THE SEMIGROUP GENERATED BY A SIMILARITY ORBIT OR A UNITARY ORBIT OF AN OPERATOR

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Abstract. Let T be an invertible operator that is not a scalar modulo the ideal of compact operators. We show that the multiplicative semigroup generated by the similarity orbit of T is the group of all invertible operators. If, in addition, T is a unitary operator, then the multiplicative semigroup generated by the unitary orbit of T is the group of all unitary operators.

INTRODUCTION

Let H be a separable infinite-dimensional complex Hilbert space and let $\mathcal{B}(H)$ be the algebra of all bounded operators on H. We consider the following question: What is the multiplicative semigroup generated by the similarity orbit of an invertible operator on H? An analogous question for the unitary group is: What is the multiplicative semigroup generated by the unitary orbit of a unitary operator?

Let us call a subset $\mathcal{S}$ of of a group $\mathcal{G}$ *conjugation invariant*, or simply *invariant* if $g^{-1}\mathcal{S}g \subseteq \mathcal{S}$ for every $g \in \mathcal{G}$. (An invariant group is also called a *normal subgroup.)* One may ask what are the invariant semigroups of the group $\mathcal{GL}(H)$ of invertible operators, or, respectively, of the group $\mathcal{U}(H)$ of unitary operators.

We prove that if T is an invertible operator that is not a scalar modulo the ideal $\mathcal{K}(H)$ of compact operators, then the multiplicative semigroup generated by the similarity orbit of T is the group of all invertible operators. Consequently, every proper invariant semigroup in $\mathcal{GL}(H)$ is included in $\mathbb{C}I + \mathcal{K}(H)$. This generalizes a Theorem of Radjavi [11] that asserts that every invertible operator is a product of a finite number (seven) of involutions and a theorem of the authors [7] that states that every invertible operator is a product of six unipotent operators.

Analogously, we show that if U is an unitary operator that is not a scalar modulo the compacts, then the semigroup generated by the unitary orbit of U is the group of all unitary operators. Consequently, every proper invariant semigroup in $\mathcal{U}(H)$ is included in $\mathbb{C}I+\mathcal{K}(H)$. This generalizes a Theorem of

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Halmos and Kakutani [9]; namely, that every unitary operator is a product of four symmetries (i.e., self-adjoint unitary operators).

In the last section we prove a result about invariant groups in the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$.

We end this introduction by noting that an additive version of the results in this paper is in [6]. A special case of the results in [6] is that every proper linear subspace of $\mathcal{B}(H)$ that is invariant under conjugation by all invertible operators (respectively, all unitary operators) is included in $\mathbb{C}I + \mathcal{K}(H)$. We also note that semigroups generated by a similarity orbit of a matrix have been investigated in [8].

1. STATEMENTS OF RESULTS

We start by stating the results about unitary operators.

Theorem A. *Let U be a unitary operator that is not the sum of a scalar and a compact operator. Then every unitary operator is a product of a finite number of operators each of which is unitarily equivalent to U.*

The following is an immediate corollary.

Corollary 1. *Every proper invariant semigroup in the group of unitary operators in included in $\mathbb{C}I + \mathcal{K}(H)$.*

In Theorem 1, if we take U to be a symmetry (i.e., a unitary operator U satisfying $U^2 = I$), and if we also assume that both ker $(U - I)$ and ker $(U + I)$ are infinite dimensional, then we recover the qualitative part of the Halmos-Kakutani [9] result that states that every unitary operator is a product of four symmetries.

The Theorem of Halmos and Kakutani has a "skew" version due to Radjavi [11]; namely, that every invertible operator is a product of seven involutions. (An *involution* is an operator whose square is the identity.) We also have the following "skew" version of Theorem A.

Theorem B. *Let T be an invertible operator which is not the sum of a scalar and a compact operator. Then every invertible operator is a product of a finite number of operators each of which is similar to T.*

As before, we conclude the following about invariant semigroups.

Corollary 2. *Every proper invariant semigroup in the group of invertible operators in included in $\mathbb{C}I + \mathcal{K}(H)$.*

The following are special cases of Theorem 2. First recall that an operator is said be *unipotent* if it is the sum of the identity and a nilpotent operator, and is said to be a *unipotent of order 2* if it is of the form $I + N$, where $N^2 = 0$.

Corollary 3. *Every invertible operator is a product of a finite number of*

(a) *involutions (cf. [11]);*
(b) *unipotents of order 2 (cf. [7]);*

(c) *invertible positive operators (cf. [10]).*

We again observe that in [11, 7, 10], the number of factors are seven, six and seven respectively. See also [13].

Proof. Parts (a) and (b) are obvious. To prove part (c), let P be an invertible positive operator that is not a scalar plus compact. By Theorem B, every invertible operator is a product of a finite number of operators each of which is similar to P. Each factor $S^{-1}PS$ is a product of two invertible positive operators since $S^{-1}PS = S^{-1}(S^{-1})^*(S^*PS)$. ∎

We end this section with the following remarks about the number of factors in Theorems A and B.

Remarks. The number of factors in Theorem A is unbounded. Indeed, if U is a unitary operator satisfying $\|U - I\| \leq 2^{-n}$, and if V is a product of n operators from the unitary orbit of U, then it is easy to see that $\|V - I\| \leq 1$. On the other hand, the proof of Theorem B given below establishes that 112 factors suffice for the factorization of that theorem. This is undoubtedly not a sharp estimate, but we make no attempt in the present work to investigate the minimum number of factors required.

2. PROOF OF THEOREM A

We denote the essential numerical range of an operator A by $W_e(A)$. For basic properties of the essential numerical range, the reader is referred to [5].

Lemma 1. *If U is a unitary operator and if zero is in the interior of the numerical range of U, then every unitary operator is a product of at most eight operators each of which is unitarily equivalent to U.*

Proof. We denote the interior of the numerical range of U by $W_e(U)^o$. Construct inductively an orthonormal sequence $\{e_n\}$ *such* that $(Ue_n, e_m) = 0$ for all n, m, as follows. Since $0 \in W_e(U)^o \subseteq W(U)$, there is a unit vector e_1 such that $(Ue_1, e_1) = 0$. Suppose now that we already have $e_1, \ldots, e_k$ such that $(Ue_n, e_m) = 0$ for all n, $m \leq k$. Let

$$\mathcal{M} = \{e_1, \ldots, e_k,\ Ue_1, \ldots, Ue_k,\ U^*e_1, \ldots, U^*e_k\}^\perp$$

and let V be the compression of U to $\mathcal{M}$. Since $\mathcal{M}^\perp$ is finite dimensional, we have $W_e(V) = W_e(U)$ and hence $0 \in W_e(V)^o$. Let e_{k+1} be a unit vector in $\mathcal{M}$ such that $(Ve_{k+1}, e_{k+1}) = 0$. Then $e_1, \ldots, e_{k+1}$ is a finite orthonormal sequence such that $(Ue_n, e_m) = 0$ for all n, $m \leq k+1$.

Let H_1 be the closed linear span of $\{e_n : n \text{ odd}\}$, let $H_3 = UH_1$ and let $H_2 = (H_1 \oplus H_3)^\perp$. The unitary operator U maps H_1 onto H_3 and hence it maps $H_2 \oplus H_3 = H_1^\perp$ onto $H_3^\perp = H_1 \oplus H_2$ and so the matrix of U relative to the decomposition: $H = H_1 \oplus H_2 \oplus H_3$ takes the form

$$U = \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ R & 0 & 0 \end{pmatrix}$$

where R is a unitary operator from H_1 onto H_3. We note that each of H_1, H_2 and H_3 is isomorphic to H.

Now let V be any unitary operator on H_3 and let

$$V_0 = \begin{pmatrix} 0 & 0 & R^*V \\ 0 & 1 & 0 \\ R & 0 & 0 \end{pmatrix}.$$

Then V_0 is a unitary operator on H and

$$UV_0UV_0^* = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & V \end{pmatrix} = \begin{pmatrix} V' & 0 \\ 0 & V \end{pmatrix}.$$

Identifying each of $H_1 + H_2$ and H_3 with H, the above computation shows that if V is a unitary operator on H, then there exists another unitary operator V' such that $V \oplus V'$ is a product of two operators unitarily equivalent to U. We now take V to be a bilateral shift of infinite multiplicity. The unitary operators V' can be written as a product V_1V_2 of two bilateral shifts of infinite multiplicity [9]. Let J be a unitary operator such that $V^* = JV_1J^*$ and let $S = JV_2J^*$. It follows that $V \oplus V'$ is unitarily equivalent to $V \oplus V^*S$ and so each of $V \oplus V^*S$ and $V^*S \oplus V$ is a product of two operators unitarily equivalent to U. So, there exist four operators unitarily equivalent to U whose product is the operator $(V \oplus V^*S)(V^*S \oplus V) = S \oplus V^*SV$ which is a bilateral shift of infinite multiplicity. Now the conclusion of the lemma follows by using, once again, the fact that every unitary operator is a product of two bilateral shifts of infinite multiplicity. ∎

Proof of Theorem A.

Suppose that U is a unitary operator which is not a scalar plus compact. The essential spectrum $\sigma_e(U)$ of U contains two distinct complex numbers λ_1 and λ_2. We may write U in the form

$$U = \begin{pmatrix} \lambda_1 1 & 0 & 0 \\ 0 & \lambda_2 1 & 0 \\ 0 & 0 & A \end{pmatrix} + K_1$$

where K_1 is a compact operator and where every direct summand is infinite dimensional, (see, e.g., [5, Theorem 4.2]). In view of Lemma 1, it suffices to show that there is a product V of a finite number of operators unitarily equivalent to U such that $0 \in W_e(V)^o$.

We consider two cases according as $\lambda_2 = -\lambda_1$ or not. In the first case,

$$U = \lambda_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & B \end{pmatrix} + K_1$$

which is unitarily equivalent to

$$\lambda_1 \begin{pmatrix} 0 & J & 0 \\ J^* & 0 & 0 \\ 0 & 0 & B \end{pmatrix} + K_1$$

for every unitary operator J. Now let R be a unitary operator such that $0 \in W_e(R)^o$. It follows that U is unitarily equivalent to each of the operators

$$U_1 = \lambda_1 \begin{pmatrix} 0 & R & 0 \\ R^* & 0 & 0 \\ 0 & 0 & B \end{pmatrix} + K_1$$

and

$$U_2 = \lambda_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & B \end{pmatrix} + K_1,$$

hence

$$U_1 U_2 = \lambda_1^2 \begin{pmatrix} R & 0 & 0 \\ 0 & R^* & 0 \\ 0 & 0 & B^2 \end{pmatrix} + K_2$$

where K_2 is compact. Therefore $0 \in W_e(U_1U_2)^o$. This ends the proof in this case.

Finally, we consider the case $\lambda_2 \neq -\lambda_1$. Let $\mu = \lambda_2/\lambda_1$, so $\mu \neq \pm 1$. It is easy to see that there exists a positive integer n such that 0 belongs to the interior of the convex hull of $\{1, \mu, \mu^2, \ldots, \mu^n\}$. For every positive integer m, we have

$$U^m = \lambda_1^m \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu^m & 0 \\ 0 & 0 & B^m \end{pmatrix} + K_m$$

where K_m is compact. So U^m is unitarily equivalent to the operator

$$V_m = \lambda_1^m \operatorname{diag}(1, \ldots, 1, \mu^m, 1, \ldots, 1, B^m) + K_m,$$

with $n+2$ direct summands and with μ^m in the $(m+1)$st position. Now let $V = V_1 V_2 \ldots V_n$, so

$$V = \lambda \operatorname{diag}\left(1, \mu, \mu^2, \ldots, \mu^n, C\right) + K$$

for a unimodular complex number λ, a bounded operator C and a compact operator K. Therefore $0 \in W_e(V)^o$ and V is a product of $n(n+1)/2$ operators that are unitarily equivalent to U. ∎

3. PROOF OF THEOREM B

We begin by stating a well-known result (see [12, Cor. 0.15]). Recall that $\sigma(A)$ denotes the spectrum of an operator A.

Lemma 2. *If $\sigma(A) \cap \sigma(B) = \emptyset$, then the operator $\begin{pmatrix} A & O \\ C & B \end{pmatrix}$ is similar to $A \oplus B$.*

To prove Theorem B, assume that T is an invertible operator which is not a scalar modulo the compacts. By a result of Brown and Pearcy [1, Theorem 2], T is similar to an operator of the form

$$T_0 = \begin{pmatrix} 0 & A & B \\ 0 & C & D \\ 1 & E & F \end{pmatrix}$$

acting on $H \oplus H \oplus H$. Let S be an arbitrary invertible operator, let

$$L_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} S & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and let $T_j = L_j^{-1} T_0 L_j$ for $j = 1, 2$. Then each of T_1 and T_2 is similar to T and

$$T_2 T_1 = \begin{bmatrix} F(S) & 0 \\ * & S \end{bmatrix},$$

where

$$F(S) = \begin{bmatrix} S^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} VA & VB \\ C & D \end{bmatrix} \begin{bmatrix} D & C \\ VB & VA \end{bmatrix}.$$

For every invertible operator X, we will show that $\sigma(\alpha X) \cap \sigma(F(\alpha X)) = \emptyset$ if $|\alpha|$ is either large enough or small enough. To prove this, notice that $F(\alpha X) = \begin{bmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{bmatrix} F(X)$, so $\|F(\alpha X)\| \leq \|F(X)\|$ for $|\alpha| \geq 1$ and hence we can choose $|\alpha|$ large enough so that $\sigma(\alpha X)$ lies outside the disc $\{z : |z| \leq \|F(X)\|\}$ which includes $\sigma(F(\alpha X))$. Similarly, for $|\alpha|$ small enough, $\sigma(\alpha X)$ is included in the disc $\left\{z : |z| < \left\|F(X)^{-1}\right\|^{-1}\right\}$, while $\sigma(F(\alpha X))$ lies outside the same disc since $\left\|F(\alpha X)^{-1}\right\| \leq \left\|F(X)^{-1}\right\|$ for $|\alpha| \leq 1$. Applying the above to $X = S$ and $X = 1$ and using Lemma 2, we conclude that there exists a scalar α such that each of the operators

$$\begin{bmatrix} F(\alpha S) & 0 \\ 0 & \alpha S \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} F\left(\alpha^{-1} 1\right) & 0 \\ 0 & \alpha^{-1} 1 \end{bmatrix}$$

is a product of two operators similar to T, and so $S \oplus F(\alpha S) F\left(\alpha^{-1} 1\right)$ is a product of four operators similar to T.

Now take S to be $U \oplus 1$ where U is a bilateral shift with infinite multiplicity and 1 is the identity operator on an infinite dimensional space. From the above, there exists an invertible operator Q on H such that $S \oplus Q$ is a product of four operators similar to T. The operator $S \oplus Q$ can be written as $U \oplus Q'$ where both U and Q' are operators on $\sum_{n \in Z} \oplus H_n$ with $H_n = H_0$ for all n and

$$U\left(\ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, \ldots\right) = \left(\ldots, x_{-2}, \boxed{x_{-1}}, x_0, x_1, \ldots\right),$$

$$Q' = \operatorname{diag}\left(\ldots, 1, \boxed{Q}, 1, 1, \ldots\right);$$

that is,

$$Q'\left(\ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2 \ldots\right) = \left(\ldots, x_{-2}, x_{-1}, \boxed{Qx_0}, x_1, x_2, \ldots\right).$$

(The box $\boxed{}$ is used to indicate the zero$^{\text{th}}$ position.) Now

$$\left(U \oplus Q'\right)\left(Q' \oplus U\right) = UQ' \oplus Q'U,$$

$$UQ'\left(\ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2 \ldots\right) = \left(\ldots, x_{-2}, \boxed{x_{-1}}, Qx_0, x_1, x_2, \ldots\right).$$

Let $J = \operatorname{diag}\left(\ldots 1, 1, \boxed{1}, Q, Q, \ldots\right)$. By direct computation, it follows that $J\left(UQ'\right)J^{-1} = U$. In the same way, we can show that $Q'U$ is similar to U. Therefore $\left(U \oplus Q'\right)\left(Q' \oplus U\right)$ is similar to a bilateral shift of infinite multiplicity. We have shown that a bilateral shift is a product of eight operators similar to T. Since each symmetry is a product of two bilateral shifts of infinite multiplicity, the theorem follows from Radjavi's result [11] which asserts that every invertible operator is a product at most seven involutions.

4. GROUPS IN THE CALKIN ALGEBRA

The Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$ will be denoted by $\mathfrak{A}$. The group of invertible elements and unitary elements of $\mathfrak{A}$ will be denoted by $GL(\mathfrak{A})$ and $U(\mathfrak{A})$ respectively. In this section, we make a few remarks about semigroups generated by a conjugacy class in $GL(\mathfrak{A})$ and $U(\mathfrak{A})$. Recall that two elements a and b of a group $\mathcal{G}$ are said to be conjugate if $a = g^{-1}bg$ for some $g \in \mathcal{G}$.

Before proceeding, we recall some facts about the Calkin algebra and index theory (see [4; Chapter 5]). The index of a Fredholm operator T is defined by $\operatorname{ind}(T) = \dim\ker(T) - \dim\ker(T^*)$. The index satisfies the equation $\operatorname{ind}(TS) = \operatorname{ind}(T) + \operatorname{ind}(S)$. Furthermore, it is invariant under compact perturbations. Let $\pi : \mathcal{B}(H) \to \mathfrak{A}$ be the canonical quotient map. Atkinson's theorem [4; Theorem 5.17] implies that the set of Fredholm operators is the inverse image under π of the set $GL(\mathfrak{A})$ of invertible operators in $\mathfrak{A}$. In view of this and the invariance of the index under compact perturbations, we define the index of an invertible element in $\mathfrak{A}$ by $\operatorname{ind}(a) = \operatorname{ind}(A)$ for any $A \in \pi^{-1}(a)$. This gives a homomorphism from the group $GL(\mathfrak{A})$ onto the group of integers $\mathbb{Z}$.

Two facts about operators of index 0 are needed in the sequel.

(1) For a Fredholm operator T, $\operatorname{ind}(T) = 0$ if and only if T is a compact perturbation of an invertible operator.

(2) If $\pi(T)$ is unitary and if $\operatorname{ind}(T) = 0$, then T is a compact perturbation of a unitary operator [2; Theorem 3.1].

One more fact about the Calkin algebra $\mathfrak{A}$ is that the centre of $\mathfrak{A}$ is the scalars [3]. It follows immediately that the centre of the group $GL(\mathfrak{A})$ is also the (nonzero) scalars. We can also easily establish the fact that the centre of the group $U(\mathfrak{A})$ is $\{\lambda 1 : |\lambda| = 1\}$ since every element of $\mathfrak{A}$ is a linear combination of four unitary elements. (Indeed, if a is self-adjoint with $\|a\| \leq 1$, then $a \pm \left(1 - a^2\right)^{1/2}$ are unitaries, and hence a is a convex combination of two unitaries.)

We now state two immediate consequences of Theorems A and B.

Proposition 1. *Let a be an invertible (respectively, a unitary) element of $\mathfrak{A}$ of index 0. If a is not a scalar, then the semigroup generated by the conjugacy class of a in $GL(\mathfrak{A})$ (respectively, $U(\mathfrak{A})$) is the subgroup of all elements of index* 0.

Proof. Since $\operatorname{ind}(a) = 0$, there exists an invertible operator B such that $\pi(B) = b$. Furthermore, if a is a unitary, then the operator B may be chosen to be a unitary [2; Theorem 3.1]. Since B is not a scalar modulo the compacts, the results follow from Theorems A and B. ∎

For more general elements, we consider only the group generated by the conjugacy class. First, we need a lemma.

Lemma 3. *If a is an invertible element in the Calkin algebra $\mathfrak{A}$ such that $a^{-1}u^{-1}au$ is a scalar for every unitary element u in $\mathfrak{A}$, then a is a scalar.*

Proof. Let b be a self-adjoint element in $\mathfrak{A}$. Since e^{itb} is unitary for every real number t, there exist scalars λ_t such that $a^{-1}e^{-itb}ae^{itb} = \lambda_t 1$ for every scalar t. Taking the derivative at $t = 0$, we get that $b - a^{-1}ba = \lambda 1$ for a scalar λ. Thus

$$\sigma(b) = \sigma\left(a^{-1}ba\right) = \sigma(b - \lambda 1) = \sigma(b) - \lambda.$$

This implies that $\lambda = 0$ and hence $ab = ba$; i.e., a commutes with every self-adjoint element in $\mathfrak{A}$. It follows that a commutes with every element in $\mathfrak{A}$ and so a is a scalar. ∎

Proposition 2. *Let $\mathcal{G}$ be either the group $GL(\mathfrak{A})$ of all invertible elements or the group $U(\mathfrak{A})$ of all unitary elements in the Calkin algebra. If a is an element of $\mathcal{G}$ with a nonzero index n, then the group generated by the conjugacy class of a in $\mathcal{G}$ is $\{g \in \mathcal{G} : n$ divides $\operatorname{ind}(g)\}$.*

Proof. Let $\mathcal{N}$ be the group generated by the conjugacy class of a. Since a is not a scalar, it follows from Lemma 3 that there exists a unitary element u in $\mathfrak{A}$ such that $b := a^{-1}u^{-1}au$ is not a scalar. Now $b \in \mathcal{N}$ and $\operatorname{ind}(b) = 0$. By Proposition 1, we have that $\mathcal{N} \supseteq \mathcal{G}_0 := \{g \in \mathcal{G} : \operatorname{ind}(g) = 0\}$. Since $\mathcal{G}_0$ is the kernel of the homomorphism ind: $\mathcal{G} \longrightarrow \mathbb{Z}$, the subgroup $\mathcal{N}$ is the inverse image under the index map of a subgroup of $\mathbb{Z}$, and the result follows. ∎

Corollary 4. *Let $\mathcal{G}$ be either the group $GL(\mathfrak{A})$ of all invertible elements or the group $U(\mathfrak{A})$ of all unitary elements in the Calkin algebra. Every normal subgroup of $\mathcal{G}$ is either included in the centre (i.e., the scalars) or is $\{g \in \mathcal{G} \ : n \text{ divides } \operatorname{ind}(g)\}$, for some integer n.*

Corollary 5. *Let $\mathcal{G}$ be as above, let S be the unilateral shift and* $\mathbf{s}$ *its image in the Calkin algebra. Then the group generated by the conjugacy class of* $\mathbf{s}$ *in $\mathcal{G}$ is all of $\mathcal{G}$.*

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