ON THE RANGES OF BIMODULE PROJECTIONS (PRELIMINARY VERSION)

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Abstract. We develop a symbol calculus for bimodule maps over a masa, that allows us to study bounded idempotents.

1. Introduction

Let $\mathcal{D} \subseteq B(\mathcal{H})$ be a masa acting on a separable Hilbert space $\mathcal{H}$ and let $P : B(\mathcal{H}) \to B(\mathcal{H})$ be an idempotent $\mathcal{D}$-bimodule map.

Let $\mathcal{M} = P(B(\mathcal{H}))$ be its range, and assume that $\mathcal{M}$ is w*-closed. The purpose of this note is to investigate the structure of $\mathcal{M}$.

In case P is contractive, Solel [So] conjectures that $\mathcal{M}$ must be a Ternary Ring of Operators (TRO), i.e. must satisfy $\mathcal{M}\mathcal{M}^*\mathcal{M} \subseteq \mathcal{M}$.

In case P is assumed to be w*-continuous, the conjecture has been proved by Solel [So]. We give a new proof of this, and in fact obtain the stronger conclusion that $\mathcal{M}$ must be the w*-closed sum of "full corners", $\mathcal{M} = \sum_n B(\mathcal{H}_n, \mathcal{K}_n)$ where $\{\mathcal{H}_n\}$ are pairwise orthogonal subspaces of $\mathcal{H}$, and ditto for $\{\mathcal{K}_n\}$.

In fact, representing $\mathcal{D}$ as the multiplication masa of a standard Borel space (X, μ), we show that the ω-support (see [EKS]) of the range $\mathcal{M}$ of a w*-continuous idempotent (whether contractive or not) is ω-open (as well as ω-closed). It follows [EKS] that the reflexive cover $\mathrm{Ref}(\mathcal{M})$ of $\mathcal{M}$ is strongly reflexive.

We begin by examining the case when $\mathcal{H} = \ell^2$ and $\mathcal{D} = \ell^\infty$ in some detail. In this case every $\mathcal{D}$-bimodule map Φ is well-known to be given by Schur multiplication against a fixed matrix $A = (a_{i,j})$, that is, $\Phi(X) = A * X = (a_{i,j}x_{i,j})$. We denote this map by $\Phi = S_A$.

Note that $\Phi \circ \Phi = \Phi$ is clearly equivalent to $a_{i,j}^2 = a_{i,j}$ and hence each entry of A must be either a 0 or 1. Thus every idempotent Φ can be identified in a one-to-one fashion with a subset $E \subseteq \mathbb{N} \times \mathbb{N}$ where $E = \{(i, j) : a_{i,j} = 1\}$ and we write $A = \chi_E$, where we regard the matrix A as a function of two variables.

The problem of determining ranges of bounded bimodule projections, becomes one of determining which subsets give rise to bounded bimodule projections.

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When the bimodule projection has norm less than $2/\sqrt{3}$, we present an elementary proof, based on an elementary "3 of 4" lemma, that gives a description of these sets. Our argument shows that, in contrast with ordinary projections, every bimodule projection of norm less than $2/\sqrt{3}$ is actually of norm 1. Thus, the set of possible norms of bimodule projections is not a connected subset of the reals.

Very little is known about the structure of sets such that Φ is only a bounded projection, but we give, hopefully, a little insight into this problem.

In the third section, we develop a symbol calculus for weak*-continuous bimodule maps over more general masas. One of the main advantages of our approach is that the symbol calculus allows proofs given in the discrete case to carry over to arbitrary masa's.

Note that if P is a contractive idempotent (hence $\|P\| = 1$) and its range is a $\mathcal{D}$-bimodule, then P is automatically a $\mathcal{D}$-bimodule map ([So]).

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2. The Discrete Case

In this section we develop the case where the masa is totally atomic, so our Hilbert space may be represented as ℓ^2 and the masa as ℓ^∞ acting in the usual fashion as diagonal matrices. Identifying $\ell^2 = L^2(\mathbb{N}, \mu)$, leads to the identification of $\ell^\infty = L^\infty(\mathbb{N}, \mu)$ acting as multiplication operators on this space of functions.

Definition 1. *Let X and Y be sets and let $E \subseteq X \times Y$. We say that E has the* **3 of 4 property** *provided that given any distinct pair of points $x_1 \neq x_2$ in X and any pair of distinct points $y_1 \neq y_2$ in Y, whenever 3 of the 4 ordered pairs (x_i, y_j) belong to E then the fourth ordered pair belongs to E also.*

Lemma 2. *Let X and Y be sets and let $E \subseteq X \times Y$. If E has the 3 of 4 property, then there exists an index set T, disjoint subsets $\{X_t\}_{t\in T}$ of X, and disjoint subsets $\{Y_t\}_{t\in T}$ of Y such that*

$$E = \cup_{t\in T} X_t \times Y_t.$$

Proof. Define a relation on X by $x_1 R x_2$ if and only if there exists $y \in Y$ such that (x_1, y) and (x_2, y) are both in E. The 3 of 4 property ensures that R is transitive and hence is an equivalence relation on the subset $X_0 = \{x \in X : (x, y) \in E \text{ for some } y \in Y\}$ of X. Let $\{X_t\}_{t\in T}$ denote the collection of equivalence classes of X_0.

Define $Y_t = \{y \in Y : (x, y) \in E \text{ for some } x \in X_t\}$. Again the 3 of 4 property ensures that the sets Y_t are disjoint with union equal to

$$Y_0 = \{y \in Y : (x, y) \in E \text{ for some } x \in X\}.$$

One final use of the 3 of 4 property shows that $E = \cup_{t\in T} X_t \times Y_t$. $\square$

Lemma 3. *Let* $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *and let* $S_A : M_2 \to M_2$ *denote the map given as Schur product by* A*, then* $\|S_A\| = 2/\sqrt{3}$.

Proof. Since the unitaries are the extreme points of the unit ball of M_2, it is easily seen that $\|S_A\| = \sup_\theta \left\| \begin{pmatrix} \cos\theta & \sin\theta \\ 0 & \cos\theta \end{pmatrix} \right\|$ and the result follows by computing this supremum. □

Theorem 4. *Let* $P : B(\ell^2) \to B(\ell^2)$ *be an* ℓ^∞*-bimodule map that is idempotent, let* $\mathcal{M} = P(B(\ell^2))$ *denote the range of* P *and assume that* $\|P\| < 2/\sqrt{3}$*. Then:*

(i) $P = S_{\chi_E}$ *where* $E = \cup I_m \times J_m$*, with* $\{I_m\}$ *and* $\{J_m\}$ *countable collections of disjoint subsets of* $\mathbb{N}$*,*
(ii) $\mathcal{M} = \sum_m \chi_{I_m} B(\ell^2) \chi_{J_m}$ *and* $\mathcal{M}\mathcal{M}^*\mathcal{M} \subseteq \mathcal{M}$*,*
(iii) $\|P\| = 1$.

Proof. The fact that $P = S_{\chi_E}$ for some set $E \subseteq \mathbb{N} \times \mathbb{N}$ was noted in the introduction. Choose any $x_1 \neq x_2$ and $y_1 \neq y_2$ in $\mathbb{N}$ and consider the compression of P as a map from the span of $\{e_{x_1}, e_{x_2}\}$ to the span of $\{e_{y_1}, e_{y_2}\}$. Since $\|P\| < 2/\sqrt{3}$, by the above lemma E will have the 3 of 4 property and hence by the first lemma be of the form given in (i). It is now obvious that $\mathcal{M}$ will have the form claimed in (ii). The second assertion in (ii) is immediate from this.

Alternatively, to see the second assertion in (ii), note that it is enough to assume that the matrix units $E_{i,j}, E_{k,l}$ and $E_{m,n}$ are in $\mathcal{M}$ and prove that $E_{i,j}E_{k,l}^*E_{m,n}$ is in $\mathcal{M}$. But this product will be 0 unless $j = l$ and $k = m$ in which case the product is $E_{i,n}$. However, in this case we have that $(i,j), (k,j), (k,n)$ belong to E and so again by the 3 of 4 property (i,n) is in E and so $E_{i,n} \in \mathcal{M}$.

Finally, to prove (iii), let $\{e_m\}$ denote the usual basis of ℓ^2, set $x_i = e_m$ when $i \in I_m$, set $y_j = e_m$ when $j \in J_m$ and note that $\chi_E(i,j) = \langle x_i, y_j \rangle$. Thus, $\|P\| \leq 1$ by the theorem characterizing the norms of Schur product maps, see for example [Pa]. □

It is well known that the range of a completely contractive projection is completely isometrically isomorphic to a TRO on some Hilbert space $\mathcal{H}$. This result induces a triple product on the range, but it is generally not the triple product given by the original representation of the range as a subspace of $B(\mathcal{H})$. Note that by (ii), we have that the range of P is a TRO *in the original triple product*, i.e., that the range is a sub-TRO of $B(\ell^2)$.

Remark 5. *By the above result we see that the set of possible norms of bimodule projections does not contain the interval from 1 to* $2/\sqrt{3}$*. This makes the structure of this set somewhat intriguing. Davidson has observed that the set of possible norms is closed under product and under the taking of suprema. By a result of Bhatia, Choi and Davis* [BCD]*, the number 2 is* *one of the limit points of this set. Other than these facts, not much seems to be known about this set.*

If we let $\Delta = \{E \subseteq \mathbb{N} \times \mathbb{N} : S_{\chi_E}$ is bounded $\}$, then by the results characterizing the norms of Schur product maps, it is easily seen that $E \in \Delta$ if and only if there exist bounded sequences of vectors, $\{x_i\}$ and $\{y_j\}$, such that $\chi_E(i,j) = \langle x_i, y_j \rangle$, but this characterization seems to be of little help in obtaining other conditions that characterize the sets in Δ.

It is not hard to show that Δ is an algebra of sets and so contains the algebra generated by the sets given by the above theorem. It is not currently known whether Δ equals the latter algebra.

3. A Functional Calculus

In the discrete case, every bounded bimodule map is given as s Schur product map and so is automatically weak*-continuous, but this is not the case in general. In this section we develop a functional calculus in the non-discrete case for weak*-continuous bimodule maps that allows us to treat these exactly like Shun product maps and consequently obtain exact analogues of the results of the previous section.

This functional calculus is a bit different from the one considered by Peller [Pe], and appears to have recently been discovered independently by Shulman and Kissin [KS].

Let $\mathcal{D} \subseteq B(\mathcal{H})$ be a masa acting on a separable Hilbert space $\mathcal{H}$ and let $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ be a w*-continuous $\mathcal{D}$-bimodule map. Since Φ is a bounded $\mathcal{D}$-bimodule map, a result of R. Smith [Smi] (see also [DP]) shows that Φ must be completely bounded, and in fact $\|\Phi\|_{cb} = \|\Phi\|$.

Now Haagerup [Haa] shows that a w*-continuous completely bounded $\mathcal{D}$-bimodule map such as Φ must be of the form

$$\Phi(T) = \sum_{n=1}^{\infty} F_n T G_n \quad (T \in B(\mathcal{H}))$$

for suitable $F_n, G_n \in \mathcal{D}$ satisfying $\| \sum F_n F_n^* \| < \infty$ and $\| \sum G_n^* G_n \| < \infty$.

Represent $\mathcal{D}$ as the multiplication masa of a standard (finite) Borel space (X, μ) acting on $\mathcal{H} = L^2(X, \mu)$. A standard null-set argument shows that we may choose two families $\{f_n\}, \{g_n\}$ of *Borel functions* with $F_n = M_{f_n}$ and $G_n = M_{g_n}$ for each n, and such that the series $\sum |f_n(t)|^2$ and $\sum |g_n(t)|^2$ converge for *all* $t \in X$ boundedly and in L^2 norm.

It follows that the series

$$\phi(s,t) = \sum_n f_n(s) g_n(t)$$

converges pointwise *everywhere* to a Borel function.

Conversely, let $f = (f_1, f_2, \ldots)$ and $g = (g_1, g_2, \ldots)$ be (essentially) *bounded* weakly Borel measurable functions from X into ℓ^2. Since ℓ^2 is separable, weak Borel measurability and strong Borel measurability are equivalent. Thus (i) each f_n is an essentially bounded complex-valued function and (ii)

$$\sup_{s\in X}\|f(s)\|_2^2=\sup_{s\in X}\sum_n|f_n(s)|^2\equiv B_f<\infty$$

and $B_g\equiv\sup_{s\in X}\|g(s)\|_2^2<\infty$. It follows that

$$\|f\|^2\equiv\int\|f(s)\|_2^2d\mu(s)=\sum_n\int|f_n(s)|^2d\mu(s)<\infty$$

and ditto for g, and hence the function

$$\phi(s,t)=\langle f(s),\bar{g}(t)\rangle=\sum_n f_n(s)g_n(t)$$

defines an element of the projective tensor product $L^2(X,\mu)\widehat{\otimes}L^2(X,\mu)$. Note that the series converges pointwise absolutely and boundedly and also in the projective norm. Thus the function ϕ is Borel on $X\times X$ and (essentially) bounded. Denoting by F_n (resp. G_n) the multiplication operator M_{f_n} (resp. M_{g_n}) acting on $\mathcal{H}=L^2(X,\mu)$ we observe that for every $T\in B(\mathcal{H})$ the series

$$\sum_n F_nTG_n$$

converges in the w*-topology. Indeed, denoting by $\Phi_N(T)$ the partial sum $\sum_{n=1}^N F_nTG_n$, we have, for all $\xi,\eta\in\mathcal{H}$ and $N>M$,

$$\begin{aligned}
&|\langle(\Phi_N(T)-\Phi_M(T))\xi,\eta\rangle|^2=\left|\sum_{n=M+1}^N\langle TG_N\xi,F_n^*\eta\rangle\right|^2\\
&\le\left(\sum_{n=M+1}^N\|TG_n\xi\|^2\right)\left(\sum_{n=M+1}^N\|F_n^*\eta\|^2\right)\le\|T\|^2\sum_{n=M+1}^N\|G_n\xi\|^2\sum_{n=M+1}^N\|F_n^*\eta\|^2\\
&=\|T\|^2\left(\sum_{n=M+1}^N\int|g_n(s)|^2|\xi(s)|^2d\mu(s)\right)\left(\sum_{n=M+1}^N\int|f_n(t)|^2|\eta(t)|^2d\mu(t)\right)\\
&\le\|T\|^2\left(\int\sum_{n=M+1}^N|g_n(s)|^2|\xi(s)|^2d\mu(s)\right)B_f\|\eta\|^2
\end{aligned}$$

and so

$$\|\Phi_N(T)\xi-\Phi_M(T)\xi\|^2\le\|T\|^2\left(\int\sum_{n=M+1}^N|g_n(s)|^2|\xi(s)|^2d\mu(s)\right)B_f.$$

But since the series $\int\sum_n|g_n(s)|^2|\xi(s)|^2d\mu(s)$ converges by monotone (or by dominated) convergence, it follows that the sequence $(\Phi_N(T)\xi)$ is Cauchy in L^2, hence $(\Phi_N(T))$ converges strongly. But

$$\begin{aligned}\|\Phi_N(T)\| = \left\|\sum_{n=1}^N F_n T G_n\right\| &\le \|T\| \left\|\sum_{n=1}^N F_n F_n^*\right\|^{\frac{1}{2}} \left\|\sum_{n=1}^N G_n^* G_n\right\|^{\frac{1}{2}} \\ &= \|T\| \sup_s \left(\sum_{n=1}^N |f_n(s)|^2\right)^{\frac{1}{2}} \sup_t \left(\sum_{n=1}^N |g_n(t)|^2\right)^{\frac{1}{2}} \\ &\le \|T\| \sqrt{B_f B_g}\end{aligned}$$

so the sequence $(\Phi_N(T))$ is bounded, hence the convergence is actually ultrastrong. Thus the series

$$\Phi_\phi(T) = \sum_n F_n T G_n$$

defines a bounded operator and furthermore the inequality

$$\|\Phi_\phi(T)\| \le \|T\| \sqrt{B_f B_g}$$

shows that the map

$$\Phi_\phi : B(\mathcal{H}) \to B(\mathcal{H}) : T \to \sum_n F_n T G_n$$

is continuous with norm (actually cb norm) at most $\sqrt{B_f B_g}$. Since each Φ_N is a $\mathcal{D}$-bimodule map, so is Φ_ϕ. Also, each Φ_N is clearly w*-continuous. We claim that Φ_ϕ is also w*-continuous. For this, it suffices to show that it is weak operator continuous on the unit ball $B(\mathcal{H})_1$ of $B(\mathcal{H})$. But if T is a contraction, then for all $\xi, \eta \in \mathcal{H}$, we have

$$\begin{aligned}|\langle(\Phi_\phi(T) - \Phi_N(T))\xi, \eta\rangle| &\le \|T\| \left(\int \sum_{n=N+1}^\infty |g_n(s)|^2 |\xi(s)|^2 d\mu(s)\right)^{1/2} \|\eta\| \\ &\le \left(\int \sum_{n=N+1}^\infty |g_n(s)|^2 |\xi(s)|^2 d\mu(s)\right)^{1/2} \|\eta\|.\end{aligned}$$

This shows that the function $T \to \langle \Phi_\phi(T)\xi, \eta\rangle$ is the *uniform* limit on $B(\mathcal{H})_1$ of the weak operator continuous functions $T \to \langle \Phi_N(T)\xi, \eta\rangle$ and so is itself weak operator continuous.

Note that the map Φ_ϕ acts as a multiplication operator on kernels of Hilbert Schmidt operators:

Proposition 6. *Let $\phi(s,t) = \langle f(s), \bar{g}(t)\rangle$ where f and g are (essentially) bounded (weakly) Borel measurable functions from X into ℓ^2, and let $\Phi_\phi : B(\mathcal{H}) \to B(\mathcal{H})$ be as above. The map Φ_ϕ leaves the space of Hilbert Schmidt operators invariant. If $T \in B(\mathcal{H})$ is a Hilbert Schmidt operator with kernel $k \in L^2(X \times X)$, then $\Phi_\phi(T)$ has kernel $M_\phi(k) = \phi k$. Thus Φ_ϕ acts on $L^2(X \times X)$ as multiplication by ϕ.*

Proof. Let $T = T_k$ be Hilbert Schmidt operator with kernel k. For $\xi, \eta \in \mathcal{H}$ we have

$$\begin{aligned}
\langle \Phi_\phi(T_k)\xi, \eta\rangle &= \langle \sum_{n=1}^{\infty} F_n T_k G_n \xi, \eta\rangle = \sum_{n=1}^{\infty} \langle T_k G_n \xi, F_n^* \eta\rangle \\
&= \sum_{n=1}^{\infty} \iint k(x,y) g_n(y)\xi(y) f_n(x)\overline{\eta(x)} d\mu(y) d\mu(x) \\
&= \iint \sum_{n=1}^{\infty} f_n(x) g_n(y) k(x,y)\xi(y)\overline{\eta(x)} d\mu(y) d\mu(x) \\
&= \iint \phi(x,y) k(x,y)\xi(y)\overline{\eta(x)} d\mu(y) d\mu(x) \\
&= \langle T_{\phi k}\xi, \eta\rangle
\end{aligned}$$

and so $\Phi_\phi(T_k) = T_{\phi k}$. □

Theorem 7. *Let $\phi(s,t) = \langle f(s), \bar{g}(t)\rangle$ where f and g are (essentially) bounded (weakly) Borel measurable functions from X into ℓ^2, and let $\Phi_\phi : B(\mathcal{H}) \to B(\mathcal{H})$ be as above. The following are equivalent:*

(1) *$\phi = 0$ m.a.e.*
(2) *$\phi = 0$ a.e.*
(3) *$\Phi_\phi = 0$.*

Proof. If the set

$$R = \{(s,t) \in X \times X : \phi(s,t) \neq 0\}$$

is contained in a set of the form $N \times X \cup X \times N$, where $N \subseteq X$ is null, then of course the product measure of R is 0. Thus (1) implies (2).

To show that (2) implies (3), observe that if $T = T_k$ is a Hilbert-Schmidt operator with (square-integrable) kernel k, then by Proposition 6 $\Phi_\phi(T_k) = T_{\phi k}$.

It follows that if $\phi = 0$ a.e. then $\Phi_\phi(T_k) = 0$ for any Hilbert-Schmidt operator T_k. Since Φ_ϕ is w*-continuous, we obtain $\Phi_\phi = 0$. Conversely if $\Phi_\phi = 0$ then $\phi = 0$ a.e.

It remains to prove that if the set R is null, then it must be marginally null. For this, first observe that R is (marginally equivalent to) a countable union of Borel rectangles. We use an argument of Arveson [Arv]: The set

$$\{(\xi, \eta) \in \ell^2 \times \ell^2 : \langle \xi, \eta\rangle \neq 0\}$$

is open in $\ell^2 \times \ell^2$, and hence is a countable union $\cup_n U_n \times V_n$ of open rectangles. Letting $A_n = \{s \in X : f(s) \in U_n\}$ and $B_n = \{t \in X : g(t) \in V_n\}$ we see that, since $f, g : X \to \ell^2$ are Borel functions, the sets A_n and B_n are Borel and

$$R = \{(s,t) : \langle f(s), g(t)\rangle \neq 0\} = \bigcup_n \{(s,t) : \langle f(s), g(t)\rangle \in U_n \times V_n\} = \bigcup_n A_n \times B_n$$

as claimed. Thus if the product measure of R is 0 we must have $\mu(A_n)\mu(B_n) = 0$ for all $n \in \mathbb{N}$. If $N_1 = \cup\{A_n : \mu(B_n) \neq 0\}$ and $N_2 = \cup\{B_n : \mu(A_n) \neq 0\}$

then $\mu(N_1) = \mu(N_2) = 0$ and

$$R \subseteq N_1 \times X \cup X \times N_2$$

which completes the proof. □

Definition 8. *We let $NCB_{\mathcal{D}}(B(\mathcal{H}))$ denote the algebra of weak*-continuous $\mathcal{D}$-bimodule maps from $B(\mathcal{H})$ into itself. Given a weak*-continuous $\mathcal{D}$-bimodule map Φ as above we call the m.a.e. equivalence class of the function $\phi(s,t)$ obtained above the* **symbol** *of Φ and denote it by $\Gamma(\Phi)$.*

Corollary 9. *Let $\mathcal{D}$ be represented as the multiplication masa on a standard Borel space (X, μ), let $\mathcal{B}_{mae}(X \times X)$ denote the algebra of bounded Borel functions on $X \times X$ modulo the marginally null functions. Then the map $\Gamma : NCB_{\mathcal{D}}(B(\mathcal{H})) \to \mathcal{B}_{mae}(X \times X)$ is a one-to-one homomorphism onto the subalgebra of functions that can be represented in the form $\phi(s,t) = \langle f(s), g(t) \rangle$ for any bounded Borel measurable functions f, g from X into a separable Hilbert space.*

We call the map Γ the **functional calculus** for weak*-continuous $\mathcal{D}$-bimodule maps.

Armed with the functional calculus, we can readily generalize the theorem of the previous section.

Theorem 10. *Let $P : B(\mathcal{H}) \to B(\mathcal{H})$ be a $\mathcal{D}$-bimodule map that is idempotent and weak*-continuous, let $\mathcal{M} = P(B(\mathcal{H}))$ denote the range of P and assume that $\|P\| < 2/\sqrt{3}$. Then:*

(1) *$\Gamma(P) = \chi_E$ where $E = \cup I_m \times J_m$, with $\{I_m\}$ and $\{J_m\}$ countable collections of disjoint Borel subsets of X,*
(2) *$\mathcal{M} = \sum_m \chi_{I_m} B(L^2) \chi_{J_m}$ and $\mathcal{M}\mathcal{M}^*\mathcal{M} \subseteq \mathcal{M}$,*
(3) *$\|P\| = 1$.*

Proof. Let $\Gamma(P) = \phi$, since $P \circ P = P$, by the functional calculus, $\phi^2 = \phi$ marginally almost everywhere. Thus, we can pick a Borel subset X_1 of X with $\mu(X \cap X_1^c) = 0$ such that $\phi^2 = \phi$ on $X_1 \times X_1$. Hence, there is a Borel subset E of $X_1 \times X_1$ such that $\phi = \chi_E$ as functions on $X_1 \times X_1$.

Thus, we may write $\chi_E(s,t) = \langle f(s), g(t) \rangle$ where f, g are functions into a separable Hilbert space with $\|f(s)\|\|g(t)\| < 2/\sqrt{3}$ for all s, t.

By Lemma 3, if for any $s_1 \neq s_2$ and $t_1 \neq t_2$, we have that 3 of the 4 values $\langle f(s_i), g(t_j) \rangle$ are 1, then the fourth value must also be 1.

Hence the set E satisfies the 3 of 4 property and so it must be a union of disjoint rectangles as in Lemma 2. Say, $E = \cup_{t \in T} I_t \times J_t$.

It remains to be shown that the indexing set T for the union is only countable and that each of the sets I_t and J_t are Borel. As in the proof of Theorem 7, the set $E = \{(s,t) \in X_1 \times X_1 : \phi(s,t) \neq 0\}$ can be written as a countable union of Borel rectangles, say $E = \cup_n A_n \times B_n$. Again, by the equivalence relation used to define the sets I_t and J_t, if any point in a rectangle $A_n \times B_n$ is contained in $I_t \times J_t$ then $A_n \times B_n \subseteq I_t \times J_t$. Hence, each set $I_t \times J_t$ is the union of the at most countably many Borel rectangles that are contained in it and consequently is itself a Borel rectangle. Moreover, the set T can be placed in a one-to-one correspondence with a partition of the integers, and hence is countable.

The remainder of the proof proceeds as in the proof of Theorem 4. □

As in the discrete case we have that $\mathcal{M}$ is a sub-TRO of $B(L^2)$, a result obtained by Solel [So].

Just as in the discrete case, very little is known about bimodule projections of greater norm.

More importantly, very little is known about contractive bimodule projections that are not weak*-continuous. Such projections do exist, for example projections onto the masa $\mathcal{D}$ exist and when $\mathcal{D}$ is not discrete, these cannot be weak*-continuous. Solel [So] conjectures that the range $\mathcal{M}$ of *any* contractive $\mathcal{D}$-bimodule projection satisfies $\mathcal{M}\mathcal{M}^*\mathcal{M} \subseteq \mathcal{M}$.

We now turn our attention to further properties of the symbol calculus and of bounded weak*-continuous idempotents.

Recall [EKS] that a subset $E \subseteq X \times X$ is said to be *ω-open* if it differs from a countable union of Borel rectangles by a marginally null set, and is *ω-closed* if its complement is ω-open. Thus the set E in Theorem 10 is ω-open. The following Proposition strengthens this result and provides an alternative approach:

Proposition 11. *Let $P \in NCB_{\mathcal{D}}(B(\mathcal{H}))$ be an idempotent with symbol $\Gamma(P) = \chi$. Then there exists an ω-open and ω-closed set $A \subseteq X \times X$ such that $\chi = \chi_A$ marginally almost everywhere.*

Proof. Notice first that, in the terminology of [EKS], any element ϕ of the projective tensor product $L^2(X)\widehat{\otimes}L^2(X)$ is *ω-continuous*, that is, $\phi^{-1}(U)$ is ω-open in $X \times X$ for any open set $U \subseteq \mathbb{C}$ [EKS, Theorem 6.5].

Since P is idempotent, so is its induced operator M_χ on $L^2(X \times X)$ (see Proposition 6). It follows that $\chi^2 = \chi$ almost everywhere, i.e. the set

$$B = \{(x, y) : \chi^2(x, y) - \chi(x, y) \neq 0\}$$

has product measure zero. On the other hand, since $\chi \in L^2(X)\widehat{\otimes}L^2(X)$, the function χ is ω-continuous hence so is $\chi^2 - \chi$. Thus B must be ω-open, in other words marginally equivalent to a countable union of rectangles. The fact that B has product measure zero now implies, as noted earlier, that it is actually marginally null. Replacing X by a suitable Borel subset X_1 such that $\mu(X \cap X_1^c) = 0$, we may assume that $B = \emptyset$, i.e. that $\chi^2(x, y) = \chi(x, y)$ for *all* $(x, y) \in X \times X$. Thus letting $A = \chi^{-1}(\{1\})$ we see that A is ω-closed (since χ is ω-continuous); but $A^c = \chi^{-1}(\{0\})$ is also ω-closed. □

It is shown in [EKS] that, given any space $\mathcal{S}$ of operators on $L^2(X)$, there exists an ω-closed set Ω, minimal up to marginally null sets, that *supports* all elements of $\mathcal{S}$, in the sense that if a Borel rectangle $\alpha \times \beta$ doesn't meet Ω then $M_\beta \mathcal{S} M_\alpha = \{0\}$ (here $M_\beta \in \mathcal{B}(L^2(X))$ denotes the projection onto $L^2(\beta)$). This set is called the *ω-support* of $\mathcal{S}$.

Proposition 12. *Let $P \in NCB_{\mathcal{D}}(B(\mathcal{H}))$ be an idempotent with symbol $\Gamma(P) = \chi_A$. Then the set A is (marginally equivalent to) the ω-support of $\mathcal{M} = P(B(\mathcal{H}))$.*

Proof. It is to be shown that a Borel rectangle $\alpha \times \beta$ has marginally null intersection with A if and only if $M_\beta \mathcal{M} M_\alpha = \{0\}$. Note that the relation $M_\beta \mathcal{M} M_\alpha = \{0\}$ is equivalent to $M_\beta P(T) M_\alpha = 0$ for all $T \in B(\mathcal{H})$. But, since the map $T \to M_\beta P(T) M_\alpha$ is w*-continuous, this is equivalent to $M_\beta P(T) M_\alpha = 0$ for all *Hilbert Schmidt* $T = T_k$. By Proposition 6 $M_\beta P(T_k) M_\alpha = M_\beta T_{\chi k} M_\alpha = T_h$, where $h = \chi_{\alpha\times\beta}\chi_A k$. Thus the relation $M_\beta P(T) M_\alpha = 0$ holds for all Hilbert Schmidt $T = T_k$ if and only if the set $(\alpha \times \beta) \cap A$ has product measure zero. But since this set is ω-open, as shown in the proof of the last proposition this can only happen when $(\alpha \times \beta) \cap A$ is *marginally null*. □

Since A is ω-open, it follows [EKS, Theorem 6.11] that the reflexive cover $\mathrm{Ref}(\mathcal{M})$ is in fact strongly reflexive, and is the strong closure of the linear span of the finite rank operators supported in A. In case P is actually contractive, this of course follows immediately from the fact that the range of P is a direct sum of full corners (Theorem 10).

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