Strong Axioms: Determinacy and Large Cardinals

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One basic problem from lecture 1:

Under what circumstances can the theory of the structure

$$\langle H(\omega_2), \in\rangle$$

be finitely axiomatized, over ZFC, *in* Ω-*logic?*

We require a definition.

Definition 1 Suppose that there exists a proper class of Woodin cardinals. A set

$$X \subseteq \mathbb{N}$$

is Ω-*recursive* if there exists a universally Baire set

$$A \subseteq \mathbb{R}$$

such that X is Δ_1 definable in $L(A, \mathbb{R})$ from $\{\mathbb{R}\}$. $\square$

– The Ω-recursive sets form a transfinite generalization (in terms of complexity) of the recursive (i. e. Turing computable) sets.

Lemma 2 *Suppose that* Ψ *is a sentence such that for all* ϕ *either*

(i) $\mathrm{ZFC} + \Psi \vdash_{\Omega}$ "$H(\omega_2) \models \phi$", *or*

(ii) $\mathrm{ZFC} + \Psi \vdash_{\Omega}$ "$H(\omega_2) \models \neg\phi$".

Let T *be the set of all sentences* ϕ *such that*

$$\mathrm{ZFC} + \Psi \vdash_{\Omega} \text{“}H(\omega_2) \models \phi\text{”}.$$

Then T *is* Ω*-recursive.* □

Theorem 3 *Suppose that there exists a proper class of Woodin cardinals.*

Let

$$T = \mathrm{Th}(H(\omega_2)).$$

The following are equivalent.

(1) *T is Ω-recursive.*

(2) *There exist a sentence Ψ and a cardinal κ such that*

$$V_\kappa \models \mathrm{ZFC} + \Psi$$

and such that for each sentence ϕ, either

a) $\mathrm{ZFC} + \Psi \vdash_\Omega$ "$H(\omega_2) \models \phi$", *or*

b) $\mathrm{ZFC} + \Psi \vdash_\Omega$ "$H(\omega_2) \models \neg\phi$". □

Thus the basic problem:

- Under what circumstances can the theory of the structure

  $$\langle H(\omega_2), \in \rangle$$

  be finitely axiomatized, over ZFC, in Ω-logic?

naturally leads to the problem:

- How complicated are the Ω-recursive subsets of $\mathbb{N}$?
  - such sets look potentially extremely complicated (because the definition involves universally Baire sets).

The combinatorial analysis of the Ω-recursive sets involves combining elements of *Descriptive Set Theory* with elements of *Fine Structure Theory*.

AD$^+$

Definition 4 Suppose $A \subset \mathbb{R}$. The set A is $^\infty$-*borel* if there exist a set $S \subset \mathrm{Ord}$, an ordinal α, and a formula $\phi(x_0, x_1)$ such that

$$A = \left\{ y \in \mathbb{R} \,\middle|\, L_\alpha[S, y] \models \phi[S, y] \right\}. \qquad \square$$

- $A \subset \mathbb{R}$ is $^\infty$-borel if A has a *transfinite borel* code.
  - A key feature is that the code be effective; i. e. that it be a set of ordinals.
- Assuming the *Axiom of Choice*, every set $A \subseteq \mathbb{R}$ is $^\infty$-borel.

Definition 5 Θ is the supremum of the ordinals α such that there exists a surjection

$$\pi : \mathbb{R} \to \alpha. \qquad \square$$

Definition 6 (ZF + $DC_{\mathbb{R}}$) AD^+:

1. Suppose $A \subset \mathbb{R}$. Then A is $^\infty$-borel.
2. Suppose $\lambda < \Theta$ and

$$\pi : \lambda^\omega \to \omega^\omega$$

   is a continuous function. Then for each $A \subset \mathbb{R}$ the set $\pi^{-1}[A]$ is determined. $\square$

Conjecture: AD implies AD^+.

Lemma 7 *Assume*

$$\mathrm{ZF} + \mathrm{AD} + \text{“}V = L(\mathbb{R})\text{”}.$$

Then AD^+. $\square$

Theorem 8 *Assume*

$$\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_{\mathbb{R}}.$$

Then AD^+. $\square$

Derived models

Theorem 9 *Suppose that δ is a limit of Woodin cardinals. Suppose that*

$$G \subset \mathrm{Coll}(\omega, < \delta)$$

is V-generic and let

$$\mathbb{R}_G = \cup \left\{ \mathbb{R}^{V[G|\alpha]} \,\middle|\, \alpha < \delta \right\}.$$

Let Γ_G be the set of $A \subset \mathbb{R}_G$ such that

(i) $A \in V(\mathbb{R}_G)$,

(ii) $L(\mathbb{R}_G, A) \models \mathrm{AD}^+$.

Then

$$L(\mathbb{R}_G, \Gamma_G) \models \mathrm{AD}^+. \qquad \square$$

Models which arise in this fashion are *derived models*.

Theorem 10 (AD^+) *Assume that*

$$V = L(\mathcal{P}(\mathbb{R})).$$

There is a partial order $\mathbb{P}$ *such that if* $G \subset \mathbb{P}$ *is* V*-generic then in* $V[G]$ *there exists an inner model*

$$N \subset V[G]$$

such that in $V[G]$*:*

(1) $N \models \mathrm{ZFC}$.

(2) ω_1^V *is a limit of Woodin cardinals in* N.

(3) *There exists an* N*-generic filter*

$$g \subset \mathrm{Coll}(\omega, < \omega_1^V)$$

such that

a) $\mathbb{R}_g = \mathbb{R}_V$,

b) $\Gamma_g = (\mathcal{P}(\mathbb{R}))^V$. □

The proofs of many of the deepest consequences of

$$\mathrm{ZF} + \mathrm{AD} + \text{``}V = L(\mathbb{R})\text{''}$$

are based on the presentation of $L(\mathbb{R})$ as an *inner model*; these proofs exploit the "smallness" of $L(\mathbb{R})$ in various essential ways.

Perhaps surprising then is that these consequences generalize, abstractly, to the theory:

$$\mathrm{ZF} + \mathrm{AD}^{+} + \text{``}V = L(\mathcal{P}(\mathbb{R}))\text{''}.$$

Theorem 11 (AD^+) *Assume*

$$V = L(\mathcal{P}(\mathbb{R})).$$

(1) *The pointclass* Σ^2_1 *has the scale property.*

(2) *Suppose* $\phi(x, y)$ *is a* Σ_1*-formula and there exists a set* X *such that*

$$V \models \phi[X, \mathbb{R}].$$

There there exists X^* *such that*

a) $V \models \phi[X^*, \mathbb{R}]$,

b) X^* *is coded by a set* $Z \subset \mathbb{R}$ *such that* Z *is* Δ^2_1. $\square$

The connection between the universally Baire sets and AD^+:

Theorem 12 *Assume there exists a proper class of Woodin cardinals.*

Let Γ^∞ *be the pointclass of all* $A \subset \mathbb{R}$ *such that* A *is universally Baire.*

(1) Γ^∞ *is a* σ*-algebra.*

(2) *For each* $A \in \Gamma^\infty$*,*

$$\mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R}) \in \Gamma^\infty.$$

(3) *For each* $A \in \Gamma^\infty$*,*

$$L(A, \mathbb{R}) \models \mathrm{AD}^+. \qquad \square$$

Combining the two previous theorems yields some information on the complexity of the Ω-recursive subsets of $\mathbb{N}$.

Theorem 13 *Assume there exist a proper class of Woodin cardinals.*

Suppose that

$$T \subseteq \mathbb{N}$$

is Ω-recursive.

Suppose that γ is a cardinal and there is a Woodin cardinal below γ.

Then T is definable in the structure

$$\langle H(\gamma), \in \rangle. \qquad \square$$

Improving this calculation requires a detour through *Fine Structure Theory*.

Fine structure and inner models

L is Gödel's constructible universe:

$$L = \cup \left\{ L_\eta \,\middle|\, \eta \in \mathrm{Ord} \right\}$$

where;

1. $L_0 = \emptyset$,
2. $L_{\eta+1} = \left\{ a \subset L_\eta \,\middle|\, a \text{ is definable in } \langle L_\eta, \in \rangle \right\}$,
3. If η is a limit ordinal then

$$L_\eta = \cup \left\{ L_\alpha \,\middle|\, \alpha < \eta \right\}.$$

The detailed analysis of L is the *fine structure* of L; this was initiated, and mostly developed, by Jensen.

The generalization of L to *inner models* in which various large cardinal axioms hold, is the *Inner Model Program*. The goal is to understand these inner models and their fine structure.

- *Measurable Cardinals*: The inner model was defined by Solovay. The fine structure was developed by Solovay, building on work of Kunen and Silver.

- *Woodin Cardinals and Beyond*: The inner models and their fine structure were defined and analyzed by Mitchell and Steel.

  – The Mitchell-Steel models can accommodate substantial large cardinals.

  – Existence can be proved at the level of Woodin cardinals; e. g. assume there exists a Woodin cardinal. Then a Mitchell-Steel inner model for a Woodin cardinal exists.

Suppose that M, N are transitive sets,

$$M, N \models \mathrm{ZFC} \backslash \mathrm{Powerset},$$

and that

$$j : M \to N$$

is elementary embedding with critical point $\kappa \in M \cap \mathrm{Ord}$.

Suppose that $\kappa < \eta \leq j(\kappa)$. We define the (κ, η)-M-*extender*, E, which is given by j. This extender is simply the function

$$F : \mathcal{P}(\kappa) \cap M \to V$$

given by: $F(A) = j(A) \cap \eta$.

The formal definition specifies E as a family of M-*ultrafilters*.

For each finite set $s \subset \eta$ let

$$E_s = \left\{ A \subset [\kappa]^{|s|} \,\middle|\, A \in M \text{ and } s \in j(A) \right\}.$$

Thus E_s is an M-ultrafilter. The set

$$E = \left\{ (s, A) \,\middle|\, s \in [\eta]^{<\omega} \text{ and } A \in E_s \right\}$$

is the (κ, η)-M-extender given by j. If $V_{\kappa+1} \subset M$ then E is a (κ, η)-extender.

This definition of an extender is due to Jensen based on an essentially equivalent notion due to Mitchell

The Mitchell-Steel models are of the form $L[\tilde{E}]$ where

$$\tilde{E} \subset \mathrm{Ord} \times V$$

is a predicate defining a sequence of (partial) extenders; more precisely if $\eta \in \mathrm{dom}(\tilde{E})$ then

$$L_\eta[\tilde{E}] \models \mathrm{ZFC} \backslash \mathrm{Powerset}$$

and $(\tilde{E})_\eta$ is a (κ, η)-$L_\eta[\tilde{E}]$-extender (for some $\kappa < \eta$).

- Thus if

$$\mathcal{P}(\kappa) \cap L_\eta[\tilde{E}] \neq \mathcal{P}(\kappa) \cap L[\tilde{E}],$$

  then $(\tilde{E})_\eta$ is *not* a (κ, η)-$L[\tilde{E}]$-extender.

- Because the extenders are partial the development of the fine structure of the Mitchell-Steel models is necessary in order to even define the models.

Another inner model can *always* be defined; it is HOD.

A set a belongs to HOD if there exist an ordinal α and a set

$$A \subset \alpha$$

such that

1. $a \in L[A]$,
2. A is definable in the structure

   $$\langle V_\alpha, \in \rangle$$

   from ordinal parameters.

- The *Axiom of Choice* holds in HOD;
  - *even* if the *Axiom of Choice* fails in the universe where HOD is computed.
- HOD is *not* absolute, it can change in passing from V to a generic extension of V.
- (Vopenka) V is a (class) generic extension of HOD.
  - If the *Axiom of Choice* fails, then V is a (class) symmetric generic extension of HOD.

Recall: Θ denotes the supremum of the ordinals α such that there exists a surjection

$$\pi : \mathbb{R} \to \alpha.$$

Theorem 14 *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let*

$$\delta = (\Theta)^{L(\mathbb{R})}.$$

Then in $(\mathrm{HOD})^{L(\mathbb{R})}$, δ *is a Woodin cardinal.*

A truly remarkable theorem of Steel:

Theorem 15 (Steel) *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let*

$$\delta = (\Theta)^{L(\mathbb{R})}.$$

Then $(\mathrm{HOD})^{L(\mathbb{R})} \cap V_\delta$ *is a Mitchell-Steel model.* □

The proof of Steel's theorem involves directed systems of Mitchell-Steel models; a version of this kind of construction will be discussed shortly and in a simpler context.

Corollaries of Steel's thorem include:

Theorem 16 (Steel) *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Then the* Generalized Continuum Hypothesis *holds in* $(\mathrm{HOD})^{L(\mathbb{R})}$. □

But

What is $(\mathrm{HOD})^{L(\mathbb{R})}$*?*

Theorem 17 $(\mathrm{HOD})^{L(\mathbb{R})}$ *is not a Mitchell-Steel model.* □

However:

- $(\mathrm{HOD})^{L(\mathbb{R})}$ is a fine structure model.
- It belongs to a new, quite different, hierarchy of models.

We illustrate the analysis which leads to the characterization of $(\mathrm{HOD})^{L(\mathbb{R})}$ by analyzing a simple case.

Theorem 18 (Kechris, Solovay) *Suppose that $x \in \mathbb{R}$. Then the following are equivalent.*

1. $L[x] \models \Sigma^1_2$-determinacy.
2. $L[x] \models$ All OD sets are determined. $\square$

Theorem 19 (Kechris, Woodin) *Assume for all* $x \in \mathbb{R}$, $x^{\#}$ *exists. Assume* Σ^1_2-Determinacy. *Then there exists* $x_0 \in \mathbb{R}$ *such that for all* $x \in \mathbb{R}$, *if* $x_0 \in L[x]$ *then*

$$L[x] \equiv L[x_0].$$

□

The converse is also true:

Theorem 20 *Assume for all* $x \in \mathbb{R}$, $x^{\#}$ *exists and that there exists* $x_0 \in \mathbb{R}$ *such that for all* $x \in \mathbb{R}$, *if* $x_0 \in L[x]$ *then*

$$L[x] \equiv L[x_0].$$

Then Σ^1_2-Determinacy. □

For each $x \in \mathbb{R}$ let κ_x be the least inaccessible cardinal of $L[x]$. We shall be considering the inner models, $L[x][G]$, where

$$G \subset \text{Coll}(\omega, < \kappa_x)$$

is $L[x]$-generic.

Of course by homogeneity, the inner model:

$$(\text{HOD})^{L[x][G]}$$

does not depend on the choice of G and further;

$$(\text{HOD})^{L[x][G]}$$

is simply HOD as computed in $(L(\mathbb{R}))^{L[x][G]}$.

The Kechris-Solovay Theorem generalizes to the models, $L[x][G]$.

Lemma 21 *Suppose that $x \in \mathbb{R}$ and $x^{\#}$ exists. Then the following are equivalent.*

1. $L[x][G] \models \Sigma^1_2$-determinacy.
2. $L[x][G] \models$ All OD sets are determined. $\square$

Thus the model $(L(\mathbb{R}))^{L[x][G]}$ under the hypothesis of Σ^1_2-determinacy is a "light-face" analog of $L(\mathbb{R})$ under the hypothesis of AD.

We assume that for all $x \in \mathbb{R}$, $x^\#$ exists and that Σ^1_2-determinacy holds. Our goal is to analyze the inner models, $(\mathrm{HOD})^{L[x][G]}$, for a cone of x, where as above $G \subset \mathrm{Coll}(\omega, < \kappa_x)$ is $L[x]$-generic.

In fact the analysis can be carried out just assuming

$$V = L[x][G]$$

and Σ^1_2-determinacy.

Theorem 22 *Suppose that $x \in \mathbb{R}$, $x^\#$ exists, and that*

$$L[x][G] \models \Sigma^1_2\text{-determinacy}.$$

Let $\delta = (\omega_2)^{L[x][G]}$. Then

$$(\mathrm{HOD})^{L[x][G]} \models \text{“}\delta \text{ is a Woodin cardinal”} \qquad \square$$

We let $\mathcal{D}^\infty$ be the set of inner models, $L[\tilde{E}]$, such that

1. $L[\tilde{E}]$ is a Mitchell-Steel inner model;
2. $\tilde{E} \subset \delta_{\tilde{E}}$ where $\delta_{\tilde{E}} < \omega_1$;
3. $\delta_{\tilde{E}}$ is the (unique) Woodin cardinal of $L[\tilde{E}]$;
4. There is no inner model of $L[\tilde{E}]$ with a Woodin cardinal δ with $\delta < \delta_{\tilde{E}}$;
5. $L[\tilde{E}]$ is (countably) interable.

Theorem 23 (Steel) *Suppose that for all $x \in \mathbb{R}$, $x^{\#}$ exists. Suppose that $L[\tilde{E}_0], L[\tilde{E}_1] \in \mathcal{D}^\infty$ and*

$$L[\tilde{E}_0] = L[\tilde{E}_1].$$

Then $\tilde{E}_0 = \tilde{E}_1$. $\square$

Theorem 24 (Mitchell, Steel) *Suppose that δ is a Woodin cardinal and suppose that $V_\delta^{\#}$ exists.*

Then $\mathcal{D}^\infty \neq \emptyset$. $\square$

Theorem 25 *Suppose that that for all $x \in \mathbb{R}$, $x^{\#}$ exists and supoose that Σ^1_2-determinacy holds.*

Then $\mathcal{D}^\infty \neq \emptyset$. □

Theorem 26 (Mitchell, Steel) *Suppose that that for all $x \in \mathbb{R}$, $x^{\#}$ exists and that $L[\tilde{E}_0], L[\tilde{E}_1] \in \mathcal{D}^\infty$. Then there exist $L[\tilde{E}] \in \mathcal{D}^\infty$ and iteration maps,*

$$j_{\tilde{E}_0} : L[\tilde{E}_0] \to L[\tilde{E}]$$

and

$$j_{\tilde{E}_1} : L[\tilde{E}_1] \to L[\tilde{E}].$$

Further $\tilde{E} \in L[\tilde{E}_0, \tilde{E}_1]$. □

So $\mathcal{D}^\infty$ is naturally a directed system under iteration maps. It is fundamental theorem of Mitchell and Steel this is a commutative system (which is a special property of iteration maps as opposed to arbitrary elementary embeddings).

We let $\mathcal{M}^\infty$ be the limit of this directed system.

For each pair (x, G) let

$$\mathcal{D}_x^G = \left\{ L[\tilde{E}] \in \mathcal{D}^\infty \cap L[x][G] \,\middle|\, \delta_{\tilde{E}} < (\omega_1)^{L[x][G]} \right\}.$$

If $\mathcal{D}_x^G \neq \emptyset$ then $\mathcal{D}_x^G$ is directed subsystem of $\mathcal{D}^\infty$.

We now come to the main points. Recall we are assuming that for all $x \in \mathbb{R}$, $x^{\#}$ exists and that Σ^1_2-Determinacy holds.

Fix $x_0 \in \mathbb{R}$ and G_0 such that $\mathcal{D}^{G_0}_{x_0} \neq \emptyset$.

The first main point is that for each ordinal α_0 there exist $L[\tilde{E}_0] \in \mathcal{D}^{G_0}_{x_0}$, g_0 and π_0 such that:

1. $(\omega_1)^{L[x_0][G_0]}$ is the least inaccessible cardinal of $L[\tilde{E}_0]$ above $\delta_{\tilde{E}_0}$;
2. g_0 is $L[\tilde{E}_0]$-generic for $\mathrm{Coll}(\omega, < \lambda)$ where $\lambda = (\omega_1)^{L[x_0][G_0]}$;
3. $L[x_0][G_0] = L[\tilde{E}_0][g_0]$;

and such that

4. $\pi_0 : L[\tilde{E}_0] \to \mathcal{M}^{\infty}_{G_0,x_0}$;

5. α_0 is in the range of π_0;

where $\mathcal{M}^{\infty}_{G_0,x_0}$ is the limit of the directed system, $\mathcal{D}^{G_0}_{x_0}$ which is a subdirected system of $\mathcal{D}^{\infty}$.

Of course π_0 is uniquely specified by $L[\tilde{E}_0]$.

Since $L[x_0][G_0] = L[\tilde{E}_0][g_0]$, clearly,

$$(\mathrm{HOD})^{L[x_0][G_0]} \subseteq L[\tilde{E}_0].$$

But it also follows, because there are differing choices of $L[\tilde{E}_0]$, that

$$(\mathrm{HOD})^{L[x_0][G_0]} \neq L[\tilde{E}_0].$$

In fact cofinally many elements of $\mathcal{D}_{x_0}^{G_0}$ satisfy the requirements specified and further

$$(\mathrm{HOD})^{L[x_0][G_0]} = \bigcap \left\{ L[\tilde{E}] \,\middle|\, L[\tilde{E}] \in \mathcal{D} \right\}$$

where $\mathcal{D} \subset \mathcal{D}_{x_0}^{G_0}$ is any such cofinal set.

We shift the focus of our analysis from $L[x_0]$ to $L[\tilde{E}_0]$. So for each $L[\tilde{E}] \in \mathcal{D}^\infty$ let

$$\mathcal{D}_{\tilde{E}} \subset \mathcal{D}^\infty$$

be the set of $L[\tilde{F}] \in \mathcal{D}^\infty$ such that

$$\delta_{\tilde{F}} < \kappa_{\tilde{E}}$$

where $\kappa_{\tilde{E}}$ is the least inaccessible of $L[\tilde{E}]$ above $\delta_{\tilde{E}}$.

For each $L[\tilde{E}] \in \mathcal{D}^\infty$, $\mathcal{D}_{\tilde{E}}$ is a directed subsystem of $\mathcal{D}^\infty$. Let $\mathcal{M}^\infty_{\tilde{E}}$ be the limit of this directed subsystem and let

$$\pi_{\tilde{E}} : L[\tilde{E}] \to \mathcal{M}^\infty_{\tilde{E}}$$

be the induced map.

Clearly $\mathcal{D}_{\tilde{E}_0}$ is a directed subsystem of $\mathcal{D}^{G_0}_{x_0}$ since

$$\mathcal{D}_{\tilde{E}_0} \subset \mathcal{D}^{G_0}_{x_0}.$$

The next key point is that $\mathcal{D}_{\tilde{E}_0}$ is actually cofinal in $\mathcal{D}^{G_0}_{x_0}$. Thus

$$\lim \mathcal{D}_{\tilde{E}_0} = \lim \mathcal{D}^{G_0}_{x_0},$$

and so $\mathcal{M}^{\infty}_{\tilde{E}} = \mathcal{M}^{\infty}_{G_0, x_0}$ and the natural map is the identity.

In particular, $\pi_0 = \pi_{\tilde{E}_0}$, where

$$\pi_{\tilde{E}_0} : L[\tilde{E}_0] \to \mathcal{M}^{\infty}_{\tilde{E}_0}$$

is the natural limit map (arising from the directed system) from $L[\tilde{E}_0]$ to

$$\lim \mathcal{D}_{\tilde{E}_0} = \mathcal{M}^{\infty}_{\tilde{E}_0}$$

recalling that $L[\tilde{E}_0] \in \mathcal{D}_{\tilde{E}_0}$.

Thus by the choice of $L[\tilde{E}_0]$, α_0 is in the range of $\pi_{\tilde{E}_0}$.

Let $\tilde{F}_0 = \pi_{\tilde{E}_0}(\tilde{E}_0)$ and so

$$\mathcal{M}^{\infty}_{G_0,x_0} = \mathcal{M}^{\infty}_{\tilde{E}_0} = L[\tilde{F}_0].$$

The calculations on the definablity of $L[\tilde{F}_0]$ show that

$$L[\tilde{F}_0] \subset (\mathrm{HOD})^{L[x_0][G_0]} = (\mathrm{HOD})^{L[\tilde{E}_0][g_0]}.$$

Shifting to $L[\tilde{F}_0]$, we have the natural map,

$$\pi_{\tilde{F}_0} : L[\tilde{F}_0] \to \mathcal{M}^{\infty}_{\tilde{F}_0}.$$

We have fixed α_0 and $L[\tilde{E}_0] \in \mathcal{D}^{G_0}_{x_0}$. Further

$$\pi_{\tilde{E}_0} : L[\tilde{E}_0] \to L[\tilde{F}_0] = \lim \mathcal{D}^{G_0}_{x_0}$$

is the associated embedding. The proof that

$$L[\tilde{F}_0] \subset (\mathrm{HOD})^{L[x_0][G_0]} = (\mathrm{HOD})^{L[\tilde{E}_0][g_0]}.$$

also shows that $\pi_{\tilde{F}_0} \subset (\mathrm{HOD})^{L[x_0][G_0]}$.

The final point is that

$$\mathcal{M}^{\infty}_{\tilde{F}_0} = \pi_{\tilde{E}_0}(L[\tilde{F}_0]).$$

and further since α_0 is in the range of $\pi_{\tilde{E}_0}$,

$$\pi_{\tilde{F}_0}(\alpha_0) = \pi_{\tilde{E}_0}(\alpha_0)$$

(but $\pi_{\tilde{F}_0} \neq \pi_{\tilde{E}_0}|L[\tilde{F}_0]$).

Now it follows that

$$(\mathrm{HOD})^{L[\tilde{E}_0][g_0]} = (\mathrm{HOD})^{L[x_0][G_0]} = L[\tilde{F}_0][\pi_{\tilde{F}_0}]$$

To see this note that if $\phi(x)$ is a formula then;

$$L[x_0][G_0] \models \phi[\alpha_0]$$

if and only if

$$L[\tilde{E}_0]^{\mathrm{Coll}(\omega,<\kappa_{\tilde{E}_0})} \models \phi[\alpha_0].$$

Now applying $\pi_{\tilde{E}_0}$ this is equivalent to the condition

$$L[\tilde{F}_0]^{\mathrm{Coll}(\omega,<\kappa_{\tilde{F}_0})} \models \phi[\pi_{\tilde{F}_0}(\alpha_0)]$$

since $\pi_{\tilde{E}_0}(\alpha_0) = \pi_{\tilde{F}_0}(\alpha_0)$.

We chose $L[\tilde{E}_0] \in \mathcal{D}_{x_0}^{G_0}$ and g_0 satisfying:

1. $(\omega_1)^{L[x_0][G_0]}$ is the least inaccessible cardinal of $L[\tilde{E}_0]$ above $\delta_{\tilde{E}_0}$;
2. g_0 is $L[\tilde{E}_0]$-generic for $\mathrm{Coll}(\omega, < \lambda)$ where $\lambda = (\omega_1)^{L[x_0][G_0]}$;
3. $L[x_0][G_0] = L[\tilde{E}_0][g_0]$;

and such that α_0 is in the range of the natural map

$$\pi_0 : L[\tilde{E}_0] \to \lim \mathcal{D}_{x_0}^{G_0}.$$

Varying the choice of $L[\tilde{E}_0]$ *does not change*

$$L[\tilde{F}_0] = \lim \mathcal{D}_{\tilde{E}_0}$$

since

$$\lim \mathcal{D}_{\tilde{E}_0} = \lim \mathcal{D}_{x_0}^{G_0}.$$

Therefore and this is really the key idea, for *all* ordinals α, for all formulas $\phi(x)$,

$$L[x_0][G_0] \models \phi[\alpha]$$

if and only if

$$L[\tilde{F}_0]^{\mathrm{Coll}(\omega,<\kappa_{\tilde{F}_0}))} \models \phi[\pi_{\tilde{F}_0}(\alpha)].$$

Thus

$$(\mathrm{HOD})^{L[x_0][G_0]} \subseteq L[\tilde{F}_0][\pi_{\tilde{F}_0}]$$

and so

$$(\mathrm{HOD})^{L[x_0][G_0]} = L[\tilde{F}_0][\pi_{\tilde{F}_0}].$$

The inner model, $L[\tilde{F}_0][\pi_{\tilde{F}_0}]$, is a member of the new class of inner models. In fact it is in some ways close to the inner model $L[\tilde{F}_0]$.

If $\delta_{\tilde{F}_0}$ is the Woodin cardinal of $L[\tilde{F}_0]$ then

$$V_{\delta_{\tilde{F}_0}} \cap L[\tilde{F}_0] = V_{\delta_{\tilde{F}_0}} \cap L[\tilde{F}_0][\pi_{\tilde{F}_0}],$$

and $\delta_{\tilde{F}_0}$ is a Woodin cardinal in $L[\tilde{F}_0][\pi_{\tilde{F}_0}]$.

However above $\delta_{\tilde{F}_0}$ the two models differ dramatically even though

$$L[\tilde{F}_0]^{\#} \notin L[\tilde{F}_0][\pi_{\tilde{F}_0}].$$

1. Cofinally many cardinals above $\delta_{\tilde{F}_0}$ are collapsed in passing from $L[\tilde{F}_0]$ to $L[\tilde{F}_0][\pi_{\tilde{F}_0}]$.

2. If X is *any* set of regular cardinals of $L[\tilde{F}_0][\pi_{\tilde{F}_0}]$ above $\delta_{\tilde{F}_0}$ and X has ordertype at least $\delta_{\tilde{F}_0}$ then

$$L[\tilde{F}_0][\pi_{\tilde{F}_0}] \subseteq L[\tilde{F}_0][X]$$

Thus $X \notin L[\tilde{F}_0]$ and further

$$L[\tilde{F}_0] \subset L[\tilde{F}_0][\pi_{\tilde{F}_0}] \subset L[\tilde{F}_0]^{\#}.$$

The analysis of $(\mathrm{HOD})^{L(\mathbb{R})}$ is similar. It is more complicated and based on the Mitchell-Steel inner models $L[\tilde{E}]$ for ω many Woodin cardinals.

Suppose that there exists a proper class of Woodin cardinals and that $A \subset \mathbb{R}$ is universally Baire. Then

$$L(A, \mathbb{R}) \models \mathrm{AD}^+$$

and the analysis of $(\mathrm{HOD})^{L(\mathbb{R})}$ in turn generalizes to an analysis of $(\mathrm{HOD})^{L(A,\mathbb{R})}$ except that now the analysis is not based on Mitchell-Steel inner models, $L[\tilde{E}]$, but on models in an hierarchy which include the models we have just discussed.

The detailed analysis of these models yields the following theorem.

Theorem 27 *Assume there exist a proper class of Woodin cardinals.*

Suppose that

$$T \subseteq \mathbb{N}$$

is Ω*-recursive.*

Then T *is definable in the structure*

$$\langle H(c^+), \in \rangle. \qquad \Box$$

Moreover this latter calculation can be improved which yields as a corollary the theorem on CH.

Let $\mathcal{I}_{\mathrm{NS}} \subset \mathcal{P}(\omega_1)$ be the ideal of all nonstationary subsets of ω_1.

Theorem 28 (CH) *Suppose that there exists a proper class of Woodin cardinals.*

Suppose that

$$T \subseteq \mathbb{N}$$

is Ω-recursive.

Then T is Δ_2 definable in the structure,

$$\langle H(\omega_2), \mathcal{I}_{\mathrm{NS}}, \in \rangle. \qquad \square$$

Thus by Tarski's theorem on the undefinability of truth:

Theorem 29 *Suppose that there exist a proper class of Woodin cardinals,*

$$V_\kappa \models \mathrm{ZFC} + \Psi$$

and for each sentence ϕ, either

(i) $\mathrm{ZFC} + \Psi \vdash_\Omega$ "$H(\omega_2) \models \phi$", or

(ii) $\mathrm{ZFC} + \Psi \vdash_\Omega$ "$H(\omega_2) \models \neg\phi$".

Then CH *is false.* □

Assuming the Ω Conjecture the inner models, $L(A, \mathbb{R})$, where $A \subseteq \mathbb{R}$ is universally Baire define the large cardinal hierarchy through Ω-logic and the notion of ϕ-closure.

But clearly the abstract definition of large cardinals that we have used to calibrate this hierarchy is too general.

This in turn leads to the question of what kinds of inner models might exist for very strong large cardinal axioms. While the inner models, HOD, computed in the inner models $L(A, \mathbb{R})$ provide a hierarchy of fine structural models suitable for all determinacy axioms (extending AD^+) these models do not actually have, at least relative to their specified extender sequences, any significantly large cardinals.

In the third and final lecture, I shall discuss the problem of finding inner models for large cardinal axioms far stronger than those we have discussed; for example huge cardinals and beyond.