10.4. **Proof of Theorem 10.1 (ii).** In part (ii) of Theorem 10.1 we can no longer directly apply the inversion of the integral representation since we can no longer control $I(s, U_\xi, \varphi')$ for $\varphi' \in V_{\pi'}$ for every proper automorphic representation π', rather only for those which are unramified for $v \in S$. Our first idea to get around this is to place local conditions on our vector ξ at $v \in S$ to ensure that this is all you need. For $v \in S$, let $\xi_v^\circ \in V_{\pi_v}$ be the "new vector", that is, the essentially unique vector fixed by $K_1(\mathfrak{p}^{f(\pi_v)})$ where $f(\pi_v)$ is the conductor of π_v as in Lecture 6. Note that for any t we have

$$K_1(\mathfrak{p}_v^t) = \left\{ k_v \in GL_n(\mathcal{O}_v) \middle| k_v \equiv \begin{pmatrix} * & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \pmod{\mathfrak{p}_v^t} \right\}$$
$$\supset \left\{ \begin{pmatrix} k_v' & \\ & 1 \end{pmatrix} \mid k_v' \in GL_{n-1}(\mathcal{O}_v) \right\}.$$

Set $\xi_S^\circ = \otimes_{v \in S} \xi_v^\circ \in V_{\pi_S}$. This is then fixed by

$$K_1(\mathfrak{n}) = \prod_{v \in S} K_1(\mathfrak{p}_v^{(f(\pi_v))}) \supset GL_{n-1}(\mathcal{O}_S).$$

For any $\xi^S \in V_{\pi^S} \simeq \otimes'_{v \notin S} V_{\pi_v}$ we can form $\xi = \xi_S^\circ \otimes \xi^S$ and for such restricted $\xi \in V_\pi$ we form U_ξ and V_ξ as before. Note that when we restrict these functions to $GL_{n-1}(\mathbb{A})$ we see that $U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix}$ and $V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix}$ are now unramified for $v \in S$. So when we form $I(s, U_\xi, \varphi')$ and $I(s, V_\xi, \varphi')$ for $\varphi' \in V_{\pi'}$ a proper automorphic representation of $GL_{n-1}(\mathbb{A})$ we find that either

- $I(s, U_\xi, \varphi') = 0 = I(s, V_\xi, \varphi')$ if π' is not unramified for $v \in S$, or
- $I(s, U_\xi, \varphi') = I(s, V_\xi, \varphi')$ as before if π' is unramified for $v \in S$.

Thus, arguing as before, we may now conclude that we have

$$U_\xi(g) = V_\xi(g) \quad \text{for all} \quad g \in K_1(\mathfrak{n}) G^S$$

where $G^S = \prod_{v \notin S} GL_n(k_v)$.

We now use the weak approximation theorem to get back to $GL_n(\mathbb{A})$. Note that

- $U_\xi(g)$ is left invariant under $P(\mathfrak{n}) = P(k) \cap K_1(\mathfrak{n}) G^S$

- $V_\xi(g)$ is left invariant under $Q(\mathfrak{n}) = Q(k) \cap K_1(\mathfrak{n})G^S$
- $P(\mathfrak{n})$ and $Q(\mathfrak{n})$ generate $\Gamma(\mathfrak{n}) = GL_n(k) \cap K_1(\mathfrak{n})G^S$.

Thus as we let ξ^S vary in V_{π^S} we obtain that

$$\xi^S \mapsto \xi = \xi_S^\circ \otimes \xi^S \mapsto U_\xi(g) \quad \text{embeds} \quad V_{\pi^S} \hookrightarrow \mathcal{A}^\infty(\Gamma(\mathfrak{n})\backslash K_1(\mathfrak{n})G^S).$$

Now weak approximation gives that $GL_n(\mathbb{A}) = GL_n(k)K_1(\mathfrak{n})G^S$ so that

$$\mathcal{A}^\infty(\Gamma(\mathfrak{n})\backslash K_1(\mathfrak{n})G^S) = \mathcal{A}^\infty(GL_n(k)\backslash GL_n(\mathbb{A})).$$

Then π^S determines a sub-representation of the space of automorphic forms on $GL_n(\mathbb{A})$ and for our π_1 we may take any irreducible constituent of this. Fortunately we still retain that $\pi_{1,v} \simeq \pi_v$ for $v \notin S$. This is the π_1 claimed in the Theorem.

10.5. **Theorem 2 and beyond.** What can we expect if we only assume that $L(s, \pi \times \pi')$ is nice for all π' in say $\mathcal{T}(m)$ or $\mathcal{T}^S(m)$?

If $L(s, \pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}(m)$ then we can proceed as above to invert the integral representation for $GL_n \times GL_m$. We form U_ξ as before, but must use a V_ξ which is adapted to this functional equation. To this end, we let Q_m be the mirabolic subgroup defined as the stabilizer in GL_n of the vector ${}^t e_{m+1}$, that is, the column vector all of whose entries are 0 except for the $(m+1)^{st}$ which is 1. We take for our permutation matrix the matrix

$$\alpha_m = \begin{pmatrix} & & 1 \\ I_m & & \\ & I_{n-m-1} & \end{pmatrix}.$$

Then we set

$$V_\xi(g) = \sum_{q \in N' \backslash Q_m} W_\xi(\alpha_m q g) \quad \text{where now} \quad N' = \alpha_m^{-1} N \alpha_m.$$

Then if we invert the $GL_n \times GL_m$ integral representation as before we obtain

$$\mathbb{P}_m U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} = \mathbb{P}_m V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \quad \text{for} \quad h \in SL_m(\mathbb{A}),\ \xi \in V_\pi$$

or

$$\mathbb{P}_m U_\xi(I_{m+1}) = \mathbb{P}_m V_\xi(I_{m+1}) \quad \text{for} \quad \xi \in V_\pi.$$

If we now set $m = n-2$ as in Theorem 10.2 (i), then this last equation becomes

$$\int U_\xi \begin{pmatrix} I_{n-1} & u \\ & 1 \end{pmatrix} \psi^{-1}(u_{n-1})\, du = \int V_\xi \begin{pmatrix} I_{n-1} & u \\ & 1 \end{pmatrix} \psi^{-1}(u_{n-1})\, du$$

where the integral is over $k^{n-1}\backslash\mathbb{A}^{n-1}$. We can rewrite this as

$$\int_{k^{n-1}\backslash\mathbb{A}^{n-1}} F_\xi \begin{pmatrix} I_{n-1} & u \\ & 1 \end{pmatrix} \psi^{-1}(u_{n-1})\, du = 0$$

with $F_\xi(g) = U_\xi(g) - V_\xi(g)$. Then our desired equality $U_\xi(I_n) = V_\xi(I_n)$ becomes $F_\xi(I_n) = 0$.

If we set $f_\xi(u) = F_\xi \begin{pmatrix} I_{n-1} & u \\ & 1 \end{pmatrix}$ then f_ξ is a periodic function on $k^{n-1}\backslash\mathbb{A}^{n-1}$ and we wish to know that $f_\xi(0) = F_\xi(I_n) = 0$ for all ξ. Instead, what we have from the above is that a certain Fourier coefficient of f_ξ vanishes. But we also know that $F_\xi(g)$ is left invariant under $P(k) \cap Q_{n-2}(k)$. Using this allows us to show that many more Fourier coefficients of f_ξ vanish. Eventually this analysis leads to the fact that $f_\xi({}^t(0,\ldots,0,u_{n-1}))$ is constant, and moreover this constant is

$$f_\xi({}^t(0,\ldots,0,0)) = f_\xi(0) = F_\xi(I_n).$$

To conclude, we now take any finite place v_1 and working in the local Kirillov model at the place v_1 we are able to place a local condition on the component ξ_{v_1} which guarantees that this common value is 0. Hence we may conclude $U_\xi(I_n) = V_\xi(I_n)$ for all $\xi \in V_\pi$ with ξ_{v_1} fixed.

Now we more or less proceed as in the proof of Theorem 10.1 (ii). We use weak approximation to obtain an automorphic representation π_1 which agrees with π except possibly at v_1. Then we repeat the argument with a second fixed place v_2 to get an automorphic representation π_2 which agrees with π except possibly at v_2. Then we use the Generalized Strong Multiplicity One Theorem and what we know about the entirety of the twisted L-functions to conclude that $\pi_1 = \pi_2 = \pi$ and π is cuspidal. This gives Theorem 10.2 (i).

Theorem 10.2 (ii) is then obtained by combining this method with the proof of Theorem 10.1 (ii). Once can take the place v_1 used above to lie in S, and then once you have used the weak approximation theorem, you are done.

Note that if $m < n-2$ then the unipotent integration in $\mathbb{P}_m$ is now *non-abelian* and our abelian Fourier expansion method (thus far) breaks down.

10.6. **A useful variant.** For applications, these theorems are used in the following useful variant form.

Useful Variant: *Let π be as in Theorems 10.1 and 10.2. Let $\mathcal{T}$ be the twisting set of either theorem. Let $\eta : k^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ be an fixed idele class character. Suppose that $L(s, \pi \times \pi')$ is nice for every $\pi' \in \mathcal{T} \otimes \eta$. Then we have the same conclusions for π as in those theorems.*

To see this, note that $L(s, \pi \times \pi')$ is nice for every $\pi' \in \mathcal{T} \otimes \eta$ iff $L(s, (\pi \otimes \eta) \times \pi'_0)$ is nice for every $\pi'_0 \in \mathcal{T}$. Hence $\pi \otimes \eta$ satisfies the conclusions of either Theorem 10.1 or 10.2. But since η is automorphic, π will as well.

In practice, the set of places S often is taken to be the places where π is ramified and η is taken to be highly ramified at those place so that stability of γ can be used.

10.7. **Conjectures.** The most widely held belief is the conjecture of Jacquet:

Conjecture 10.1. *Let π be as in Theorem 10.1 or Theorem10.2. Suppose that $L(s, \pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}^S\left(\left[\frac{n}{2}\right]\right)$. Then we have the same conclusions as in those theorems. In particular, if S is empty then π should be cuspidal.*

The most interesting and useful conjecture is due to Piatetski-Shapiro:

Conjecture 10.2. *Let π be as in Theorem 10.1 or Theorem10.2. Suppose that $L(s, \pi \otimes \chi)$ is nice for all $\chi \in \mathcal{T}(1)$, that is, for all idele class characters. Then there exists an automorphic representation π_1 such that $\pi_{1,v} \simeq \pi_v$ at all places where they are both unramified and*

$$L(s, \pi \otimes \chi) = L(s, \pi_1 \otimes \chi) \quad \textit{for all } \chi \in \mathcal{T}(1).$$

In particular π and π_1 have the same L-function, so that the formal Euler product defining $L(s, \pi)$ is in fact modular.

One can easily formulate a version of this conjecture for $\mathcal{T}^S(1)$.

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11. Introduction to Functoriality

In this lecture we would like to give a brief introduction to functoriality and how one uses the Converse Theorem to attack the problem of functoriality from reductive groups G to GL_n.

11.1. **The Weil-Deligne group.**

Local functoriality is mediated by admissible maps of the Weil-Deligne group into the Langlands dual group or the L-group.

Let k be a local field. We have defined the Weil group W_k when $k = \mathbb{R}$ or $\mathbb{C}$. So we will let k denote a non-achimedean local field (of characteristic 0 as usual). Let $\mathcal{O}$ be the ring of integers of k, $\mathfrak{p}$ its unique prime ideal, and $\kappa = \mathcal{O}/\mathfrak{p}$ its residue field. Let p be the characteristic of κ and $q = |\kappa|$. Let $\overline{k}$ denote the algebraic closure of k.

Reduction mod $\mathfrak{p}$ gives a surjective map $Gal(\overline{k}/k)$ to $Gal(\overline{\kappa}/\kappa)$ and we let I denote its kernel:

$$1 \longrightarrow I \longrightarrow Gal(\overline{k}/k) \longrightarrow Gal(\overline{\kappa}/\kappa) \longrightarrow 1.$$

I is called the inertia group. We know that $Gal(\overline{\kappa}/\kappa)$ is cyclic and generated by the Frobenius automorphism. Let $\Phi \in Gal(\overline{k}/k)$ be any inverse image of the inverse of Frobenius (a so-called geometric Frobenius).

Since the Galois group $Gal(\overline{k}/k)$ is a pro-finite compact group, to obtain a sufficiently rich class of representations to hopefully classify admissible representation of GL_n, we need to relax this topology. So we let W_k denote the subgroup of $Gal(\overline{k}/k)$ generated by I and Φ, but we topologize W_k so that I retains its induced topology from the Galois group, I is open in W_k, and multiplication by Φ is a homeomorphism. W_k with this topology is the *Weil group of* k. (It carries the structure of a group scheme over $\mathbb{Q}$.) W_k has a natural character $|| \; || : W_k \to q^{\mathbb{Z}} \subset \mathbb{Q}^\times$ given by $||w|| = 1$ for $w \in I$ and $||\Phi|| = q^{-1}$.

The topology on W_k, being essentially pro-finite on I, is still too restrictive to have a sufficiently interesting theory of complex representations. However it has many interesting $\overline{\mathbb{Q}}_\ell$-representations and these are the ones that arise in arithmetic geometry. In order to free the representation theory from incompatible topologies, Deligne introduced the *Weil-Deligne group* W_k'. Following Deligne and Tate we take

W_k' to be the semi-direct product $W_k \ltimes \mathbb{G}_a$ of the Weil group with the additive group where W_k acts on $\mathbb{G}_a$ by $wxw^{-1} = ||w||x$.

What is important about W_k' is not so much its structure but its representation theory. A representation ρ' of W_k' is a pair $\rho' = (\rho, N)$ consisting of

(i) an n-dimensional vector space V and a group homomorphism $\rho : W_k \to GL(V)$ whose kernel contains an open subgroup of I (so it is continuous with respect to the discrete topology on V);
(ii) a nilpotent endomorphism N of V such that $\rho(w)N\rho(w)^{-1} = ||w||N$.

The representation ρ' is called semi-simple if ρ is. This category of representations is independent of the (characteristic 0) coefficient field.

[Often one sees $W_k' = W_k \times SL_2$. This can be made consistent in terms of the representation theory via the Jacobson-Morozov Theorem. However it is the nilpotent endomorphism N that arises naturally as a monodromy operator in the theory of ℓ-adic Galois representations (Grothendieck) so I have chosen to retain this formulation.]

When k is $\mathbb{R}$ or $\mathbb{C}$, we simply take $W_k' = W_k$.

11.2. **The dual group.** Now let k be either local or global and let G be a connected reductive algebraic group over k. For simplicity we will take G split, so things behave as it k were algebraically closed.

Recall from Kim's lectures that over an algebraically closed field G is determined by its root data. If T is a maximal split torus in G then the rood data for G is $\Psi = (X^*(T), \Phi, X_*(T), \Phi^\vee)$ where

$X^*(T)$ is the set of rational characters of T
$\Phi \subset X^*(T)$ is the root system $\Phi(G,T)$
$X_*(T)$ is the set of rational co-characters (one parameter subgroups)
$\Phi^\vee \subset X_*(T)$ is the co-root system).

If we dualize this to obtain $\Psi^\vee = (X_*(T), \Phi^\vee, X^*(T), \Phi)$ then this dual data determines a complex group ${}^LG^\circ = {}^LG$ which is the Langlands dual group or the (connected component of the) L-group of G.

11.3. **The local Langlands conjecture.** Let k be a local field and let G be a reductive algebraic group over k, assumed split as before. The local Langlands conjecture essentially says that the irreducible admissible representations of $G(k)$ are parameterized by admissible homomorphisms of the Weil-Deligne group W_k' to the L-group ${}^LG$. To be more precise, for this lecture let us set $\mathcal{A}(G)$ denote the equivalence classes of irreducible admissible (complex) representations of $G(k)$ and let $\Phi(G)$ denote the set of all admissible homomorphisms $\phi : W_k' \to {}^LG$ (module inner automorphisms). We won't worry about the precise definition of admissible, but just note that for $G = GL_n$ an admissible homomorphism is simply a semi-simple representation as above.

Local Langlands Conjecture: *There is a surjective map $\mathcal{A}(G) \longrightarrow \Phi(G)$ with finite fibres which partitions $\mathcal{A}(G)$ into finite sets $\mathcal{A}_\phi(G)$, called L-packets, satisfying certain naturality conditions.*

This is known in the following cases which will be of relevance to us. (This list is not exhaustive.)

1. If $k = \mathbb{R}$ or $\mathbb{C}$ this was completely established by Langlands. His naturality conditions were representation theoretic in nature.

2. If k is non-archimedean (recalling that G is split) then one knows how to parameterize the unramified representations of $G(k)$ by unramified admissible homomorphisms. This is the Satake classification.

3. If k is non-archimedean and $G = GL_n$ this is known and due to Harris-Taylor and then Henniart (remember we have taken k of characteristic 0) and in fact the map is a bijection. In these works the naturality conditions were phrased in terms of matching twisted L- and ε-factors for the Weil-Deligne representations with those we presented here for GL_n.

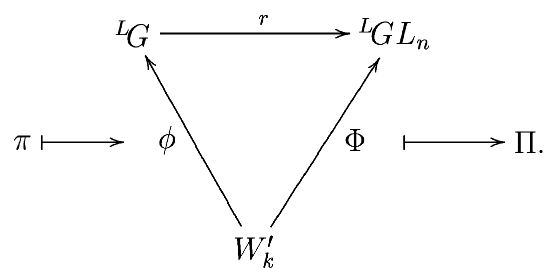
Note that there is at present no similar formulation of a global Langlands conjecture for global fields of characteristic 0. To obtain one, one would need to replace the local Weil-Deligne group by the conjectural Langlands group $\mathcal{L}_k$ that Jim Arthur talked about in the Shimura Variety Workshop. With $\mathcal{L}_k$ in hand it would be relatively easy to formulate a conjecture like the one above.

11.4. **Local Functoriality.** We still take k to be a local field. Let G be a split reductive algebraic group over k. Let $r : {}^LG \to GL_n(\mathbb{C})$ be a complex analytic representation. Since ${}^LGL_n = GL_n(\mathbb{C})$, the map r

is an example of what Langlands referred to as an L-homomorphism. Langlands' Principle of Functoriality can then be roughly stated as saying:

Principle of Functoriality: *Associated to the L-homomorphism $r : {}^LG \to {}^LGL_n$ there should be associated a natural lift or transfer of admissible representations from $\mathcal{A}(G)$ to $\mathcal{A}(GL_n)$.*

If we assume the local Langlands conjecture for G, this is easy to formulate. We begin with $\pi \in \mathcal{A}(G)$. Associated to π we have a parameter $\phi \in \Phi(G)$. Then via the diagram



we obtain a parameter $\Phi \in \Phi(GL_n)$ and hence a representation Π of $GL_n(k)$. We refer to Π as the local functorial lift of π. As part of the formalism we obtain

$$L(s, \pi, r) = L(s, r \circ \phi) = L(s, \Phi) = L(s, \Pi)$$

with similar equalities for ε-factors and twisted versions.

[To Be Continued]