CATEGORICAL STRUCTURES AND COMMUTATOR THEORY

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1. MALTSEV CATEGORIES AND CONNECTORS

D. Bourn - M. Gran, *Centrality and connectors in Maltsev categories*, to appear in *Algebra Universalis*

2. MODULARITY AND PSEUDOGROUPOIDS

D. Bourn - M. Gran, *Categorical aspects of modularity*, Preprint

1. Maltsev categories and Connectors

Definition (Smith 1976)
A variety $\mathcal{V}$ of universal algebras is a *Maltsev variety* if $\exists\, p(x,y,z)$ such that $p(x,y,y)=x$ and $p(x,x,y)=y$.

Theorem (Maltsev 1954)
A variety $\mathcal{V}$ is Maltsev if and only if

$$R \circ S = S \circ R$$

for any R, S in $Congr(X)$ and any X in $\mathcal{V}$.

Examples

- Groups

$$p(x,y,z) = x \cdot y^{-1} \cdot z$$

- Abelian groups
- Rings, Commutative rings
- Commutative algebras, Lie algebras
- Crossed modules
- Quasigroups

$$p(x,y,z) = (x/(y\backslash y))\cdot(y\backslash z)$$

- Heyting Algebras

$$p(x,y,z) = ((x \to y) \to z) \wedge ((z \to y) \to x)$$

Definition (Carboni, Lambek, Pedicchio 1990)

A finitely complete category $\mathcal{C}$ is a **Maltsev category** if any internal reflexive relation R on any X in $\mathcal{C}$ is an equivalence relation.

Examples

- Any Maltsev variety
- Any Maltsev quasivariety: for example Ab_{tf} torsion-free abelian groups
- Topological groups, Hausdorff groups
- $\mathcal{E}^{op}$ if $\mathcal{E}$ is a topos
- Any localization of a Maltsev variety
- Any abelian category, any semi-abelian category

The theory of **centrality** for varieties was developed in

J.D.H. Smith, *Mal'cev Varieties*,
Lect. Notes Math. 554 (1976)

Definition

If R and S are equivalence relations on X, let $R \times_X S$ be the pullback

$$\begin{array}{ccc} R \times_X S & \xrightarrow{p_1} & S \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ R & \xrightarrow[d_1]{} & X \end{array}$$

$$R \times_X S = \{(x, y, z) \in X \times X \times X \mid xRySz\}$$

A **connector** between R and S is an arrow $p : R \times_X S \to X$ such that

1. $p(x, x, y) = y$ $\quad$ 1^*. $p(x, y, y) = x$

2. $p(x, y, z)Sx$ $\quad$ 2^*. $p(x, y, z)Rz$

3. $p(x, y, p(y, u, v)) = p(x, u, v)$ $\quad$ 3^*. $p(p(x, y, u), u, v) = p(x, y, v)$

Remark

The notion of connector weakens the notion of internal pregroupoid (Pedicchio).

MAIN FACTS : if $\mathcal{C}$ is Maltsev, then

- there is **at most one** connector p between R and S

- the axioms 1 and 1^* imply all the others

- when $\mathcal{C}$ is a Maltsev variety:

$$\exists!\,\text{connector}\, p\colon R \times_X S \to X \quad \Leftrightarrow \quad [R, S] = 0_X$$

- when $\mathcal{C}$ is a Maltsev category:

we write $[R,S]=0_X$ to indicate that there is a connector between R and S.

Theorem Let $\mathcal{C}$ be a regular Maltsev category. Then

1. **Symmetry**: $[R,S]=0_X \Leftrightarrow [S,R]=0_X$
2. **Inclusion in the intersection**:

$$R\cap S=0_X \Rightarrow [R,S]=0_X$$

3. **Monotonicity**: if $S_1\subseteq S_2$, then

$$[R,S_2]=0_X \Rightarrow [R,S_1]=0_X$$

4. Stability with respect to **products**:

$[R_1,S_1]=0_X$ and $[R_2,S_2]=0_Y \Rightarrow [R_1\times R_2, S_1\times S_2]=0_{X\times Y}$

5. Stability with respect to **regular images**: if $f\colon X\to Y$ is a regular epimorphism, then

$$[R,S]=0_X \Rightarrow [f(R),f(S)]=0_Y$$

6. Stability with respect to **joins**:

$$[R,S_1]=0_X \text{ and } [R,S_2]=0_X \Leftrightarrow [R,S_1\cup S_2]=0_X$$

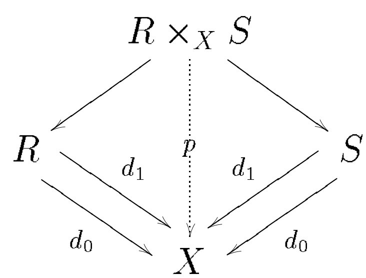
Let 2-$Eq(\mathcal{C})$ be the category whose <u>objects</u> are

$$R \underset{d_0}{\overset{d_1}{\rightrightarrows}} X \underset{d_0}{\overset{d_1}{\leftleftarrows}} S$$

and <u>arrows</u> are triples of arrows (f_R, f, f_S) in $\mathcal{C}$ as in

$$\begin{array}{ccccc} R & \underset{d_0}{\overset{d_1}{\rightrightarrows}} & X & \underset{d_0}{\overset{d_1}{\leftleftarrows}} & S \\ {\scriptstyle f_R}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_S} \\ \overline{R} & \underset{d_0}{\overset{d_1}{\rightrightarrows}} & Y & \underset{d_0}{\overset{d_1}{\leftleftarrows}} & \overline{S} \end{array}$$

Let $Conn(\mathcal{C})$ be the category whose <u>objects</u> are



and <u>arrows</u> those in 2-$Eq(\mathcal{C})$ which preserve the connector.

Lemma When $\mathcal{C}$ is Maltsev, then the forgetful functor

$$U\colon Conn(\mathcal{C}) \to 2 - Eq(\mathcal{C})$$

is a full inclusion.

Theorem Let $\mathcal{C}$ be a finitely complete category. The following conditions are equivalent:

1. $\mathcal{C}$ is Maltsev
2. the forgetful functor $U\colon Conn(\mathcal{C}) \to 2-Eq(\mathcal{C})$ is closed under subobjects
3. the forgetful functor $V\colon Grpd(\mathcal{C}) \to RG(\mathcal{C})$ is closed under subobjects

2. Modularity and Pseudogroupoids

In more general varieties of universal algebras

$$\exists!\,\text{connector}\, p\colon R \times_X S \to X \quad \not\Leftrightarrow \quad [R,S] = 0_X$$

Definition A variety $\mathcal{V}$ of universal algebras is *congruence modular* if the lattice $Congr(X)$ on any X in $\mathcal{V}$ is modular:

$$T \subseteq R \;\Rightarrow\; R \cap (S \cup T) = (R \cap S) \cup T$$

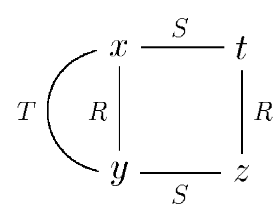
Examples

- Any Maltsev variety
- Any distributive variety: lattices, median algebras
- Any Goursat variety (i.e. $R \circ S \circ R = S \circ R \circ S$): generalized right complemented semigroups

Theorem (Gumm 1982)
A variety $\mathcal{V}$ is congruence modular

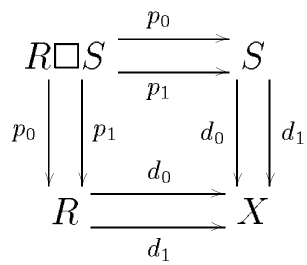
$$\Updownarrow$$

the *Shifting Lemma* holds in $\mathcal{V}$: for any X in $\mathcal{V}$, for any R, S, T in $Congr(X)$ with $R \cap S \subseteq T$, then

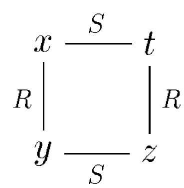


implies that tTz.

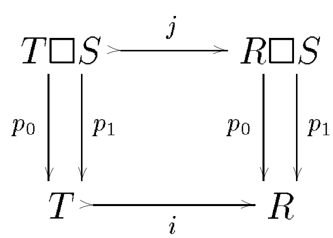
Given R and S, let $R\Box S$ be the largest double equivalence relation on R and S:



In *Sets*: $R\Box S$ is formed by all R-S rectangles:



Lemma Let $\mathcal{C}$ be a finitely complete category. TFCAE:
1. the Shifting Lemma holds in $\mathcal{C}$
2. for any X in $\mathcal{C}$, for any equivalence relations R, S, T on X with $R \cap S \subseteq T \subseteq R$ the canonical inclusion

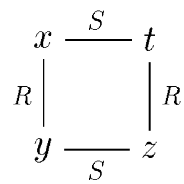


is a discrete fibration.

Definition
A finitely complete category satisfies the *Shifting property* if it satisfies the equivalent conditions in the Lemma.

Examples
Any modular variety, any regular Maltsev category, any Goursat category.

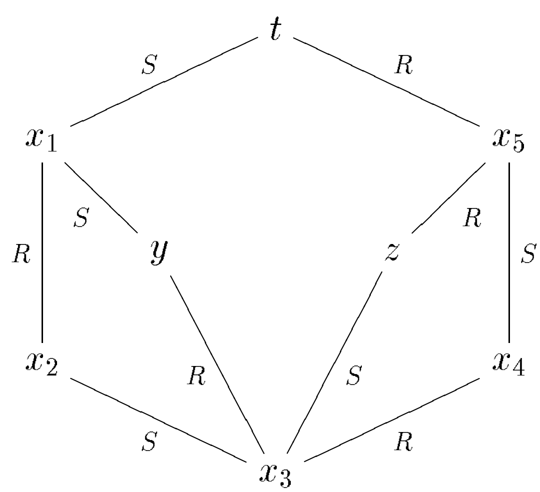
Definition (Janelidze - Pedicchio 2001)
An internal **pseudogroupoid** on R and S is an arrow $m\colon R\Box S \to X$ in $\mathcal{C}$, written as $m(x,y,t,z)$ for any



in $R\Box S$, with the following properties:

1. $m(x,x,t,y) = y$ $\quad$ 1*. $m(x,y,t,y) = x$
2. $m(x,y,t,z)Sx$ $\quad$ 2*. $m(x,y,t,z)Rz$
3. $m(x,y,t,z) = m(x,y,t',z)$
4. $m(m(x_1,x_2,y,x_3),x_4,t,x_5) = m(x_1,x_2,t,m(x_3,x_4,z,x_5))$

for any



Remark
Any connector p is a pseudogroupoid: $m(x,y,t,z) = p(x,y,z)$

Theorem (Janelidze - Pedicchio)
If $\mathcal{C}$ is a congruence modular variety then

$$\exists!\,\text{pseudogroupoid } m\colon R\Box S \to X \quad \Leftrightarrow \quad [R,S]_{mod} = 0_X$$

Consequences of the Shifting property

Let $\mathcal{C}$ be a category satisfying the Shifting property:

Theorem

1. There is **at most one** internal pseudogroupoid on two equivalence relations R and S
2. $1, 1^*, 2, 2^*, 3$ imply 4
3. The forgetful functor $W\colon Psgrd(\mathcal{C}) \to 2\text{-}Eq(\mathcal{C})$ is a full inclusion

Corollary

1. There is **at most one** connector between R and S
2. $1, 1^*, 2, 2^*$ imply $3, 3^*$
3. The forgetful functor $U\colon Conn(\mathcal{C}) \to 2\text{-}Eq(\mathcal{C})$ is a full inclusion.

Remark

For two equivalence relations having a pseudogroupoid (or a connector) structure becomes a **property**. We write

$$< R, S > = 0_X \quad \Leftrightarrow \quad \exists!\,\text{pseudogroupoid } m\colon R\Box S \to X$$

$$[R, S] = 0_X \quad \Leftrightarrow \quad \exists!\,\text{connector } p\colon R \times_X S \to X$$

Corollary

1. Given X in $\mathcal{C}$, there is at most one internal Maltsev operation
$$p\colon X \times X \times X \to X$$
2. p is associative: $p(x, y, p(z, u, v)) = p(p(x, y, z), u, v)$
3. p is commutative: $p(x, y, z) = p(z, y, x)$
4. $Mal(\mathcal{C})$ is naturally Maltsev

In the case of modular varieties:

G. Janelidze - M.C. Pedicchio
Internal categories and groupoids in congruence modular varieties, Journal of Algebra (1997)

Internal categories

Let $\mathcal{C}$ be a category satisfying the Shifting property.

Theorem
a) For an internal reflexive graph X

$$X_1 \underset{d_1}{\overset{d_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}} X_0 \quad (s_0 \colon X_0 \to X_1)$$

in $\mathcal{C}$, the following conditions are equivalent:

1. there is a (necessarily unique) **category** structure on X

2. there is a (necessarily unique) multiplication

$$m\colon X_1 \times_{X_0} X_1 \to X_1$$

satisfying the axioms

$$m(1_{d_0(f)}, f) = f = m(f, 1_{d_1(f)})$$

and

$$d_0(m(f,g)) = d_0(f) \qquad d_1(m(f,g)) = d_1(g)$$

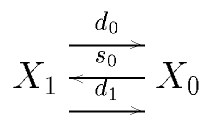
b) The forgetful functor $Cat(\mathcal{C}) \to RG(\mathcal{C})$ is a full inclusion.

When $\mathcal{C}$ is regular and satifies the Shifting property:

Proposition For two equivalence relations R and S on X the following conditions are equivalent:

1. $[R, S] = 0_X$
2. $< R, S >= 0_X$ and $R \circ S = S \circ R$

Corollary For an internal reflexive graph X



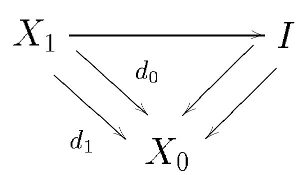
in $\mathcal{C}$ the following conditions are equivalent:

1. there is a (necessarily unique) **groupoid** structure on X

2. $[R[d_0], R[d_1]] = 0_{X_1}$

3. $< R[d_0], R[d_1] >= 0_{X_1}$ and $R[d_0] \circ R[d_1] = R[d_1] \circ R[d_0]$

4. $< R[d_0], R[d_1] >= 0_{X_1}$ and the regular image I of X



is an equivalence relation.

Central Extensions

A **central extension** is a regular epimorphism

$$f\colon A \to B$$

such that

$$[R[f], 1_A] = 0_A.$$

Let $Centr(\mathcal{C})$ be the category of central extensions in $\mathcal{C}$.

An internal groupoid

$$X_1 \times_{X_0} X_1 \ \underset{p_0}{\overset{p_1}{\underset{m}{\rightrightarrows\!\!\!\!\!\!\rightarrow}}}\ X_1 \ \underset{d_1}{\overset{d_0}{\underset{\overleftarrow{s_0}}{\rightrightarrows}}}\ X_0$$

is **connected** if the arrow

$$(d_0, d_1)\colon X_1 \to X_0 \times X_0$$

is a regular epi.

Let $ConnGrpd(\mathcal{C})$ be the category of internal connected groupoids in $\mathcal{C}$.

Theorem
If $\mathcal{C}$ is a pointed Barr-exact category satisfying the Shifting property, then

$$Centr(\mathcal{C}) \simeq ConnGrpd(\mathcal{C}).$$