F(\rho)}$, the central character of $\pi_F(\rho)$*

(3) $\widetilde{\pi_F(\rho)} = \pi_F(\tilde{\rho})$

(4) $L(s, \rho_1 \otimes \rho_2) = L(s, \pi_F(\rho_1) \times \pi_F(\rho_2))$, $\epsilon(s, \rho_1 \otimes \rho_2, \psi) = \epsilon(s, \pi_F(\rho_1) \times \pi_F(\rho_2), \psi)$. *Here the left hand sides are Artin L and ϵ-factors defined by Deligne. The right hand sides are Rankin-Selberg L and ϵ-factors defined by Jacquet, Piatetski-Shapiro and Shalika.*

Fact. *$\pi_F(\rho)$ is supercuspidal if and only if ρ is an irreducible representation of W_F.*

Lemma 10.3. *Irreducible representations of W_F are of the form $\tau \otimes | \, |^u$, where τ is a representation of $Gal(\bar{F}/F)$, and $u \in \mathbb{C}$.*

Example 10.4 (Supercuspidal representations of $GL_2(F)$). By the local Langlands' correspondence, supercuspidal representations of $GL_2(F)$ correspond to irreducible 2-dimensional representations of W_F. By the above lemma, it is enough to consider irreducible representations $\tau : Gal(\bar{F}/F) \longrightarrow GL_2(\mathbb{C})$. It is well-known that the image of $\bar{\tau}$, where $\bar{\tau} : Gal(\bar{F}/F) \longrightarrow PGL_2(\mathbb{C})$, is one of the following: D_{2n} (dihedral group), A_4 (tetrahedral), S_4 (octahedral), and A_5 (icosahedral). However, since any Galois group $Gal(E/F)$ is solvable, A_5 cannot occur. If the image is D_{2n}, it corresponds to $Ind_F^E \chi$, where E/F is a quadratic extension and χ is a character of $E^\times$. We call it monomial cuspidal representation, or tame case. The other cases are called extraordinary supercuspidal representations, and they occur only when $v|2$.

Recall Jacobson-Morozov theorem: any equivalence class of homomorphism $SL_2(\mathbb{C}) \longrightarrow {}^L G$ corresponds to a unipotent orbit in ${}^L G$.

By the theory of Jordan normal forms, unipotent classes in $GL_n(\mathbb{C})$ are in one to one correspondence with partitions λ of n. Let $\phi_1 : W_F \longrightarrow GL_m(\mathbb{C})$ be the parametrization of a supercuspidal representation σ of $GL_m(F)$, and $\phi_2 : SL_2(\mathbb{C}) \longrightarrow GL_p(\mathbb{C})$ be the homomorphism given by the unipotent orbit (p). Then

$$\phi_1 \otimes \phi_2 : W_F \times SL_2(\mathbb{C}) \longrightarrow GL_{mp}(\mathbb{C}),$$

parametrizes the Steinberg representation, denoted by $St(\sigma, p)$, which is the unique subrepresentation of

$$Ind \, |det|^{\frac{p-1}{2}} \sigma \otimes |det|^{\frac{p-1}{2}-1} \sigma \otimes \cdots \otimes |det|^{-\frac{p-1}{2}} \sigma.$$

The unique quotient of the above induced presentation is parametrized by

$$|det|^{\frac{p-1}{2}} \phi_1 \oplus \cdots \oplus |det|^{-\frac{p-1}{2}} \phi_1 : W_F \times SL_2(\mathbb{C}) \longrightarrow GL_m(\mathbb{C}) \times \cdots \times GL_m(\mathbb{C}) \subset GL_{mp}(\mathbb{C}).$$

Special case of unramified principal series (spherical representations). Let π be a spherical representation such that $\pi \hookrightarrow I(\chi)$, where χ is an unramified character. Then we have associated a semi-simple conjugacy class $\{\hat{t}\} \subset {}^L G$ such that $\chi \circ \beta^\vee(\varpi) = \beta^\vee(\hat{t})$. Then the Langlands' parameter for π is

$$\phi : F^\times \longrightarrow {}^L T \subset {}^L G, \quad \varpi \longmapsto \hat{t}.$$

For example, let $\pi = \pi(\mu_1, ..., \mu_n)$ be a spherical representation of $GL_n(F)$, where μ_i's are unramified characters. Then the semi-simple conjugacy class is $diag(\mu_1(\varpi), ..., \mu_n(\varpi))$, and

$$\phi : F^\times \longrightarrow {}^L T \subset GL_n(\mathbb{C}), \quad \varpi \longmapsto diag(\mu_1(\varpi), ..., \mu_n(\varpi)).$$