Galois theory for corings and cleft entwining structures

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1. Survey of descent theory

The problem of descent theory

$B \to A$ an extension of (commutative) rings.
1) N an "object" (module, algebra,...) defined over A
Does there exist an object M over B such that $M \otimes_B A \cong N$?
2) Classify all these forms.

1) Galois descent theory (Serre, LNM 5, 1965)
l/k Galois field extension.
N descends to M iff there exists a Galois descent datum:

$$\varphi : \ G \to Aut_k(N) \quad \text{with} \quad \varphi(\sigma) \ \sigma\text{-semilinear}$$

The descended module is $M = N^G$.

2) Grothendieck (1959): descent theory for schemes

3) Knus-Ojangruren (LNM 389, 1974): affine version of Grothendieck descent.
Let $i : B \to A$ be a morphism of commutative rings.
Descent datum: (N, g), with $N \in \mathcal{N}_A$ and

$$g: \; A \otimes_B N \to N \otimes_B A \; \text{ in } \; \mathcal{M}_{A \otimes_B A}$$

such that

$$g2 = g_3 \circ g_1 : \; A \otimes_B A \otimes_B N \to N \otimes_B A \otimes_B A$$

$$\mu_N(g(1 \otimes_B m)) = m$$

Adjoint pair of functors (F, G) between $\mathcal{M}_B$ and $\underline{\text{Desc}}(A/B)$.

$$G(N, g) = \{n \in N \mid g(1 \otimes n) = n \otimes 1\}$$

Equivalence of categories if and only if $B \to A$ is pure as a morphism of B-modules. A/B faithfully flat is a sufficient condition.

4) Cippola (Discesa fedelmente piatta dei moduli, 1976), Ph. Nuss (1997)

$i : B \to A$ be a morphism of noncommutative rings.

"Descent datum": (N, ρ) with $N \in \mathcal{M}_A$, $\rho : N \to N \otimes_B A$ such that, with

$$\rho(n) = \sum_i n_i \otimes a_i$$

$$\sum_i n_i a_i = n \quad ; \quad \sum_i \rho(m_i) \otimes a_i = \sum_i n_i \otimes 1 \otimes a_i$$

Again we have an adjoint pair of functors between $\mathcal{M}_B$ and $\underline{\text{Desc}}(A/B)$. It is an equivalence if A/B is faithfully flat.

5) Graded ring theory

Let A be a G-graded module, and $A_e = B$. We have an adjoint pair of funtors between $\mathcal{M}_B$ and gr_A^G. It is an equivalence if and only if A is strongly graded.

6) H a Hopf algebra, A an H-comodule algebra, $B = A^{\mathrm{co}H}$. Relative Hopf module N

$$\rho(nh) = \rho(n)\rho(h)$$

We have an adjoint pair of functors between $\mathcal{M}_B$ and $\mathcal{M}_A^H$.
A is an H-Galois extension of B if

$$can: \ A \otimes_B A \to A \otimes H$$

$$can(a \otimes a') = a\rho(a') = aa'_{[0]} \otimes a'_{[1]}$$

is an isomorphism. If moreover A/B is faithfully flat, then we have an equivalence of categories.

7) Brzeziński and Majid introduced the notion of coalgebra Galois extension (1996).

2. Generalized Hopf modules

Doi-Hopf modules (Doi-Koppinen 1992-1995)

Let

- H a Hopf algebra
- A a (right) H-comodule algebra
- C a (right) H-module coalgebra

N is called a Doi-Hopf module if C coacts on M, A acts on N, and

$$\rho(na) = n_{[0]}a_{[0]} \otimes n_{[1]}a_{[1]}$$

Special cases: modules, comodules, (relative) Hopf modules, graded modules, Yetter-Drinfeld modules.

Entwined modules (Brzeziński-Majid (1996)

Let A be an algebra, C a coalgebra, and $\psi : C \otimes A \to A \otimes C$. Write

$$\psi(c \otimes a) = a_\psi \otimes c^\psi = a_\Psi \otimes c^\Psi$$

$$\begin{aligned}
&(ab)_\psi \otimes c^\psi = a_\psi b_\Psi \otimes c^{\psi\Psi} \\
&1_\psi \otimes c^\psi = 1 \otimes c \\
&a_\psi \otimes \Delta(c^\psi) = a_{\psi\Psi} \otimes c^\Psi_{(1)} \otimes c^\psi_{(2)} \\
&\varepsilon(c^\psi) a_\psi = \varepsilon(c) a
\end{aligned}$$

(A, C, ψ) is called a right-right entwining structure. N is called an entwined module if

$$\rho(na) = n_{[0]} a_\psi \otimes n^\psi_{[1]}$$

Doi-Hopf modules are a special case:

$$\psi : \ C \otimes A \to A \otimes C, \quad \psi(c \otimes a) = a_{[0]} \otimes c a_{[1]}$$

Takeuchi (1999): Entwined modules can be viewed as comodules over a coring.

3. Corings

A is a ring. An A-coring is a coalgebra $\mathcal{C}$ in ${}_A\mathcal{M}_A$. This means that we have (A, A)-bimodule maps

$$\Delta_{\mathcal{C}} : \ \mathcal{C} \to \mathcal{C} \ \text{ and } \ \varepsilon_{\mathcal{C}} : \ \mathcal{C} \to A$$

such that the usual coassociativity and counit properties hold. We write

$$\Delta_{\mathcal{C}}(c) = c_{(1)} \otimes_A c_{(2)}$$

Right $\mathcal{C}$-comodule: right A-module, with right A-linear map

$$\rho^r : \ M \to M \otimes_A \mathcal{C}$$

satisfying the usual coassociativity and counit properties.

The category of right $\mathcal{C}$-comodules is denoted by $\mathcal{M}^{\mathcal{C}}$.

Examples

1) let $B \to A$ be a ring morphism, and take $\mathcal{D} = A \otimes_B A$.

$$\Delta_{\mathcal{D}} : \ A \otimes_B A \to A \otimes_B A \otimes_A A \otimes_B A \cong A \otimes_B A \otimes_B A$$

$$\Delta_{\mathcal{D}}(a \otimes b) = a \otimes 1 \otimes b$$

$$\varepsilon_{\mathcal{D}}:\ A \otimes_B A \to A, \quad \varepsilon_{\mathcal{D}}(a \otimes a') = aa'$$

Let N be a right A-module, and $\rho^r : \ N \to N \otimes_A A \otimes_B A \cong N \otimes_B A$.
(N, ρ^r) is a right $\mathcal{D}$-comodule if and only if it is a "descent datum" in the sense of Cipolla.
$\mathcal{D}$ is called the canonical coring.

2) Let (A, C, ψ) be a right-right entwining structure. Put

$$\mathcal{C} = A \otimes C$$

$\mathcal{C}$ is an A-bimodule:

$$a'(a'' \otimes c)a = a'a''a_\psi \otimes c^\psi$$

Comultiplication and counit:

$$\Delta_{\mathcal{C}}:\ A \otimes C \to A \otimes C \otimes_A A \otimes C \cong A \otimes C \otimes C$$

$$\Delta_{\mathcal{C}}(a \otimes c) = a \otimes c_{(1)} \otimes c_{(2)}$$

$$\varepsilon_{\mathcal{C}}(a \otimes c) = \varepsilon(c)a$$

A right $A \otimes C$-comodule is nothing else then an entwined module
(identify $M \otimes C$ and $M \otimes_A A \otimes C$).

$x \in \mathcal{C}$ is grouplike $\Longleftrightarrow$ $\Delta_{\mathcal{C}}(x) = x \otimes_A x$ and $\varepsilon_{\mathcal{C}}(x) = 1$. $G(\mathcal{C})$ will be the set of grouplike elements.

Take $i: B \to A$, $\mathcal{D} = A \otimes_B A$ and $\mathcal{C}$ an arbitrary A-coring.

Lemma

$$\begin{aligned} \text{Hom}_{\text{coring}}(\mathcal{D}, \mathcal{C}) &\cong \text{G}(\mathcal{C})^{\text{B}} \\ &= \{x \in G(\mathcal{C}) \mid bx = xb, \text{ for all } b \in B\} \end{aligned}$$

Proof: The homomorphism can corresponding to x is given by

$$can(a \otimes_B a') = axa'$$

Lemma $G(\mathcal{C})$ is in bijective correspondence with maps $\rho : A \to A \otimes_A \mathcal{C}$ making A into a $\mathcal{D}$-comodule.

Proof. Put $\rho(a) = xa$.

Corollary Let $i : A \to B$ be a ring morphism, and $\mathcal{C}$ an A-coring. $\mathcal{C} \cong A \otimes_B A$ if and only if there exists $x \in G(\mathcal{D})^B$ such that can is bijective.

Definition (Brzezińksi) Let $(\mathcal{C}, x)$ be an A-coring with a fixed grouplike, and

$$\begin{aligned} B = A^{\mathrm{co}\mathcal{C}} &= \{b \in A \mid \rho(a) = ax\} \\ &= \{b \in A \mid xa = ax\} \end{aligned}$$

$(\mathcal{C}, x)$ is called a Galois coring if can is bijective.

Proposition (Wisbauer) TFAE

- $(\mathcal{C}, x)$ is a Galois coring

- $\varphi_C :$ $\mathsf{Hom}^{\mathcal{C}}(\mathsf{A}, \mathcal{C}) \otimes_{\mathsf{B}} \mathsf{A} \to \mathcal{C}$,
  $\varphi_C(f \otimes a) = f(a)$, is an isomorphism

- $\varphi_N :$ $\mathsf{Hom}^{\mathcal{C}}(\mathsf{A}, \mathsf{N}) \otimes_{\mathsf{B}} \mathsf{A} \to \mathsf{N}$,
  $\varphi_N(f \otimes a) = f(a)$, is an isomorphism for every $(\mathcal{C}, A)$-injective $N \in \mathcal{M}^{\mathcal{C}}$.

Remark that $\mathsf{Hom}^{\mathcal{C}}(\mathsf{A}, \bullet) \cong (\bullet)^{\mathsf{co}\mathcal{C}}$.

If $(\mathcal{C}, x)$ is Galois, then (obviously) $\mathcal{M}^{\mathcal{C}}$ and $\mathcal{M}^{\mathcal{D}}$ are isomorphic. Assume that we also know that $\mathcal{M}^{\mathcal{D}}$ and $\mathcal{M}_B$ are equivalent (e.g. if ${}_BA$ is faithfully flat). Then we have an equivalence of categories

$$(F, G) : \mathcal{M}_B \to \mathcal{M}^{\mathcal{C}}$$

$$F(M) = M \otimes_B A \quad ; \quad G(N) = N^{\mathrm{co}\mathcal{C}}$$

Proposition (Wisbauer) TFAE

- $(\mathcal{C}, x)$ is Galois and ${}_BA$ is faithfully flat
- ${}_A\mathcal{C}$ is flat and A is a projective generator in $\mathcal{M}^{\mathcal{C}}$
- ${}_A\mathcal{C}$ is flat and (F, G) is an equivalence

Examples

1) Let (A, C, ψ) be an entwining structure, and $x \in C$ grouplike. Then $1 \otimes x \in G(A \otimes C)$. We recover the definition of coalgebra Galois extension.

2) Assume that a finite group G act as a group of automorphisms on a k-algebra A, such that $A^G = k$. Put

$$\mathcal{C} = \oplus_{\sigma \in G} A v_\sigma$$

with

$$av_\sigma b = a\sigma(b)v_\sigma, \quad \varepsilon(v_\sigma) = \delta\sigma, e$$

$$\Delta(v_\sigma) = \sum_{\tau\in G} v_\tau \otimes v_{\tau^{-1}\sigma}$$

Then $x = \sum_{\sigma\in G} v_\sigma$ is grouplike.

$$can: \ A \otimes A \to \mathcal{C}, \quad can(a \otimes b) = \sum_{\sigma\in G} a\sigma(b)v_\sigma$$

We recover the classical definition of Galois extension.

3) Let H be a Hopf algebra, and A an H-comodule algebra, $B = A^{\mathrm{co}H}$. Put

$$\mathcal{C} = A \otimes H$$

$$a'(a'' \otimes h)a = a'a''a_{[0]} \otimes ha_{[1]}$$

Take $x = 1 \otimes 1$.

$$can: \ A \otimes_B A \to A \otimes H, \quad can(a \otimes b) = ab_{[0]} \otimes b_{[1]}$$

4. Morita Theory (SC, J. Vercruysse, Shuan-hong Wang)

$$R = {}^*\mathcal{C} = {}_A\mathsf{Hom}\,(\mathcal{C}, \mathsf{A})$$

is a ring:

$$(f\#g)(c) = g(c_{(1)}f(c_{(2)}))$$

We have a ring homomorphism $i:\ A \to {}^*\mathcal{C}$

$$i(a)(c) = \varepsilon_{\mathcal{C}}(c)a$$

and a functor

$$F:\ \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_{^*\mathcal{C}}$$

$$m \cdot f = m_{[0]}f(m_{[1]})$$

F is an isomorphism if ${}_A\mathcal{C}$ is finitely generated projective.
Fix $x \in G(\mathcal{C}$, and let $B = A^{\mathrm{co}\mathcal{C}}$. we have an adjunction between $\mathcal{M}_B$ and $\mathcal{M}^{\mathcal{C}}$.
Weak structure theorem (WST): if counit is an isomorphism
Strong structure theorem (SST): if adjunction is an equivalence

Let $\mathcal{D} = A \otimes_B A$. Then ${}^*\mathcal{D} \cong {}_B\text{End}\,(\text{A})^{\text{op}}$, and

$$ {}^*can: \; {}^*\mathcal{C} \to {}^*\mathcal{D} \cong {}_B\text{End}\,(\text{A})^{\text{op}} $$

$$ {}^*can(f)(a) = f(xa) $$

Obvious facts:

- $(\mathcal{C}, x)$ Galois $\Longleftrightarrow$ ${}^*can$ is iso
- We have the converse if $\mathcal{C}$ and $\mathcal{D}$ are reflexive
- If $(\mathcal{C}, x)$ is Galois and $(\mathcal{D}, 1 \otimes 1)$ satisfies the SST, then $(\mathcal{C}, x)$ also satisfies SST
- If $(\mathcal{C}, x)$ satisfies WST, then $(\mathcal{C}, x)$ is Galois

A is a right $\mathcal{C}$-comodule, hence a right R-module. It is also a left B-module, and a (B, R)-bimodule.

${}_A\text{End}\,(\mathcal{C})$ is a left R-module:

$$(f\#\varphi)(c) = \varphi(c_{(1)}f(c_{(2)})$$

Put

$$Q = \{q \in {}^*\mathcal{C} \mid c_{(1)}q(c_{(2)}) = q(c)x, \text{ for all } c \in \mathcal{C}\}$$

Lemma: Q is an (R,B)-bimodule.

Theorem We have a Morita context

$$(B,R,A,Q,\tau,\mu)$$

$$\mu:\ Q \otimes_B A \to R, \quad \mu(q \otimes_B a) = q\#i(a)$$

$$\tau:\ A \otimes_R Q \to B, \quad \tau(a \otimes_R q) = q(xa)$$

Theorem TFAE

- τ is surjective
- $\exists \Lambda \in Q : \quad \Lambda(x) = 1$

- for every right R-module M

$$\begin{aligned}\omega_M : \; & M \otimes_R Q \to M^R \\ = \; & \{m \in M \mid m \cdot f = mf(x), \forall f \in R\}\end{aligned}$$

$$\omega_M(m \otimes_R q) = m \cdot q$$

  is bijective

Theorem Assume that ${}_A\mathcal{C}$ is finitely generated projective. TFAE

- τ is surjective

- $(\mathcal{C}, x)$ satisfies WST

- ${}_BA$ is projective and $(\mathcal{C}, x)$ is Galois

5. Cleft entwining structures (SC, J. Vercruysse, Shuanhong Wang)
Entwining structure (A, C, ψ), coring $\mathcal{C} = A \otimes C$.
Fix a grouplike $x \in c$. $1 \otimes x$ is a grouplike in $A \otimes C$.

$$^*\mathcal{C} = {}_A\mathrm{Hom}\,(\mathrm{A} \otimes \mathrm{C}, \mathrm{A}) \cong \mathrm{Hom}\,(\mathrm{C}, \mathrm{A})$$

as k-module. The multiplication is

$$(f\#g)(c) = f(c_{(2)})_\psi g(c^\psi_{(1)})$$

This algebra is denoted $\#(C, A)$.
The bimodule Q takes the following form:

$$Q = \{q \in \#(C, A) \mid q(c_{(2)})_\psi \otimes c^\psi_{(1)} = q(c) \otimes x\}$$

Remark that there is another algebra structure on $\mathrm{Hom}\,(\mathrm{C}, \mathrm{A})$, namely the usual convolution:

$$(f * g)(c) = f(c_{(1)})g(c_{(2)})$$

Proposition Assume that $\lambda : C \to A$ is convolution invertible. TFAE

- $\lambda \in Q$

- for all $c \in C$:

$$\lambda^{-1}(c_{(1)})\lambda(c_{(3)})_\psi \otimes c_{(2)}^\psi = \varepsilon(c)1_A \otimes x$$

- λ^{-1} is right C-colinear:

$$\lambda^{-1}(c_{(1)}) \otimes c_{(2)} = \lambda^{-1}(c)_\psi \otimes x^\psi$$

If such a λ exists, then we call (A, C, ψ, x) cleft.

Proposition If (A, C, ψ, x) is cleft, then τ is surjective.

Definition (A, C, ψ, x) satisfies the right normal basis property (RNB) if and only if $B \otimes C$ and A are isomorphic as left B-modules and right C-comodules.

Theorem TFAE

- (A, C, ψ, x) is cleft
- (A, C, ψ, x) satisfies SST and RNB
- (A, C, ψ, x) is Galois and satisfies RNB
- ${}^*can$ is bijective and (A, C, ψ, x) satisfies RNB