The fixed point on compacta property of topological groups

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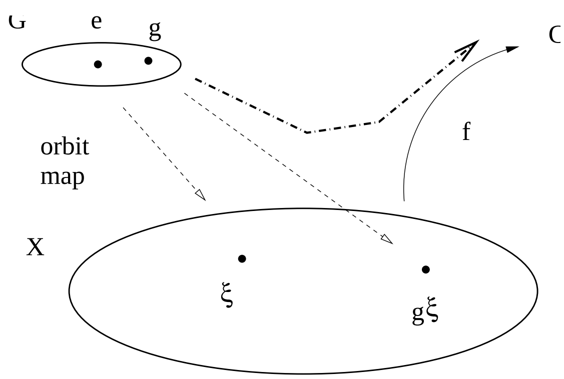
def. A topological group G has the *fixed point on compacta property* (*f.p.c.*) (= is *extremely amenable*) if every continuous action of G on a compact space has a fixed point.

Veech thm.: *every locally compact group acts freely on some compact space* ($\Rightarrow$ fails the f.p.c.)

remark: Let G act on a compactum X, $f \in C(X)$, $\xi \in X$, $\Rightarrow$
$G \ni g \mapsto f(g \cdot \xi) \in \mathbb{C}$ is right uniformly continuous bounded (RUCB):



$\forall \epsilon > 0$, $\exists V \ni e_G$, $gh^{-1} \in V \Rightarrow |f(g) - f(h)| < \epsilon$
Every $f \in \mathrm{RUCB}\,(G)$ is obtained in this way:
$X = S(G)$, the space of maximal ideals of $\mathrm{RUCB}\,(G)$, the *greatest ambit* of G.

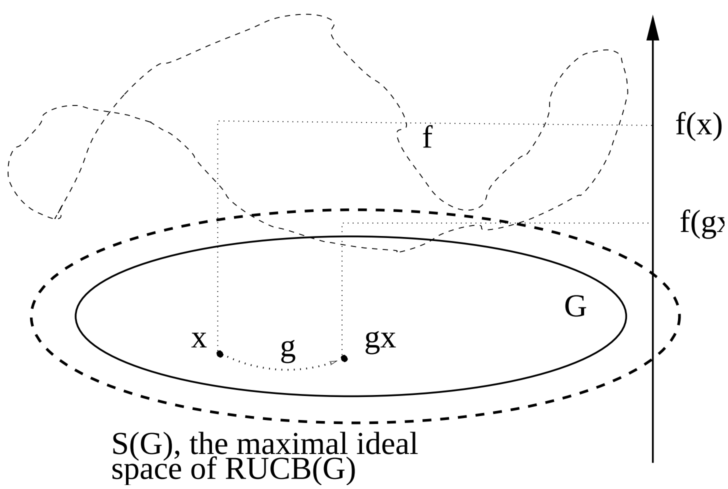
Proof of Veech thm. For every $g \in G$ construct a right uniformly continuous bounded (RUCB) $f\colon G \to \mathbb{R}^n$ with

$|f(x) - f(gx)| \geq 1$ for all $x \in G$.

[for G discrete — obvious, how.]

f extended over the maximal ideal space, $\mathcal{S}(G)$, of $\mathrm{RUCB}\,(G, \mathbb{R})$ has the same property

$\Rightarrow G$ acts freely on $\mathcal{S}(G)$.

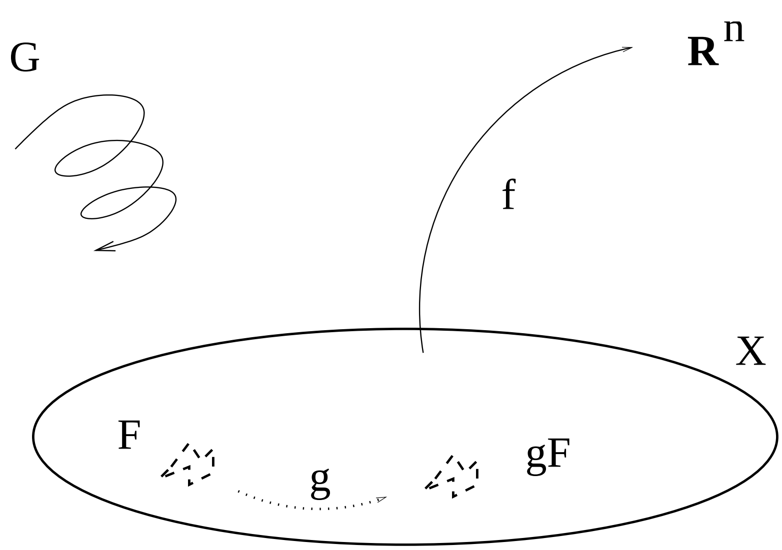


$\square$

G has the fixed point on compacta property $\Leftrightarrow$
for every $f \in \mathrm{RUCB}\,(G, \mathbb{R}^N)$, every $\epsilon > 0$ and every $g_1, g_2, \ldots, g_k \in G$,
$\exists g \in G$, s. that $|f(g) - f(g_i g)| < \epsilon$ for $i = 1, 2, \ldots, k$.
$\Updownarrow$
for every $f \in \mathrm{LUCB}\,(G, \mathbb{R}^N)$ and every finite $F \subseteq G$,
$\exists g \in G$, s. that $|f(g) - f(gx)| < \epsilon$ for $x \in F$
$\Updownarrow$
if G acts transitively by isometries on a metric sp. X,

> for every uniformly continuous bounded $f\colon X \to \mathbb{R}^N$, for each finite $F \subseteq X$, there is $g \in G$ such that $\mathrm{Osc}\,(f|_{gF}) < \epsilon$.

(Ramsey–Dvoretzky–Milman property of (X, G))



ex. 0. S_∞, symmetric group of ω,

Polish topology: $S_\infty \subset (\omega_{discrete})^\omega$.

S_∞ does *not* have the f.p.c.:

$X = \omega^{(2)} := \omega^2 \setminus \Delta_\omega$, trivial metric (0 or 1),

$F = \{(0,1),(1,0)\}$,

$$f\colon \omega^{(2)} \ni (x,y) \mapsto \begin{cases} 1, & \text{if } x < y, \\ 0, & \text{otherwise.} \end{cases},$$

For each $\sigma \in S_\infty$, $\mathrm{Osc}\,(f|_{gF}) = 1$.

ex. 1 η - a dense linear order on ω, without min and max.

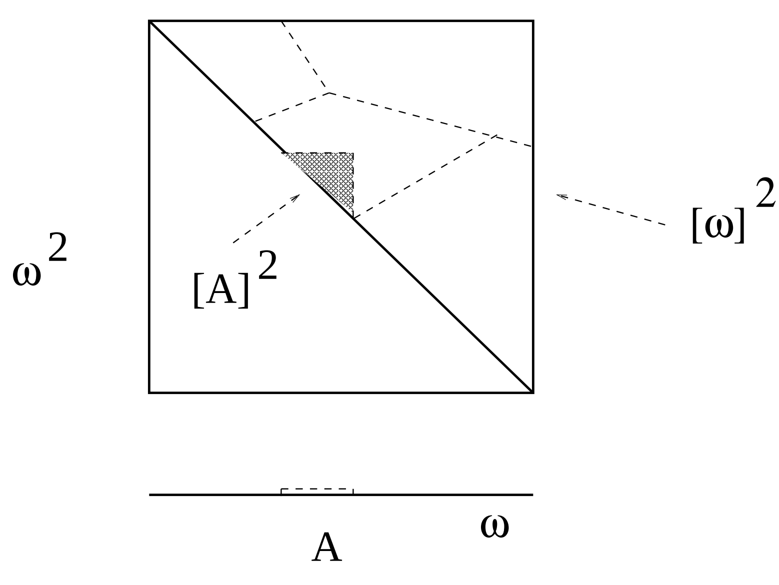
$\mathrm{Aut}\,(\eta) \subset S_\infty$: η-preserving bijections.

A closed subgroup.

Key fact: $\mathrm{Aut}\,(\eta)$ *has the f.p.c.* —

and this is a reformulation of the classical

Finite Ramsey theorem: if $[\omega]^n$ is partitioned into finitely many subsets, then for every k there is an $A \subseteq \omega$ with $|A| = k$ and $[A]^n$ monochromatic.



• St_F, for finite $F \subset \omega$, form open nbhd basis at e.

• $\mathrm{Aut}(\eta)/\mathrm{St}_F \cong [\omega]^n$, where $n = |F|$, with the discrete metric.

$\triangleleft$ G has f.p.c.

$\Leftrightarrow$

$\forall n$, $([\omega]^n, \mathrm{Aut}(\eta))$ has the R-D-M property

$\Leftrightarrow$

[enough to consider functions with finite range]

for each finite colouring of $([\omega]^n$ and each finite $B \subseteq [\omega]^n$, there is a $\sigma \in \mathrm{Aut}(\eta)$ with $\sigma(B)$ monochromatic

$\Leftrightarrow$

apply Ramsey thm. to $B = [A]^n$ and notice that $\mathrm{Aut}(\eta)$ acts transitively on k-subsets of ω. $\triangleright$

(the speaker, 1998.)

corol.: $\mathrm{Homeo}_+[0,1]$ and $\mathrm{Homeo}_+\mathbb{R}$ have f.p.c.

$\triangleleft$ $\mathrm{Aut}(\eta) \to \mathrm{Homeo}_+[0,1]$ has dense range. $\triangleright$

Lemma. *If $H < G$ is a closed subgroup, H has f.p.c., and G/H is compact, then G/H is a universal minimal compact G-space.*

$\triangleleft$ G acts on $X \Rightarrow \exists$ G-fixed $\xi \in X \Rightarrow G \ni g \mapsto g\xi \in X$ factors through G/H. $\triangleright$

corol.: *$\mathbb{S}^1$ is the universal minimal flow for $\mathrm{Homeo}_+(\mathbb{S}^1)$.*

For closed manifolds X in $\dim X > 1$ this is no longer true:

thm. (V.V. Uspenskij) *The universal minimal flow is never 3-transitive.*

Q.: what is $\mathrm{M}(\mathrm{Homeo}(X))$, X is a compact manifold or the Hilbert cube?

Q.: is $\mathrm{diff}^k[0,1]$ amenable ($k \geq 1$)?

ex. 2. $U(l_2)_s$, the full unitary group of the infinite dimensional Hilbert space with the strong operator topology ($\hookrightarrow (l_2)^{l_2}$), has the f.p.c. property [Gromov and Milman, 1983]

$\Rightarrow$ R–D–M property of $(U(l_2), \mathbb{S}^\infty)$ — at the heart of Milman's proof of Dvoretzky theorem on almost spherical sections of convex bodies.

ex. 3. (Glasner; indep., Furstenberg and Weiss.) $L_1(X, U(1))$ has f.p.c. property.

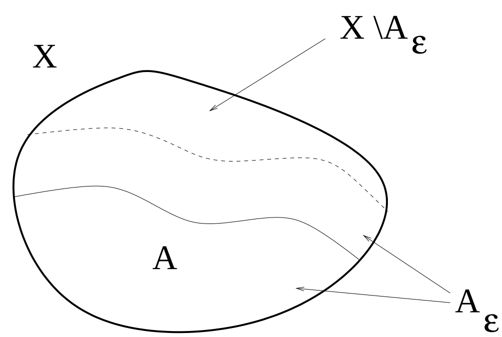
In most examples, the proofs go thus:

concentration of measure $\Rightarrow$ Fixed point on compacta property $\Rightarrow$ Ramsey-type results

The phenomenon of concentration of measure on high-dimensional structures

$X = (X, d, \mu)$ is a metric space with probability measure. ('mm-space')

Let $A \subseteq X$, $\mu(A) \geq \frac{1}{2}$, and $\epsilon > 0$.



• *For high-dimensional X, the measure of the 'cap' $\mu(X \setminus A_\epsilon) \approx 0$ even for small $\epsilon > 0$.*

('a simple but nontrivial observation' — Gromov)

def. a family (X_n, d_n, μ_n) of mm-spaces is a *Lévy family* if whenever $A_n \subseteq X_n$ and $\liminf \mu_n(A_n) > 0$, then for all $\epsilon > 0$

$$\lim \mu_n((A_n)_\epsilon) = 1.$$

ex. 1: unit spheres $\mathbb{S}^n$ with rotation-invariant probability measures and Euclidean (or geodesic) distances (Paul Lévy);

ex. 2: groups $\mathrm{SU}(n)$ with Hilbert–Schmidt (or uniform operator) distance and Haar measure (Gromov's isoperimetric ineq.);

ex. 3: permutation groups S_n with Hamming distance,

$$d(\sigma, \tau) = \frac{1}{n} \left| \{ i \colon \sigma_i \neq \tau_i \} \right|,$$

and normalized counting measure (Maurey);

ex. 4: the finite powers X^n of a probability space $X = (X, \mu)$ with the Hamming-type distance

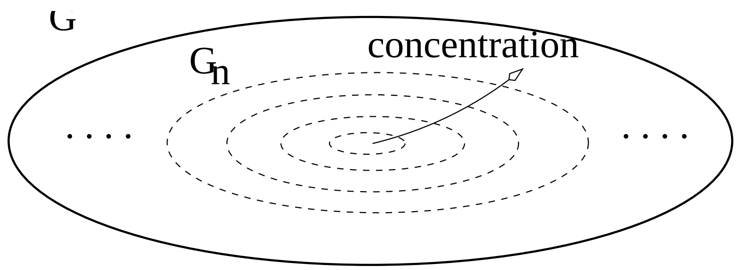
$$d(x, y) = \frac{1}{n} \left| \{ i \colon x_i \neq y_i \} \right|$$

and the product measure $\mu^{\otimes n}$ (Schechtman, Talagrand).

def. (Gromov and Milman) A topological group G is *Lévy* if there are compact subgroups

$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq \cdots \subseteq G$$

such that
(i) $\cup G_n$ is everywhere dense in G,
(iii) (G_n, d, μ_n) form a Lévy family, where
- μ_n is a normalized Haar measures on G_n,
- d is a right-invariant metric on G.



ex. 1: $G = U(l_2)_s$, $G_n = \mathrm{SU}\,(n)$.

ex. 2: $G = L_1([0,1], U(1))$, $G_n = U(1)^n$, tori (formed by simple functions on elements of a refining seq. of partitions of $[0,1]$).

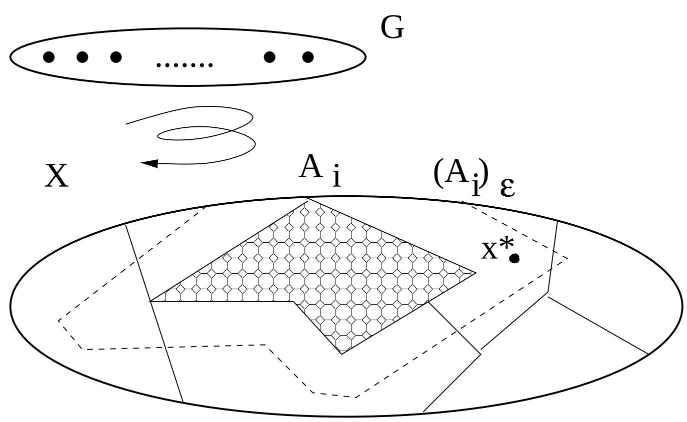
thm. (Gromov—Milman) Every Lévy group G has f.p.c.

Idea behind the proof:
Let a group G act on an mm-space $X = (X, d, \mu)$, preserving metric and measure.

Let $\alpha(\epsilon)$ drop off sharply ('X is concentrated').
E.g., $\mu(A) \geq \frac{1}{7} \Rightarrow \mu(A_{1/10}) > 0.99$.
Partition $A = A_1 \cup A_2 \cup \ldots \cup A_7$,
choose $g_1, g_2, \ldots, g_{100} \in G$,
for some i, $\mu(A_i) \geq 1/7$, $\Rightarrow$ the translates

$$g_1(A_i)_{1/10},\ g_2(A_i)_{1/10},\ \cdots,\ g_{100}(A_i)_{1/10}$$

have a point x^* in common.



ex.: The topological groups $U(l_2)_s$ and $L_1([0,1], U(1))$ have f.p.c.

von Neumann and C^*-algebras

Equip the unitary group $U(M)$ of a von Neumann algebra M with the $\sigma(M, M_*)$-topology. (It is a group topology.)

thm. (Thierry Giordano and the speaker, 2001)
A von Neumann algebra M is approximately finite dimensional $\Leftrightarrow$
the unitary group $U(M)_*$ is the direct product of a compact group and group with f.p.c. property.

ex. 1: $M = \mathcal{B}(H), U(M)_* = U(\mathcal{H})_s$.

ex. 2: $M = L^\infty(0,1), U(M)_* = L_1((0,1), U(1))$.

For a C^*-algebra A, the unitary group $U(A)$ with the $\sigma(A, A^*)$-topology is a topological group.

thm. (Th. G. – V.P.)
A C^*-algebra A is nuclear $\Leftrightarrow$
every minimal compact $U(A)$-flow is equicontinuous.

The Urysohn space

The *Urysohn metric space,* $\mathbb{U}$:

• a complete separable metric space,

• ω-homogeneous (every isometry between two finite subspaces extends to an isometry of $\mathbb{U}$),

• contains an isometric copy of every separable metric space.

This $\mathbb{U}$ is unique up to an isometry.

(Vershik: the completion of ω w.r.t. a 'sufficiently random' metric is almost surely $\cong \mathbb{U}$.)

thm. (the speaker, 2000)
The isometry group $\mathrm{Iso}\,(\mathbb{U})$ *(with the compact-open topology) has the f.p.c. property.*

Non-separable case: The same, but no uniqueness.

thm. (Uspenskij) *Every top. group* $\hookrightarrow \mathrm{Iso}\,(U')$ *for a generalized Urysohn space* U'*.*

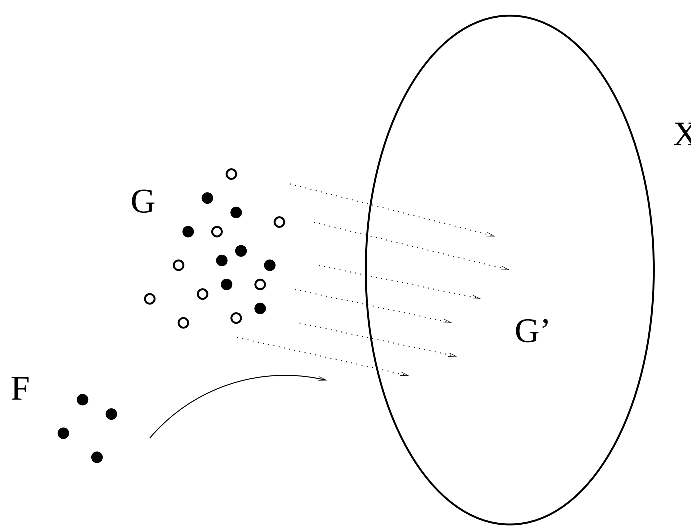
corol. *Every top. group* $\hookrightarrow$ *an amenable top. group.*

(Cf.: a closed subgroup of an amenable LC is amenable.

De la Harpe: $F_2 \hookrightarrow U(\mathcal{H})$.)

Ramsey-type results for metric spaces

thm. *Let X be an ω-homogeneous metric space.* $\mathrm{Aut}(X)$ *has f.p.c.* $\Leftrightarrow$
for every finite metric space $F \subseteq X$, if the space of isometric embeddings $X^{\hookleftarrow F}$ is coloured using finitely many colours, then for every finite metric subspace $G \subseteq X$ and every $\epsilon > 0$ there is an isometric copy of G, $G' \subseteq X$, such that all isometric embeddings $F \hookrightarrow X$ that factor through G' are monochromatic up to ϵ.



corollaries: *for $\mathbb{U}$, l_2, and for the sphere in l_2.*

Using concentration in (S_n)

Recall: (S_n) is a Lévy family (Maurey).
One cannot use it to prove that S_∞ has f.p.c.

Let X be a non-atomic standard Borel measure space, either finite or sigma-finite.
$\mathrm{Aut}\,(X)$ — the group of measure-preserving automorphisms.
$\mathrm{Aut}^*(X)$ — the group of measure class preserving automorphisms.
The weak topology: induced via the quasi-regular representation of $\mathrm{Aut}^*(X)$ in $L^2(X)$.

thm. (Thierry Giordano and the author) The groups $\mathrm{Aut}\,(X)$ and $\mathrm{Aut}^*(X)$ with the weak topology has f.p.c. property.

$\triangleleft$ $\mathrm{Aut}\,(X)$: Rokhlin lemma, approximation with groups of interval exchange transformations. $\triangleright$

$\mathrm{Aut}\,(X)$ with the uniform topology does not have f.p.c.

Dynamics of S_∞

Since S_∞ does not have f.p.c. $\Rightarrow$ it admits minimal actions on non-trivial compacta.

thm. (Eli Glasner—Benji Weiss, 2001)
The universal minimal compact flow for S_∞ is the set of all linear orders on ω, considered as a top. subspace of $\{0,1\}^{\omega\times\omega}$, with the natural action of S_∞.
This action is uniquely ergodic, with the inv. measure supported on all dense orders.

(and of course, once the result is proved, you can do it simpler!)

$\triangleleft$ Let $\mathcal{M}(S_\infty)$ be the universal minimal flow.
$\mathrm{Aut}\,(\eta)$ has f.p.c. $\Rightarrow$ it has a fixed pt, $\xi \in \mathcal{M}(S_\infty)$.
$S_\infty/\mathrm{Aut}\,(\eta)$ is the set of all dense orderings on ω.
$S_\infty \ni \tau \mapsto \tau(\xi)$ factors through an equivariant, right u.c. map $S_\infty/\mathrm{Aut}\,(\eta) \to \mathcal{M}(S_\infty)$.
Easy to check that the equivariant compactification of $S_\infty/\mathrm{Aut}\,(\eta)$ is the set of all linear orders. $\triangleright$

Concentration to a non-trivial subspace

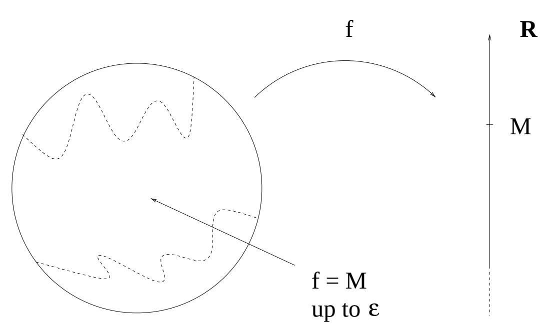
X is an mm-space, $f\colon X \to \mathbb{R}$ Lipschitz-1, M is a median value of f:

$$\mu\{f(x) \leq M\} \geq \frac{1}{2}, \mu\{f(x) \geq M\} \geq \frac{1}{2},$$

then with high probability f differs from M very little:

$$\mu\{|f(x) - M| > \epsilon\} \leq 2\alpha(\epsilon).$$

A 'nice' function f on a high-dimensional space 'concentrates near one value' ($=$ is, probabilistically, 'almost constant.')



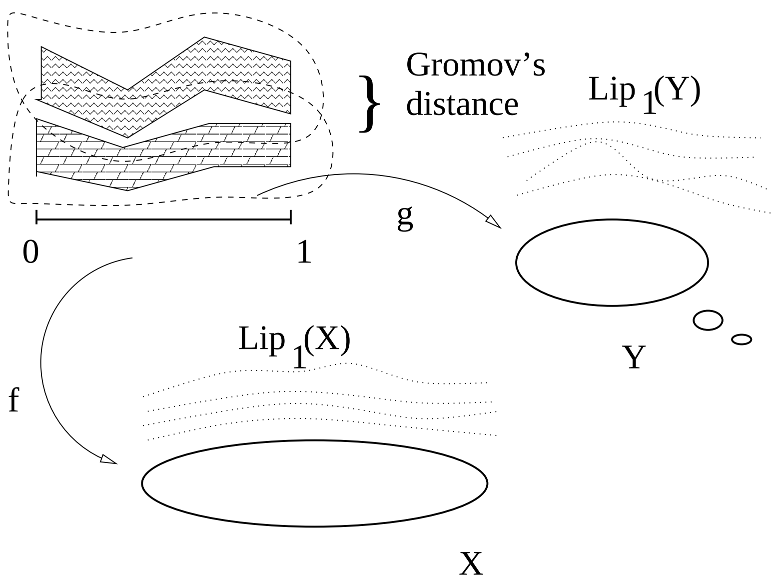
$\therefore$ One can define a metric $H\mathcal{L}_1$ on the isomorphism classes of mm-spaces (Gromov)so that

(X_n) is a Lévy family $\Leftrightarrow X_n \to \{*\}$.

Parametrize $f\colon (0,1) \to X$, $g\colon (0,1) \to Y$,

pull back $\mathrm{Lip}_1(X)$ and $\mathrm{Lip}_1(Y)$,

$H\mathcal{L}_1(X,Y)$ = the Hausdorff distance between the pulled-back spaces [w.r.t. a metric determining convergence in measure].



Concentration to a nontrivial space: $X_n \to X$.

Very few natural examples.

Q.: theory, technique: concentration of subobjects of G to the universal minimal flow?

Some more questions

Q. Is $C^\infty(X,G)$ amenable, X is a compact manifold, G is a simple compact Lie group?

($\in$ *math. physics*)

to begin with: what is the minimal flow for $C([0,1],G)$, G is a compact group?

F.p.c. property $\Rightarrow$ in a particular case the R-D-M property of the unit sphere in ℓ^2.

Q.: (Vitali Milman): does $\exists$ a property of top. groups which, for $G = U(\ell^2)$, would 'correspond to' (and imply) the distortion property of ℓ^2?

Distortion property of ℓ^2 (Odell and Schlumprecht): $\exists$ an equivalent norm $\|\cdot\|$ and $\lambda > 1$ s.t. for every infinite-dimensional subspace $V \subseteq \ell^2$, $\exists x,y \in V$, $\|x\|_2 = \|y\|_2 = 1$, $\|y\|/\|x\| > \lambda$.

examples! — what concrete 'large' ('massive,' 'infinite dimensional') groups have f.p.c.? what are their universal minimal flows? any subobjects that concentrate to those flows? ...

Reading suggestions:

The introductory notes *and* references therein:

V. Pestov, mm*-Spaces and group actions,*
to appear in L'Enseignement Mathématique,

`http://arXiv.org/abs/math.FA/0110287`

Have several hard copies with me right now.

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