14. Langlands' functoriality.

Langlands' functoriality can be roughly formulated as: If $\mathbf{H}, \mathbf{G}$ are two quasi-split reductive groups over a number field F, then to each homomorphism of L-groups, $\phi : {}^LH \longrightarrow {}^LG$, there is associated a lift (transfer) of automorphic representations of $\mathbf{H}(\mathbb{A})$ to automorphic representations of $\mathbf{G}(\mathbb{A})$.

More precisely, let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $\mathbf{H}(\mathbb{A})$. Let $\{t_v\}$ be the semi-simple conjugacy class of π_v for $v \notin S$. Then there exists an automorphic representation $\Pi = \otimes_v \Pi_v$ of $\mathbf{G}(\mathbb{A})$ such that $\{\phi(t_v)\}$ is the semi-simple conjugacy class of Π_v for $v \notin S$.

Example 14.1. Let $\mathbf{H} = \{e\}$ and $\mathbf{G} = GL_2$. Take ${}^LH = Gal(\bar{F}/F)$ and ${}^LG = GL_2(\mathbb{C})$. Then Langlands' functoriality becomes the strong Artin conjecture, namely, given $\phi : Gal(\bar{F}/F) \longrightarrow GL_2(\mathbb{C})$, there exists an automorphic representation π of $GL_2(\mathbb{A})$ such that $\phi(Frob_v)$ is the semi-simple conjugacy class of π_v for $v \notin S$. Strong Artin conjecture remains open for icosahedral Galois representations.

Example 14.2. Let $\mathbf{H} = Sp(2n), SO(2n+1), SO(2n)$, and $\mathbf{G} = GL_N$, where $N = 2n+1$ or $2n$, and $\phi : {}^LH \longrightarrow {}^LG$ is the embedding. This case is proved by Cogdell, Kim, Piatetski-Shapiro, and Shahidi.

Reformulation of Langlands' functoriality using Langlands' hypothetical group L_F. The global Langlands' conjecture is that automorphic representations of $\mathbf{G}(\mathbb{A})$ are parametrized by admissible homomorphisms $\psi : L_F \longrightarrow {}^LG$;

(1) There exists an L-packet Π_ψ which consists of automorphic representations of $\mathbf{G}(\mathbb{A})$ attached to ψ
(2) Every automorphic representation belongs to Π_ψ for a unique ψ.
(3) Π_ψ's are disjoint.

Using the global Langlands' conjecture, Langlands' functoriality can be formulated as: Given an automorphic representation π of $\mathbf{H}(\mathbb{A})$, attach $\psi : L_F \longrightarrow {}^LH$. It gives rise to $\phi \circ \psi : L_F \longrightarrow {}^LG$. There is a L-packet attached to $\phi \circ \psi$.

Reformulation of Langlands' functoriality using local Langlands correspondence.

Let $\pi = \otimes_v \pi_v$ be an automorphic representation of $\mathbf{H}(\mathbb{A})$. By the local Langlands' correspondence, to each π_v, there is associated an admissible homomorphism $\psi_v : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow {}^LH$. It gives rise to $\phi \circ \psi_v : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow {}^LG$. It gives rise to an irreducible admissible representation Π_v of $\mathbf{G}(F_v)$. Then $\Pi = \otimes_v \Pi_v$ is an irreducible admissible representation of $\mathbf{G}(\mathbb{A})$.

Langlands' functoriality conjecture. Π *is automorphic.*

Example 14.3. Let $\phi : GL_{n_1}(\mathbb{C}) \times GL_{n_2}(\mathbb{C}) \longrightarrow GL_{n_1 n_2}(\mathbb{C})$ be the map given by the tensor product. Let π_1, π_2 be cuspidal representations of $GL_{n_i}(\mathbb{A})$, resp. Let $\psi_{iv} : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_{n_i}(\mathbb{C})$ be the parametrization of π_{iv} for $i = 1, 2$. Then we obtain a map $\phi \circ (\psi_{1v} \otimes \psi_{2v}) : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_{n_1 n_2}(\mathbb{C})$. Let $\pi_{1v} \boxtimes \pi_{2v}$ be the irreducible admissible representation of $GL_{n_1 n_2}(F_v)$ attached to $\phi \circ (\psi_{1v} \otimes \psi_{2v})$ by the local Langlands correspondence [H-T, He2, La4]. Let $\pi_1 \boxtimes \pi_2 = \otimes_v (\pi_{1v} \boxtimes \pi_{2v})$.

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Langlands' functoriality conjecture is that $\pi_1 \boxtimes \pi_2$ is an automorphic representation of $GL_{n_1 n_2}(\mathbb{A})$.

Example 14.4. Let $Sym^m : GL_2(\mathbb{C}) \longrightarrow GL_{m+1}(\mathbb{C})$ be the m^{th} symmetric power representation of $GL_2(\mathbb{C})$ on the space of symmetric tensors of rank m. Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $GL_2(\mathbb{A})$. By the local Langlands' correspondence [H-T, He2, La4], $Sym^m(\pi_v)$ is a well-defined representation of $GL_{m+1}(F_v)$ for all v; Let $\psi_v : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_2(\mathbb{C})$ be the parametrization of π_v. Then we have a map $Sym^m(\psi_v) : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_{m+1}(\mathbb{C})$. Then $Sym^m(\pi_v)$ is the representation of $GL_{m+1}(F_v)$, corresponding to $Sym^m(\psi_v)$. Let $Sym^m(\pi) = \otimes_v Sym^m(\pi_v)$. It is an irreducible admissible representation of $GL_{m+1}(\mathbb{A})$. Langlands' functoriality is that $Sym^m(\pi) = \otimes_v Sym^m(\pi_v)$ is an automorphic representation of $GL_{m+1}(\mathbb{A})$.

Example 14.5. Let $\wedge^2 : GL_n(\mathbb{C}) \longrightarrow GL_N(\mathbb{C})$, where $N = \frac{n(n-1)}{2}$, be the map given by exterior square. Let $\pi = \otimes_v \pi_v$ be a cuspidal (automorphic) representation of $GL_n(\mathbb{A})$. By the local Langlands correspondence, $\wedge^2 \pi_v$ is well-defined as an irreducible admissible representation of $GL_N(F_v)$ for all v (the work of Harris-Taylor and Henniart [H-T, He2] on p-adic places, and of Langlands [La4] on archimedean places). Let $\wedge^2 \pi = \otimes_v \wedge^2 \pi_v$. It is an irreducible admissible representation of $GL_N(\mathbb{A})$. Then Langlands' functoriality is that $\wedge^2 \pi$ is automorphic.

There are three methods in establishing Langlands' functoriality:

(1) Trace formula (Selberg-Arthur); this method was successfully used in establishing base change and automorphic induction (Arthur-Clozel) and certain unitary groups (Flicker, Rogawski).
(2) Theta correspondence (Howe, Li, Kudla, Rallis, Soudry, Ginzburg, Jiang,...); this method does not give Langlands functoriality in the above sense. Rather it gives a correspondence between cuspidal representations of different groups such as between Sp_{2n} and O_m.
(3) Converse theorem of Cogdell and Piatetski-Shapiro

The converse theorem is especially well suited when $\mathbf{G} = GL_N$.

15. Converse theorem of Cogdell and Piatetski-Shapiro.

Theorem 15.1. *Suppose $\Pi = \otimes_v \Pi_v$ is an irreducible admissible representation of $GL_n(\mathbb{A})$ such that $\omega_\Pi = \otimes_v \omega_{\Pi_v}$ is a grössencharacter of F. Let S be a finite set of finite places and let $\mathcal{T}^S(m)$ be a set of cuspidal representations of $GL_m(\mathbb{A})$ that are unramified at all places $v \in S$. Suppose $L(s, \sigma \times \Pi)$ is nice (i.e., entire, bounded in vertical strips and satisfies a functional equation) for all cuspidal representations $\sigma \in \mathcal{T}^S(m)$, $m < n-1$. Then there exists an automorphic representation Π' of $GL_n(\mathbb{A})$ such that $\Pi_v \simeq \Pi'_v$ for all $v \notin S$.*

Remark. In all our applications of the converse theorem, we need only the weak version of twisting up to $n-1$. We used the stronger version only for the convenience.

Suppose $\mathbf{H}$ be a split reductive group and $\phi : {}^L H \longrightarrow GL_N(\mathbb{C})$. Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $\mathbf{H}(\mathbb{A})$, and S be a finite set of finite places such that

π_v is unramified for $v \notin S$, $v < \infty$. We assume the local Langlands' correspondence for $\mathbf{H}$, and we obtain the irreducible admissible representation $\Pi = \otimes_v \Pi_v$, where Π_v is attached to $\phi \circ \psi_v$, and $\psi_v : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow {}^L H$.

We apply the converse theorem to Π and S. We need to do the following:

(1) For all v,

$$\gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \phi, \psi_v) = \gamma(s, \sigma_v \times \Pi_v, \psi_v),$$
$$L(s, \sigma_v \otimes \pi_v, \rho_m \otimes \phi) = L(s, \sigma_v \times \Pi_v),$$

for all σ_v, where $\sigma = \otimes_v \sigma_v \in \mathcal{T}^S(m)$, $m = 1, ..., N-2$. Here ρ_m is the standard representation of $GL_m(\mathbb{C})$.

(2) Prove the analytic continuation and functional equation of the L-functions $L(s, \sigma \otimes \pi, \rho_m \otimes \phi)$.

(3) Prove that $L(s, \sigma \otimes \pi, \rho_m \otimes \phi)$ is entire for $\sigma \in \mathcal{T}^S(m)$, $m = 1, ..., N-2$

(4) Prove that $L(s, \sigma \otimes \pi, \rho_m \otimes \phi)$ is bounded in vertical strips for $\sigma \in \mathcal{T}^S(m)$, $m = 1, ..., N-2$

We have to first see whether the L-functions $L(s, \sigma \otimes \pi, \rho_m \otimes \phi)$ are available from Langlands-Shahidi method. If they are, then (2), (3), and (4) are consequences of very general results; (2) is a result of Shahidi. (4) is a result of Gelbart-Shahidi. (3) is not true in general. We need to replace $\mathcal{T}^S(m)$ by $\mathcal{T}^S(m) \otimes \chi$, where χ is a grössencharacter such that χ_v is highly ramified for $v \in S$. (1) is not obvious: The left hand side is defined as normalizing factors of intertwining operators. The right hand side is defined by integral representations.

In the case of $Sp(2n), SO(2n), SO(2n+1)$, Π_v is not even defined due to the lack of the local Langlands' correspondence. We use the stability of γ-factors.

Proposition 15.2. $L(s, (\sigma_v \otimes \chi_v) \times \Pi_v) = 1$ *for every highly ramified character* χ_v; $L(s, (\sigma_v \otimes \chi_v) \otimes \pi_v, \rho_m \otimes \phi) = 1$ *if the* L*-functions show up in the Langlands-Shahidi method.*

Proposition 15.3 (Jacquet-Shalika [J-S2]). *Suppose* Π_{1v}, Π_{2v} *be representations of* $GL_N(F_v)$ *such that* $\omega_{\Pi_{1v}} = \omega_{\Pi_{2v}}$. *Then for every highly ramified character* χ_v,

$$\gamma(s, (\sigma_v \otimes \chi_v) \times \Pi_{1v}, \psi_v) = \gamma(s, (\sigma_v \otimes \chi_v) \times \Pi_{2v}, \psi_v).$$

Conjecture 15.4 (stability of γ-factors). *Let* π_{1v}, π_{2v} *be representations of* $\mathbf{H}(F_v)$ *such that* $\omega_{\pi_{1v}} = \omega_{\pi_{2v}}$. *Then for every highly ramified character* χ_v,

$$\gamma(s, (\sigma_v \otimes \chi_v) \otimes \pi_{1v}, \rho_m \otimes \phi, \psi_v) = \gamma(s, (\sigma_v \otimes \chi_v) \otimes \pi_{2v}, \rho_m \otimes \phi, \psi_v).$$

Assume the stability of γ-factors, and we explain how (1) can be obtained. If $v \in S$, take Π_v to be arbitrary so that $\omega_{\Pi_v} = \omega_{\pi_v}$. Now we use the set $\mathcal{T}^S(m) \otimes \chi$, where χ is a grössencharacter of F such that χ_v is highly ramified at $v \in S$. Then

$$L(s, (\sigma_v \otimes \chi_v) \otimes \pi_v, \rho_m \otimes \phi) = L(s, (\sigma_v \otimes \chi_v) \times \Pi_v) = 1.$$

Take π'_v to be a spherical representation such that $\pi'_v \hookrightarrow Ind_B^H \mu_1 \otimes \cdots \otimes \mu_r$. For a spherical representation π'_v, we have a lift Π'_v. Then by multiplicativity of γ-factors,

$$\gamma(s, (\sigma_v \otimes \chi_v) \otimes \pi'_v, \rho_m \otimes \phi, \psi_v) = \gamma(s, (\sigma_v \otimes \chi_v) \times \Pi'_v, \psi_v),$$

is a product of γ-factors for $\gamma(s, \sigma_v \otimes \eta, \psi_v)$ for various η. Hence by Proposition 14.3 and Conjecture 14.4,

$$\gamma(s, (\sigma_v \otimes \chi_v) \otimes \pi_v, \rho_m \otimes \phi, \psi_v) = \gamma(s, (\sigma_v \otimes \chi_v) \otimes \pi'_v, \rho_m \otimes \phi, \psi_v) = \gamma(s, (\sigma_v \otimes \chi_v) \times \Pi'_v, \psi_v) = \gamma(s, (\sigma_v \otimes \chi_v) \otimes \Pi_v, \psi_v).$$

By applying the converse theorem, we obtain an automorphic representation $\Pi'' = \otimes_v \Pi''_v$ such that $\Pi_v \simeq \Pi''_v$ for all $v \notin S$. We make the following definition.

Definition 15.5. *Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $\mathbf{H}(\mathbb{A})$. We say that an automorphic representation Π' of $GL_N(\mathbb{A})$ is a strong exterior square lift of π if for every v, Π'_v is a local lift of π_v, in the sense that*

$$\gamma(s, \sigma_v \otimes \pi_v, \rho_m \otimes \phi, \psi_v) = \gamma(s, \sigma_v \times \Pi'_v, \psi_v),$$
$$L(s, \sigma_v \otimes \pi_v, \rho_m \otimes \phi) = L(s, \sigma_v \times \Pi'_v),$$

for all generic irreducible representation σ_v of $GL_m(F_v)$, $1 \le m \le N-2$.

If the above equality holds for almost all v, then Π' is called weak lift of π.

Often proving that $\Pi'_v \simeq \Pi_v$ is very difficult, especially at primes dividing 2.