WEIGHT-MONODROMY CONJECTURE FOR p-ADICALLY UNIFORMIZED VARIETIES

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Abstract. The aim of this note is to explain the main idea of the author's proof of the weight-monodromy conjecture (Deligne's conjecture on the purity of monodromy filtration) for varieties with p-adic uniformization by the Drinfeld upper half spaces of any dimension in mixed characteristic (for details with complete proofs, see [It3] (math.NT/0301201)). The ingredients of the proof are to prove a special case of the Hodge standard conjecture, and apply an argument of Steenbrink, M. Saito to the weight spectral sequence of Rapoport-Zink. As an application, by combining our results with the results of Schneider-Stuhler, we compute the local zeta functions of p-adically uniformized varieties in terms of representation theoretic invariants.

1. Introduction

Let p be a prime number, K a finite extension of the p-adic field $\mathbb{Q}_p$ with finite residue field $\mathbb{F}_q$, $\mathscr{O}_K$ the ring of integers of K, and l a prime number different from p. Let X be a proper smooth variety over K, and $V := H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ the l-adic cohomology of $X_{\overline{K}} := X \otimes_K \overline{K}$ on which the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ acts. We define the inertia group I_K of K by the exact sequence :

$$1 \longrightarrow I_K \longrightarrow \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \longrightarrow 1.$$

Let $\mathrm{Fr}_q \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ be the geometric Frobenius element, which is the inverse of the q-th power map on $\overline{\mathbb{F}}_q$.

We define the monodromy filtration $M_\bullet$ on V as follows (for details, see [Il]). The pro-l-part of I_K is isomorphic to $\mathbb{Z}_l(1)$ by $t_l\colon I_K \ni \sigma \mapsto \big(\sigma(\pi^{1/l^m})/\pi^{1/l^m}\big)_m \in \varprojlim \mu_{l^m} =: \mathbb{Z}_l(1)$, where π is a fixed uniformizer of K, and μ_{l^m} is the group of l^m-th roots of unity. By Grothendieck's monodromy theorem, there exist an open subgroup $J \subset I_K$ and a unique nilpotent map $N\colon V(1) \to V$ called the

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monodromy operator such that $\rho(\sigma) = \exp\big(t_l(\sigma)N\big)$ for all $\sigma \in J \subset I_K$, where $V(1)$ denotes the Tate twist of V. The *monodromy filtration* $M_\bullet$ on V is a unique increasing filtration such that $M_k V = V$, $M_{-k}V = 0$ for sufficiently large k, $N\big(M_k V(1)\big) \subset M_{k-2}V$ for all k, and the induced maps $N^r\colon \mathrm{Gr}^M_r V(r) \to \mathrm{Gr}^M_{-r} V$ are isomorphisms for all $r \geq 0$, where $\mathrm{Gr}^M_k V := M_k V / M_{k-1} V$.

The weight-monodromy conjecture claims the coincidence of weight and monodromy filtrations up to some shift ([De], [Il], [RZ], [Ra]).

Conjecture 1.1 (Weight-monodromy conjecture)**.** *For any lift* $\widetilde{\mathrm{Fr}}_q \in \mathrm{Gal}(\overline{K}/K)$ *of* Fr_q*, all eigenvalues of the action of* $\widetilde{\mathrm{Fr}}_q$ *on* Gr^M_k *are algebraic integers whose all conjugates have complex absolute value* $q^{(w+k)/2}$*.*

If X has a proper smooth model over $\mathscr{O}_K$, Conjecture 1.1 is nothing but the Weil conjecture proved by Deligne ([De]). If X is an abelian variety over K or $\dim X \leq 2$, Conjecture 1.1 was known to hold ([SGA7-I], [RZ]). In [It2], the author proved Conjecture 1.1 for certain threefolds with strictly semistable reduction. However, in mixed characteristic and in dimension ≥ 3, Conjecture 1.1 is still open up to now.

It is worth noting that a characteristic $p > 0$ analogue of Conjecture 1.1 was known ([De], [Te], [It1]). In fact, in [De], I, 1.8.3, Deligne proved an l-adic sheaf version of Conjecture 1.1. Also, a Hodge analogue over $\mathbb{C}$ was known by Steenbrink, M. Saito ([St], [Sa], 4.2.5).

The aim of this note is to explain the main idea of the author's proof of the following theorem in [It3].

Theorem 1.2 ([It3])**.** *Let* $\widehat{\Omega}^d_K$ *be the Drinfeld upper half space of dimension* d *over* K*, and* $\Gamma \subset \mathrm{PGL}_{d+1}(K)$ *be a cocompact torsion free discrete subgroup. Then, Conjecture 1.1 holds for the algebraization* X_Γ *of the rigid analytic quotient* $\Gamma\backslash\widehat{\Omega}^d_K$ *(for details, see §2).*

We can also prove a p-adic analogue by using the D_{st}-functor of Fontaine and the weight spectral sequence of Mokrane [1].

Needless to say, there is a very important theory so called the theory of p-adic uniformization of Shimura varieties established by Čerednik, Drinfeld, Varshavsky, Rapoport-Zink. We strongly expect that Theorem 1.2 has applications to the zeta functions of Shimura varieties with p-adic uniformization ([Ra], [RZ]). We also expect that this will establish a special case of the compatibility of the global and local Langlands correspondences ([Ha], Problem 1).

[1] After this work was completed, Ehud de Shalit informed the author that he also obtained a p-adic weight-monodromy conjecture by different methods ([dS]). In his proof, he crucially used a combinatorial results on harmonic cochains of Alon-de Shalit and Große-Klönne's results on log-rigid cohomology of Drinfeld upper half spaces.

2. Review of the theory of p-adic uniformization

2.1. **Construction of $\widehat{\Omega}^d_K$.** Let $\widehat{\Omega}^d_K$ be a rigid analytic space obtained by removing all K-rational hyperplanes from $\mathbb{P}^d_K$. We have a natural action of $\mathrm{PGL}_{d+1}(K)$ on $\widehat{\Omega}^d_K$. As a formal scheme, $\widehat{\Omega}^d_K$ can be constructed as follows ([Mum], [Mus], [Ku]). Take the projective space $\mathbb{P}^d_{\mathscr{O}_K}$ over $\mathscr{O}_K$. Then, take successive blowing-ups of $\mathbb{P}^d_{\mathscr{O}_K}$ along all linear subvarieties in the special fiber $\mathbb{P}^d_{\mathbb{F}_q}$. By continuing this process for *all* exceptional divisors appearing in the blowing-ups, we obtain a formal scheme $\widehat{\Omega}^d_{\mathscr{O}_K}$ locally of finite type over $\mathrm{Spf}\,\mathscr{O}_K$. By construction, the rigid analytic space associated with $\widehat{\Omega}^d_{\mathscr{O}_K}$ is isomorphic to $\widehat{\Omega}^d_K$. In other words, $\widehat{\Omega}^d_K$ is the "generic fiber" of $\widehat{\Omega}^d_{\mathscr{O}_K}$ in the sense of Raynaud.

2.2. **Construction of X_Γ.** Let Γ be a cocompact torsion free discrete subgroup of $\mathrm{PGL}_{d+1}(K)$. We take a quotient $\widehat{\mathfrak{X}}_\Gamma := \Gamma\backslash\widehat{\Omega}^d_{\mathscr{O}_K}$ as a formal scheme. Since the relative dualizing sheaf $\omega_{\widehat{\mathfrak{X}}_\Gamma/\mathscr{O}_K}$ is invertible and ample ([Mus], [Ku]), by Grothendieck's algebraization theorem, $\widehat{\mathfrak{X}}_\Gamma$ can be algebraized to a projective scheme $\mathfrak{X}_\Gamma$ over $\mathscr{O}_K$. The generic fiber $X_\Gamma := \mathfrak{X}_\Gamma \otimes_{\mathscr{O}_K} K$ is a projective smooth variety over K whose associated rigid analytic space is the rigid analytic quotient $\Gamma\backslash\widehat{\Omega}^d_K$. By construction, the special fiber of $\mathfrak{X}_\Gamma$ is described by the cell complex $\Gamma\backslash\mathfrak{T}$, where $\mathfrak{T}$ denotes the Bruhat-Tits building of $\mathrm{PGL}_{d+1}(K)$.

2.3. **The variety B^d.** All irreducible components of the special fiber of $\widehat{\Omega}^d_{\mathscr{O}_K}$ are isomorphic to the variety B^d constructed as follows :

$$B^d = Y_{d-1} \xrightarrow{g_{d-2}} Y_{d-2} \xrightarrow{g_{d-3}} \cdots \xrightarrow{g_1} Y_1 \xrightarrow{g_0} Y_0 = \mathbb{P}^d_{\mathbb{F}_q},$$

where $g_k\colon Y_{k+1} \to Y_k$ is the blow-up of Y_k along the union of all proper transforms of $\mathbb{F}_q$-rational linear subvarieties of dimension k in $\mathbb{P}^d$. We put $f := g_0 \circ \cdots \circ g_{d-2}\colon B^d \to \mathbb{P}^d$. Note that the above construction is equivariant with respect to a natural $\mathrm{PGL}_{d+1}(\mathbb{F}_q)$-action.

2.4. **Divisors on B^d.** For $0 \le k \le d-2$, let D_k be a divisor on B^d which is the proper transform of the exceptional divisor of the blow-up $g_k\colon Y_{k+1} \to Y_k$. For $k = d-1$, let D_{d-1} be the proper transform of the union of all $\mathbb{F}_q$-rational hyperplanes in $\mathbb{P}^d$. From the construction, it is easy to see that each D_k is isomorphic to a disjoint union of $B^k \times B^{d-k-1}$. In particular, each connected component of D_{d-1} is isomorphic to B^{d-1}. Furthermore, intersections of irreducible components of the special fiber of $\widehat{\Omega}^d_{\mathscr{O}_K}$ are isomorphic to products of B^k. A divisor D on B^d is called $\mathrm{PGL}_{d+1}(\mathbb{F}_q)$-*invariant* if D can be written as $D = \alpha \cdot f^*\mathscr{O}_{\mathbb{P}^d}(1) + \sum_{k=0}^{d-1} a_k D_k$, where the equality means that two divisors are linearly equivalent. Finally, we also see that the restriction of the "relative dualizing sheaf" $\omega_{\widehat{\Omega}^d_{\mathscr{O}_K}/\mathscr{O}_K}$ to B^d is $-(d+1)f^*\mathscr{O}_{\mathbb{P}^d}(1) + \sum_{k=0}^{d-1}(d-k)D_k$, which is an ample $\mathrm{PGL}_{d+1}(\mathbb{F}_q)$-invariant divisor on B^d.

3. Weight spectral sequence of Rapoport-Zink

Here we recall the weight spectral sequence of Rapoport-Zink ([RZ]). Assume that a proper smooth variety X over K has a *proper strictly semistable model* $\mathfrak{X}$ over $\mathscr{O}_K$. This means that $\mathfrak{X}$ is a regular scheme which is proper and flat over $\mathscr{O}_K$ such that the generic fiber $\mathfrak{X} \otimes_{\mathscr{O}_K} K$ is isomorphic to X and the special fiber $\mathfrak{X} \otimes_{\mathscr{O}_K} \mathbb{F}_q$ is a divisor of $\mathfrak{X}$ with simple normal crossings.

Let $X_1, \ldots, X_m$ be the irreducible components of the special fiber of $\mathfrak{X}$, and

$$X^{(k)} := \coprod_{1 \leq i_1 < \cdots < i_k \leq m} X_{i_1} \cap \cdots \cap X_{i_k}.$$

Then $X^{(k)}$ is a disjoint union of proper smooth varieties of dimension $d-k+1$ over $\mathbb{F}_q$. The *weight spectral sequence of Rapoport-Zink* is as follows :

$$E_1^{-r,\,w+r} = \bigoplus_{k \geq \max\{0,-r\}} H^{w-r-2k}_{\text{ét}}\big(X^{(2k+r+1)}_{\overline{\mathbb{F}}_q},\, \mathbb{Q}_l(-r-k)\big) \Longrightarrow H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l).$$

This spectral sequence is $\mathrm{Gal}(\overline{K}/K)$-equivariant. The differentials $d_1^{i,j} \colon E_1^{i,j} \to E_1^{i+1,j}$ can be described in terms of restriction morphisms and Gysin morphisms explicitly (see [RZ], 2.10 for details).

The action of the monodromy operator N on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is induced from a natural map $N \colon E_1^{i,j}(1) \to E_1^{i+2,j-2}$ satisfying

$$N^r \colon E_1^{-r,\,w+r}(r) \xrightarrow{\ \cong\ } E_1^{r,\,w-r}$$

for all r, w. We can describe $N \colon E_1^{i,j}(1) \to E_1^{i+2,j-2}$ explicitly (for details, see [Il], [RZ], 2.10). Since $E_1^{i,j}$ has weight j in the sense of [De], this spectral sequence degenerates at E_2. Therefore, the weight-monodromy conjecture (Conjecture 1.1) is equivalent to the following conjecture.

Conjecture 3.1 ([RZ], [Il])**.** *N^r induces an isomorphism*

$$N^r \colon E_2^{-r,\,w+r}(r) \xrightarrow{\ \cong\ } E_2^{r,\,w-r}$$

on E_2-terms for all r, w.

4. Proof of Theorem 1.2

4.1. **Outline of the proof.** First of all, by replacing Γ by its finite index subgroup, we may assume that $\mathfrak{X}_\Gamma$ is a proper strictly semistable model of X_Γ (see §2). Therefore, it is enough to prove Conjecture 3.1 for $\mathfrak{X}_\Gamma$. The idea is essentially the same as in [Sa], 4.2.5, where M. Saito used polarized Hodge structures to prove a Hodge analogue of Conjecture 3.1.

One of the biggest obstruction to follow a Hodge theoretic argument is that there doesn't exist a good analogue of *polarized* Hodge structures for l-adic cohomology. Namely, the notion of *signature* doesn't make sense over $\mathbb{Q}_l$.

The first point here is that all l-adic cohomology groups of B^d are generated by algebraic cycles. Hence the E_1-terms of the weight spectral sequence of Rapoport-Zink have natural $\mathbb{Q}$-structure, and all maps $d_1^{i,j}$, N are compatible with it. Therefore, it is enough to show that there exists an analogue of *polarization* for the $\mathbb{Q}$-structure of the cohomology of B^d, which is nothing but the *Grothendieck's Hodge standard conjecture* for B^d. In the followings, we shall prove this conjecture for certain choice of an ample divisor (Key Lemma 4.3).

4.2. **Notation.** Since we will discuss the signature of the cup product pairings, it is convenient to work over $\mathbb{R}$ instead of $\mathbb{Q}$ or $\mathbb{Q}_l$. Let $N^k(B^d)$ be the group of algebraic cycles on B^d of codimension k modulo numerical equivalence, which is a finitely generated free $\mathbb{Z}$-module. We put

$$H^k(B^d) = \begin{cases} N^{k/2}(B^d) \otimes_{\mathbb{Z}} \mathbb{R} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

Namely, we consider $H^k(B^d)$ as a virtual cohomology group with "coefficients in $\mathbb{R}$". We have a natural "cup product" $\cup$ on $H^*(B^d) := \bigoplus_{k=0}^d H^{2k}(B^d)$ which comes from the intersection product on $N^*(B^d) := \bigoplus_{k=0}^d N^k(B^d)$.

4.3. **Conjectures.** Let L be an ample $\mathbb{R}$-divisor on B^d, which is a formal linear combination of ample divisors on B^d with positive real coefficients. We can naturally think of the "cohomology class" $[L] \in H^2(B^d)$ and the "Lefschetz operator" $L\colon H^k(B^d) \to H^{k+2}(B^d)$.

Conjecture 4.1 (Hard Lefschetz conjecture)**.** *For each k, L^k induces an isomorphism*

$$L^k\colon H^{d-k}(B^d) \xrightarrow{\cong} H^{d+k}(B^d).$$

If L is an ample $\mathbb{Q}$-divisor, Conjecture 4.1 holds by Deligne ([De]). However, it seems that the case of ample $\mathbb{Q}$-divisors doesn't automatically imply the case of ample $\mathbb{R}$-divisors.

Assume Conjecture 4.1 holds for L. We define the *primitive part* by

$$P^k(B^d) := \operatorname{Ker}(L^{d-k+1}\colon H^k(B^d) \longrightarrow H^{2d-k+2}(B^d)).$$

Then, we have the *primitive decomposition* of as $H^k(B^d) = \bigoplus_{i\geq 0} L^i P^{k-2i}(B^d)$.

Conjecture 4.2 (Hodge standard conjecture)**.** *For even k with $0 \leq k \leq d$, the following pairing*

$$\langle , \rangle\colon P^k(B^d) \times P^k(B^d) \longrightarrow H^{2d}(B^d) \cong \mathbb{R}, \quad \langle x, y\rangle = (-1)^{k/2} L^{d-k} x \cup y$$

is positive definite.

4.4. **Main results and proofs.** The main result of this section is the following Key Lemma 4.3. As we already explained in §4.1, Theorem 1.2 follows from this (for details, see [It3]).

Key Lemma 4.3. *If L is an ample $\mathrm{PGL}_{d+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on B^d, then Conjecture 4.1 and Conjecture 4.2 hold for L.*

Key Lemma 4.3 follows easily from the following 4 lemmas (Lemma 4.4, Lemma 4.5, Lemma 4.6, Lemma 4.7) by induction on d.

Lemma 4.4. *Conjecture 4.1 and Conjecture 4.2 hold for*

$$L = f^*\mathscr{O}_{\mathbb{P}^d}(1) - \sum_{k=0}^{d-2} a_k D_k \quad \textit{with} \quad 1 \gg a_0 \gg a_1 \gg \cdots \gg a_{d-2} > 0.$$

Proof. This follows from an explicit computation of the "limit of cup product pairings" of blow-ups. ☐

Lemma 4.5. *If L_1, L_2 are ample $\mathbb{R}$-divisor on B^d such that Conjecture 4.1 holds for $tL_1 + (1-t)L_2$ for all $t \in [0,1]$. Then, Conjecture 4.2 for L_1, L_2 are equivalent to each other.*

Proof. This follows from the fact that the signature is a "homotopy invariant" and doesn't change in a continuous family. Note that, this argument works only over $\mathbb{R}$. ☐

Lemma 4.6. *Let $L = \sum_{k=0}^{d-1} a_k D_k$ be an ample $\mathrm{PGL}_{d+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on B^d, then $a_k > 0$.*

Proof. We prove the assertion by induction on d. We have

$$f^*\mathscr{O}_{\mathbb{P}^d}(1) = \sum_{k=0}^{d-1} \frac{|\mathbb{P}^{d-k-1}(\mathbb{F}_q)|}{|\mathbb{P}^d(\mathbb{F}_q)|} D_k,$$

where $|\mathbb{P}^m(\mathbb{F}_q)| = \frac{q^{m+1}-1}{q-1}$ denotes the number of $\mathbb{F}_q$-rational points on $\mathbb{P}^m$. Note that $|\mathbb{P}^d(\mathbb{F}_q)|$ is the total number of $\mathbb{F}_q$-rational hyperplanes in $\mathbb{P}^d$, and $|\mathbb{P}^{d-k-1}(\mathbb{F}_q)|$ is the number of $\mathbb{F}_q$-rational hyperplanes in $\mathbb{P}^d$ containing a fixed $\mathbb{F}_q$-rational linear subvariety of $\mathbb{P}^d$ of dimension k. Hence L can be rewritten as

$$L = a_{d-1}|\mathbb{P}^d(\mathbb{F}_q)| \cdot f^*\mathscr{O}_{\mathbb{P}^d}(1) + \sum_{k=0}^{d-2} \left(a_k - a_{d-1}|\mathbb{P}^{d-k-1}(\mathbb{F}_q)| \right) \cdot D_k.$$

From this, we see that $a_{d-1} > 0$. Since D_{d-1} is isomorphic to a disjoint union of B^{d-1}, we consider the restriction $\mathscr{O}(L)|_{B^{d-1}}$ of L to B^{d-1}. Then, $\mathscr{O}(L)|_{B^{d-1}}$ has the same expression as above. We rewrite $\mathscr{O}(L)|_{B^{d-1}}$ in the following form

$$\mathscr{O}(L)|_{B^{d-1}} = \sum_{k=0}^{d-2} \left\{ a_k - a_{d-1} \left(|\mathbb{P}^{d-k-1}(\mathbb{F}_q)| - \frac{|\mathbb{P}^d(\mathbb{F}_q)| \cdot |\mathbb{P}^{d-k-2}(\mathbb{F}_q)|}{|\mathbb{P}^{d-1}(\mathbb{F}_q)|} \right) \right\} \cdot D_k.$$

By induction hypothesis, all coefficients of D_k are positive. Since $a_{d-1} > 0$ and

$$|\mathbb{P}^{d-k-1}(\mathbb{F}_q)| \cdot |\mathbb{P}^{d-1}(\mathbb{F}_q)| - |\mathbb{P}^{d}(\mathbb{F}_q)| \cdot |\mathbb{P}^{d-k-2}(\mathbb{F}_q)| > 0$$

by an explicit computation, we have $a_k > 0$ as desired. □

Lemma 4.7. *If Key Lemma 4.3 holds in dimension $< d$, then Conjecture 4.1 holds in dimension d.*

Proof. Assume that there exist $k < d$ and $x \in H^k(B^d)$ such that $x \neq 0$, $L^{d-k}x = 0$. Firstly, we observe that, for each i, the restriction of x to each connected component of D_i is primitive. Let L be written as $L = \sum_{k=0}^{d-1} a_k D_k$ with $a_k > 0$ by Lemma 4.6. Then we compute as follows :

$$\begin{aligned} 0 &= (-1)^{k/2} L^{d-k} x \cup x \\ &= \sum_{i=0}^{d-1} a_i \cdot \Big\{ (-1)^{k/2} \big(\mathscr{O}(L)|_{D_i}\big)^{d-k-1} \ (x|_{D_i}) \cup (x|_{D_i}) \Big\}. \end{aligned}$$

Since D_i is isomorphic to a disjoint union of $B^i \times B^{d-i-1}$, by induction hypothesis,

$$(-1)^{k/2} \big(\mathscr{O}(L)|_{D_i}\big)^{d-k-1} \ (x|_{D_i}) \cup (x|_{D_i}) \geq 0$$

and the equality holds if and only if $x|_{D_i} = 0$. Since $a_i > 0$ for all i, we have $x|_{D_i} = 0$ for all i. However, it is easy to see from the construction of B^d that this implies $x = 0$. This is a contradiction. □

5. Applications

Here we combine Theorem 1.2 with Schneider-Stuhler's results on the cohomology of $\widehat{\Omega}^d_K$ and its quotient $\Gamma\backslash\widehat{\Omega}^d_K$ in [SS]. For a cocompact torsion free discrete subgroup $\Gamma \subset \mathrm{PGL}_{d+1}(K)$, let $\mathrm{Ind}_\Gamma := C^\infty\big(\mathrm{PGL}_{d+1}(K)/\Gamma, \mathbb{C}\big)$ be the $\mathrm{PGL}_{d+1}(K)$-representation induced from the trivial character on Γ. Let $\mu(\Gamma)$ be the multiplicity of the Steinberg representation in Ind_Γ. In [SS], §5, Schneider-Stuhler explicitly computed the E_2-terms of a Hochschild-Serre type spectral sequence :

$$E_2^{r,s} = H^r\big(\Gamma, H^s_{\text{ét}}(\widehat{\Omega}^d_K \otimes_K \overline{K}, \mathbb{Q}_l)\big) \Longrightarrow H^{r+s}_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_l)$$

([SS], §5, Proposition 2), and proved

$$H^k_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_l) \cong \begin{cases} \mathbb{Q}_l\big(-\frac{k}{2}\big) & \text{if } k \text{ is even, } 0 \leq k \leq 2d,\ k \neq d \\ 0 & \text{if } k \text{ is odd, } k \neq d. \end{cases}$$

For $k = d$, they proved that this spectral sequence defines a decreasing filtration $F^\bullet$ on $V := H^d_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_l)$ such that $F^0V = V$, $F^{d+1}V = 0$ and

$$F^rV/F^{r+1}V \cong \begin{cases} \mathbb{Q}_l(r-d)^{\oplus \mu(\Gamma)} & \text{if } 0 \leq r \leq d,\ r \neq \frac{d}{2} \\ \mathbb{Q}_l\big(-\frac{d}{2}\big)^{\oplus(\mu(\Gamma)+1)} & \text{if } r = \frac{d}{2} \\ 0 & \text{otherwise.} \end{cases}$$

They conjectured that $F^\bullet$ essentially coincides with the monodromy filtration $M_\bullet$ on V (see [SS], introduction and a remark following Theorem 5). We can prove this conjecture in the following form.

Theorem 5.1 (Schneider-Stuhler's conjecture on the filtration $F^\bullet$)**.** *Define a filtration $F'_\bullet$ on V by $F'_i V = F^{-\lfloor i/2 \rfloor} V$. Then, we have $M_i V = F'_{i-d} V$ for all i.*

Consequently, we compute the local zeta function $\zeta(s, X_\Gamma)$ of X_Γ as follows :

$$\begin{aligned}\zeta(s, X_\Gamma) &:= \prod_{k=0}^{2d} \det\left(1 - q^{-s} \cdot \mathrm{Fr}_q \,;\, H^k_{\text{ét}}(X_\Gamma \otimes_K \overline{K},\, \mathbb{Q}_l)^{I_K}\right)^{(-1)^{k+1}} \\ &= (1 - q^{-s})^{\mu(\Gamma)\cdot(-1)^{d+1}} \cdot \prod_{k=0}^{d} \frac{1}{1 - q^{k-s}},\end{aligned}$$

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