PEL-type Shimura varieties

They arise as moduli spaces for polarized abelian varieties endowed with the action of a simple algebra over $\mathbb{Q}$ and a level structure.

Thm.(Shimura) They are smooth and projective varieties defined over a number field.

The PEL-data:
• $(\mathbf{B}, *)$ is a finite dimensional simple algebra over $\mathbb{Q}$, together with a positive involution;
• $(\mathbf{V}, \langle,\rangle)$ is a nonzero finitely generated left B-module, together with a nondegenerate $*$-hermitian $\mathbb{Q}$-valued alternanting form.

• $\mathbf{G}/\mathbb{Q}$ is the algebraic group of the B-linear automorphisms of V preserving the pairing $\langle,\rangle$ up to a scalar multiple;
• $\mathbf{G_1} \subset \mathbf{G}$ is the subgroup of the B-linear automorphisms of V preserving the pairing $\langle,\rangle$.

Assumptions

• $E = Z(B)$ is a quadratic imaginary extension of $\mathbb{Q}$;

• $*_{|E}$ is complex conjugation,
i.e. $*$ is an involution of the II kind;

• p is a totally split prime in E,
i.e. $p = u \cdot u^c$ and $E_u = \mathbb{Q}_p$;

• B splits at the prime u,
i.e. $G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times GL_n(\mathbb{Q}_p)$.

• $G_1(\mathbb{R}) \simeq U(q, n-q)$, with $1 \leq q \leq n$.

Remark (*in progress*): The following results generalize to all PEL-type Shimura varieties which are <u>unramified</u> at p,
i.e. $B_{\mathbb{Q}_p} = \prod_i M_{r_i}(L_i)$ for $L_i/\mathbb{Q}_p$ unramified $\forall i$.

The cohomology of the Shimura varieties

Let $U \subset G(\mathbb{A}^\infty)$ be a sufficiently small open compact subgroup.
Let X_U/E be the *Shimura variety* of level U.

$$\forall U \subset V : \qquad X_U \to X_V,$$

$$\forall g \in G(\mathbb{A}^\infty) : \qquad g : X_U \to X_{g^{-1}Ug}.$$

Let $l \neq p$ be a prime.
Def. $H^i(X, \mathbb{Q}_l) = \varinjlim_U H^i_{et}(X_U \times_E \bar{E}, \mathbb{Q}_l)$
They are representations of $G(\mathbb{A}^\infty) \times Gal_E$.

We study the above representations when restricted to $G(\mathbb{A}^\infty) \times W_{E_u}$, where $u|p$, i.e.
$H^i(X, \mathbb{Q}_l)_{|G(\mathbb{A}^\infty) \times W_{E_u}} = \varinjlim_U H^i_{et}(X_U \times_{E_u} \bar{E}_u, \mathbb{Q}_l)$.

The main theorem

Thm. There is an equality of virtual representations of $G(\mathbb{A}^\infty) \times W_{E_u}$

$$\sum_n (-1)^n H^n(X, \mathbb{Q}_l)^{\mathbb{Z}_p^\times} = \sum_{\alpha,d,e,f} (-1)^{d+e+f} \varinjlim_M \mathcal{E}^{d,e,f}_{\alpha,M}$$

and

$$\mathcal{E}^{d,e,f}_{\alpha,M} = Ext^d_{T_\alpha-smooth}(H^e_c(\mathcal{M}^{\mathrm{rig}}_{\alpha,M}, \mathbb{Q}_l(-D)), H^f_c(J_\alpha, \mathbb{Q}_l))$$

where

- $D = \dim X_U$,
- $\mathbb{Z}_p^\times \subset \mathbb{Q}_p^\times \subset \mathbb{Q}_p^\times \times GL_n(\mathbb{Q}_p) = G(\mathbb{Q}_p)$,
- the T_α's are abstract p-adic groups;
- the cohomology of the Igusa varieties $H^f_c(J_\alpha, \mathbb{Q}_l)$ is a representation of

$$T_\alpha \times G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times / \mathbb{Z}_p^\times \times W_{E_u}/I_u,$$

- the cohomology of the Rapoport-Zink spaces $\varinjlim_M H^e_c(\mathcal{M}^{\mathrm{rig}}_{\alpha,M}, \mathbb{Q}_l)$ is a representation of

$$T_\alpha \times GL_n(\mathbb{Q}_p) \times W_{E_u}.$$

Sketch of the proof

• construct integral models X of the Shimura varieties over $\mathcal{O}_{E_u} = \mathcal{O}_u$:

$$R\Gamma(X \times_{\mathcal{O}_u} \overline{k(u)}, R\Psi(\mathbb{Z}/l^r\mathbb{Z})) \simeq R\Gamma(X \times_{\mathcal{O}_u} \overline{E}_u, \mathbb{Z}/l^r\mathbb{Z}).$$

• stratify the reduction $\bar{X}$ of the Shimura varieties by locally closed subschemes $\bar{X}^{(\alpha)}$:

$$\sum_p (-1)^p H^p(\bar{X}, \mathcal{F}) = \sum_{\alpha, j} (-1)^j H_c^j(\bar{X}^{(\alpha)}, \mathcal{F}_{|\bar{X}^{(\alpha)}}).$$

• define a foliation of $\bar{X}^{(\alpha)}$ by closed smooth subschemes C_Σ.

• define the Igusa varieties as finite étale Galois covers $J_\alpha \to C_\alpha = C_{\Sigma_\alpha} \subset \bar{X}^{(\alpha)}$, for a fixed Σ_α.

• construct a finite surjective morphism

$$\pi : J_\alpha \times \bar{\mathcal{M}}_\alpha \to \bar{X}^{(\alpha)}$$

invariant under the action of the p-adic group T_α, s.t.

$$\text{``}(J_\alpha \times \bar{\mathcal{M}}_\alpha)/T_\alpha \sim \bar{X}^{(\alpha)}\text{``}$$

• deduce the existence of a spectral sequence

$$H_p(T_\alpha, H_c^q(J_\alpha \times \bar{\mathcal{M}}_\alpha, \pi^*\mathcal{F})) \Rightarrow H_c^{p+q}(\bar{X}^{(\alpha)}, \mathcal{F}).$$

• compare the pullbacks of vanishing cycles sheaves

$$\pi^* R\Psi(\mathbb{Z}/l^r\mathbb{Z}_{/X}) \simeq p_2^* R\Psi(\mathbb{Z}/l^r\mathbb{Z}_{/\mathcal{M}}).$$

• observe that

$$H_.(T_\alpha, H_c^{\cdot}(J_\alpha \times \bar{\mathcal{M}}_\alpha, p_2^* R\Psi(\mathbb{Z}/l^r\mathbb{Z})) =$$

$$= Tor_{T_\alpha}^{\cdot}(H_c^{\cdot}(\bar{\mathcal{M}}_\alpha, R\Psi(\mathbb{Z}/l^r\mathbb{Z})), H_c^{\cdot}(J_\alpha, \mathbb{Z}/l^r\mathbb{Z}))$$

$$= Ext_{T_\alpha}^{\cdot}(H_c^{\cdot}(\mathcal{M}_\alpha^{\text{rig}}, \mathbb{Z}/l^r\mathbb{Z}(-D)), H_c^{\cdot}(J_\alpha, \mathbb{Z}/l^r\mathbb{Z})).$$

The reduction of the Shimura varieties

Prop. If the level $U \subset G(\mathbb{A}^\infty)$ is of the form

$$U = U^p(M) = U^p \times \mathbb{Z}_p^\times \times \{A \equiv \mathbb{I}_n \operatorname{mod} p^M\} \subset$$

$$\subset G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times GL_n(\mathbb{Q}_p) = G(\mathbb{A}^\infty),$$

then X_U has a proper model over $\mathcal{O}_u$.
Moreover, if $M = 0$, X_U is smooth.

Rmk. As representations of $G(\mathbb{A}^\infty) \times W_{E_u}$

$$H^i(X, \mathbb{Z}/l^r\mathbb{Z})^{\mathbb{Z}_p^\times} = \varinjlim_{U^p,M} H^i_{et}(X_{U^p(M)} \times_{\mathcal{O}_u} \overline{E_u}, \mathbb{Z}/l^r\mathbb{Z}).$$

Fix $U^p \subset G(\mathbb{A}^{\infty,p})$:

- $\bar{X} = X_{U^p(0)} \times_{\mathcal{O}_u} k(u)$,
- $\mathcal{A}/\bar{X}$ the universal abelian variety.

Prop. There exists a Barsotti-Tate group

$$\mathcal{G} \subset \mathcal{A}[p^\infty]/\bar{X}$$

of dimension q and height n s.t.

$$\forall x \in |\bar{X}| : \mathcal{O}^\wedge_{x,\bar{X}} \text{ is the deformation ring of } \mathcal{G}_x.$$

Serre-Tate Thm. $\mathcal{O}^{\wedge}_{x,\bar{X}}$ is the deformation ring of $\mathcal{A}_x[p^\infty]$ together with a polarization and an action of $\mathcal{O}_{B_p}$ (maximal $\mathbb{Z}_{(p)}$-order of $B_{\mathbb{Q}_p}$).

$$\mathcal{A}[p^\infty] = \mathcal{A}[u^\infty] \oplus \mathcal{A}[(u^c)^\infty]$$

and the polarization on $\mathcal{A}$ gives rise to an isomorphism

$$\mathcal{A}[u^\infty] \simeq \mathcal{A}[(u^c)^\infty]$$

as $\mathcal{O}_{B_u}$-modules.

We fix an isomorphism

$$B_u \simeq M_n(E_u) = M_n(\mathbb{Q}_p)$$

s.t. $\mathcal{O}_{B_u} \subset B_u$ corresponds to the maximal order $M_n(\mathcal{O}_{E_u}) \subset M_n(E_u)$.
Let $\epsilon \in \mathcal{O}_{B_u}$ be the *idempotent* corresponding to the matrix with $a_{1,1} = 1$ and 0 elsewhere.

Let $\mathcal{G} = \epsilon\mathcal{A}[u^\infty]$. Then $\mathcal{G}$ is a BT group and

$$\mathcal{A}[u^\infty] \simeq \mathcal{O}_{B_u} \otimes_{\mathcal{O}_{E_u}} \mathcal{G}.$$

The NP stratification and Oort's foliation

Let α be a Newton polygon of dim q and ht n.
Prop. (Grothendieck) The set

$$\bar{X}^{(\alpha)} = \{x \in |\bar{X}| | \mathcal{N}(\mathcal{G}_x) = \alpha\} \subset \bar{X}$$

is locally closed.

Let Σ be a BT group, $\mathcal{N}(\Sigma) = \alpha$.
Prop. (Oort) The set

$$C_{\Sigma} = \{x \in |\bar{X}| | \mathcal{G}_x \simeq \Sigma \times k(x)\} \subset \bar{X}^{(\alpha)}$$

is closed and as a reduced closed subscheme is smooth.

Rmk. The action of $G(\mathbb{A}^{\infty,p})$ on $\bar{X}$ preserves both the NP stratification and Oort's foliation.

Complete slope divisible BT groups

Prop.(Grothendieck, Zink) Let S be a smooth scheme of char p, and $\mathcal{H}/S$ a BT group with constant Newton polygon.
(1) $\mathcal{H}$ is isogenous to a c.s.d. BT group endowed with a *slope filtration*.
(2) If $S = \operatorname{Spec} k$, for k a perfect field the slope filtration of a c.s.d. BT group canonically splits.

Let α be a Newton polygon of dim q and ht n:
- $1 \geq \lambda_1 \geq \cdots \lambda_k \geq 0$ the slopes of α,
- $\Sigma_\alpha/\mathbb{F}_p$ a c.s.d. BT group with $\mathcal{N}(\Sigma_\alpha) = \alpha$.

- $\Sigma_\alpha = \oplus_i \Sigma^i$, for Σ^i slope divisible isoclinic BT groups of slope λ_i.
- $T_\alpha = \mathrm{QIsog}(\Sigma_\alpha) = \mathrm{Isog}(\Sigma_\alpha) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$,

$T_\alpha = \prod_i M_{r_i}(D_i)$, where $D_i/\mathbb{Q}_p$ are finite dimensional division algebras.

The Igusa varieties

Let $C_\alpha = C_{\Sigma_\alpha}$ be the leaf associated to Σ_α.

Prop.(Zink) The universal BT group $\mathcal{G}/C_\alpha$ is completely slope divisible with slope filtration

$$0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_k = \mathcal{G}$$

where
- $\mathcal{G}^i = \mathcal{G}_i/\mathcal{G}_{i-1}$ is s.d. isoclinic of slope λ_i,
- $\forall x \in |C_\alpha| : \mathcal{G}^i_x \simeq \Sigma^i \times k(x)$.

Def. The *Igusa variety* $J_{\alpha,m}$ of level $m \geq 1$ is the universal space for the existence of isomorphisms

$$\phi^i_m : \Sigma^i[p^m] \simeq \mathcal{G}^i[p^m] \qquad \forall i = 1, \ldots, k$$

étale locally extendable to any level $m' \geq m$.

Prop. (1) $J_{\alpha,m} \to C_\alpha$ are finite étale Galois.
(2) $\Gamma_\alpha = \mathrm{Aut}(\Sigma_\alpha)$ acts on the Igusa varieties

$$\forall \gamma = (\gamma^i) \in \Gamma_\alpha : \quad (A, \phi^i_m) \mapsto (A, \phi^i_m \circ \gamma^i_{|[p^m]}).$$

Let $J_{\alpha,U^p,m}$ be the Igusa varieties of level m over $C_{\alpha,U^p} \subset \bar{X}_{U^p(0)}$.

Prop. (1) There exists an action of the group $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times$ on the Igusa varieties $J_{\alpha,U^p,m}$,

$$\forall g \in G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times : J_{\alpha,U^p,m} \to J_{\alpha,g^{-1}U^p g,m}$$

compatible with the action on the Shimura varieties.

(2) There exists a submonoid $\Gamma_\alpha \subset S_\alpha \subset T_\alpha$ s.t.

(i) the action of Γ_α on the Igusa varieties extends to an action of S_α,

(ii) $T_\alpha = \langle S_\alpha, p, fr \rangle$ and $p^{-1}, fr^{-1} \in S_\alpha$.

Prop. The cohomology of the Igusa varieties

$$H_c^j(J_\alpha, \mathbb{Z}/l^r\mathbb{Z}) = \varinjlim{}_{U^p,m} H_c^j(J_{\alpha,U^p,m} \times_{k(u)} \overline{k(u)}, \mathbb{Z}/l^r\mathbb{Z})$$

is a representation of

$$T_\alpha \times G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times/\mathbb{Z}_p^\times \times W_{E_u}/I_u.$$

The Rapoport-Zink spaces

For each α, we consider the RZ spaces associated to the BT group Σ_α.

They are rigid analytic spaces over E_u^{nr} which arise as moduli spaces for BT groups endowed with a *quasi-isogeny* from Σ_α and a level structure.

Let $V \subset GL_n(\mathbb{Q}_p)$ an open compact subgroup. Let $\mathcal{M}_{\alpha,V}^{\text{rig}}$ be the *RZ space* of level V.

$T_\alpha = \mathrm{QIsog}(\Sigma_\alpha)$ acts on the RZ spaces

$$\forall \rho \in T_\alpha : (H, \beta) \mapsto (H, \beta \circ \rho)$$

Def. The cohomology of Rapoport-Zink spaces

$$\varinjlim_V H_c^k(\mathcal{M}_{\alpha,V}^{rig} \times_{E_u^{ur}} \overline{E_u}, \mathbb{Z}/l^r\mathbb{Z})$$

is a representation of $T_\alpha \times GL_n(\mathbb{Q}_p) \times W_{E_u}$.

Prop. Let $V = \{A \equiv \mathbb{I}_n \bmod p^M\}$, $M \geq 0$. Then $\mathcal{M}_{\alpha,V}^{rig}$ has an model $\mathcal{M}_{\alpha,M}$ over Spf $\mathcal{O}_{E_u^{ur}}$. Moreover, if $M = 0$, the formal scheme $\mathcal{M}_{\alpha,0}$ is formally smooth.

The morphism π

Let $\bar{\mathcal{M}}_\alpha = \mathcal{M}_{\alpha,0} \times_{\mathcal{O}_{E_u^{ur}}} \overline{k(u)}$.
Prop. (Rapoport-Zink) Let $x \in |C_\alpha|$ and fix an isomorphism $\phi : \Sigma_\alpha \simeq \mathcal{G}_x$. There exists a map

$$f_\phi : \bar{\mathcal{M}}_\alpha \twoheadrightarrow Isog_x \subset \bar{X}^{(\alpha)} \times \overline{k(u)}$$

s.t. $(\Sigma_\alpha, \mathbb{I}) \in |\bar{\mathcal{M}}_\alpha|$ maps to $x \in |\bar{X}^{(\alpha)} \times \overline{k(u)}|$.

Idea: The "isomorphism" leaves C_x and the "isogeny" leaves $Isog_x$ are two orthogonal directions inside $\bar{X}^{(\alpha)}$, i.e.

$$\text{“}C_x \times Isog_x \sim \bar{X}^{(\alpha)}\text{“}$$

Def. (truncated RZ spaces) $\forall n, d$, the set $\bar{\mathcal{M}}_\alpha^{n,d} = \{(H, \beta) \in |\bar{\mathcal{M}}_\alpha| \,|\, p^n\beta,\, p^{d-n}\beta^{-1}$ are isogenies$\}$ is closed in $\bar{\mathcal{M}}_\alpha$.

Rmk: $f_{\phi_{|\bar{\mathcal{M}}_\alpha^{n,d}}}$ depends only on the restriction

$$\phi_{|[p^d]} : \Sigma_\alpha[p^d] \simeq \mathcal{G}_x[p^d].$$

Lemma. Let $\mathcal{G}$ be a c.s.d. BT group with slope filtration $0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_k = \mathcal{G}$.
Then $\forall\, d \geq 1$, $\exists\, N_d \geq 1$ s.t. *canonically*

$$\mathcal{G}^{(p^N)}[p^d] \simeq \prod_i \mathcal{G}^{i\,(p^N)}[p^d] \qquad \forall N \geq N_d.$$

Main Constr. $\forall m, n, d, N$ s.t. $m \geq d$, $N \geq d$,

$$\exists \quad \pi_N : J_{\alpha,m} \times \bar{\mathcal{M}}_\alpha^{n,d} \to \bar{X}^{(\alpha)} \times \overline{k(u)} \quad \text{s.t.}$$

- π_N is finite and surjective for $m, n, d \gg 0$,
- $\forall m' \geq m$, $\pi_N \circ (q_{m',m} \times 1) = \pi_N$,
- $\forall n' - n \geq d' - d \geq 0$, $\pi_N \circ (1 \times i_{n,d}^{n',d'}) = \pi_N$,
- $\pi_{N+1} = (Fr^B_{\bar{X}} \times 1) \circ \pi_N$,
- $\forall \rho \in S_\alpha$, $\pi_N \circ (\rho \times \rho) = \pi_N$.
- π_N commute with the action of Frobenius.

Prop. $\forall x \in |\bar{X}^{(\alpha)}|$, the fiber $\Pi^{-1}(x) = \{\pi_N^{-1}(x)\}_N$ is a free T_α-principal homogeneous space.

The spectral sequence

Let $\mathcal{L}$ be an abelian torsion sheaf over $\bar{X}^{(\alpha)}$, with torsion orders prime to p (e.g. $\mathcal{L} = \mathbb{Z}/l^r\mathbb{Z}$, $R\Psi(\mathbb{Z}/l^r\mathbb{Z})$).

Prop. (1) For all m, n, d the sheaves

$$\mathcal{F}_m^{n,d} = (Fr_{\bar{X}^{(\alpha)}}^{NB} \times 1)^*(\pi_N)_!(\pi_N)^*(Fr_{\bar{X}^{(\alpha)}}^{NB} \times 1)_!(\mathcal{L})$$

form a direct limit;

(2) the sheaf $\mathcal{F} = \varinjlim_{m,n,d} \mathcal{F}_m^{n,d}$ is endowed with a smooth action of T_α and a morphism $\mathcal{F} \to \mathcal{L}$;

(3) for all points x in $\bar{X}^{(\alpha)}$

$$\mathcal{F}_x = C^\infty(\Pi^{-1}(x), \mathcal{L}_x) \simeq \mathrm{c-Ind}_{\{1\}}^{T_\alpha}(\mathcal{L}_x);$$

(4) If $\mathcal{L}$ is endowed with an acton of $W_{\mathbb{Q}_p}$, then $\mathcal{F}$ is also and the two actions are compatible.

Prop. There is a $W_{\mathbb{Q}_p}$-equivariant spectral sequence

$$E_2^{p,q} = H_p(T_\alpha, H_c^q(\bar{X}^{(\alpha)}, \mathcal{F})) \Rightarrow H_c^{p+q}(\bar{X}^{(\alpha)}, \mathcal{L}).$$

Suppose $\mathcal{L}$ is a l^r-torsion sheaf.
Let $p_2 : J_{\alpha,m} \times \bar{\mathcal{M}}_\alpha^{n,d} \to \bar{\mathcal{M}}_\alpha$ be the projection.

Prop. (Künneth formula) If $\pi^*\mathcal{L} \simeq p_2^*\mathcal{D}$, for a sheaf $\mathcal{D}/\bar{\mathcal{M}}_\alpha$, then

$$\bigoplus_{s+t=q} Tor^p_{T_\alpha}(H^s_c(\bar{\mathcal{M}}_\alpha, \mathcal{D}), H^t_c(J_\alpha, \mathbb{Z}/l^r\mathbb{Z})) \Rightarrow$$

$$\Rightarrow H_c^{p+q}(\bar{X}^{(\alpha)}, \mathcal{L})$$

(e.g. $\mathcal{L} = \mathbb{Z}/l^r\mathbb{Z}$ and $\mathcal{D} = \mathbb{Z}/l^r\mathbb{Z}$.)

Let $f_M : X_M \to X$ between Shimura varieties, and $\mathcal{L} = R\Psi(f_{M*}\mathbb{Z}/l^r\mathbb{Z}_{/X_M})_{|\bar{X}^{(\alpha)}}$.
Let $g_M : \mathcal{M}^{\text{rig}}_{\alpha,M} \to \mathcal{M}^{\text{rig}}_\alpha$ between the RZ spaces.

Prop. There exists a system of $W_{\mathbb{Q}_p}$-equivariant isomorphisms

$$\pi^* R\Psi(f_{M*}\mathbb{Z}/l^r\mathbb{Z}_{/X_M})_{|\bar{X}^{(\alpha)}} \simeq p_2^* R\Psi(g_{M*}\mathbb{Z}/l^r\mathbb{Z}_{/\mathcal{M}_{\alpha,M}}).$$

Lifting to formal schemes over $\widehat{\mathbb{Z}}_p^{nr}$

Let $\mathfrak{X} = X^{\wedge}_{/\bar{X}}$ and $\mathfrak{C}_\alpha = X^{\wedge}_{/C_\alpha}$.
Let $\mathcal{J}_{\alpha,m} \to \mathfrak{C}_\alpha$ be the finite étale Galois covers corresponding to $J_{\alpha,m} \to C_\alpha$.
Let $\mathfrak{X}_M \to \mathfrak{X}$ and $\mathcal{M}_{\alpha,M} \to \mathcal{M}_{\alpha,M}$ be the spaces with structure of level M at p.

Let $Y/\mathrm{Spf}\,\widehat{\mathbb{Z}}_p^{nr}$, $\mathcal{I}$ an ideal of definition of Y, $p \in \mathcal{I}$. Then $Y(t) = Z(\mathcal{I}^t)$ over $\widehat{\mathbb{Z}}_p^{nr}/p^t$.

- For any t, when $m, N \gg d, t$, the morphisms $\pi_N : J_{\alpha,m} \times \mathcal{M}_\alpha^{n,d} \to \bar{X}^{(\alpha)}$ lift to some morphisms

$$\pi_N(t) : (\mathcal{J}_{\alpha,m} \times \mathcal{M}_\alpha^{n,d})(t) \to \mathfrak{X}(t)$$

s.t. $\pi_N(t)^*(\mathfrak{X}_{t/2}) \simeq p_2(t)^*(\mathcal{M}_{\alpha,t/2})$, and also $\pi_N(t)(t-1) = \pi_N(t-1)$.

- For any affine open $V \subset \mathcal{J}_{\alpha,m} \times \mathcal{M}_\alpha^{n,d}$, the morphism $\pi_N(t)_{|V}$ lifts to a morphism

$$\pi_{V,t} : V \to \mathfrak{X}$$

s.t. $\pi_{V,t}^*(\mathfrak{X}_{t/2}) \simeq p_{2|V}^*(\mathcal{M}_{\alpha,t/2})$.

Comparing vanishing cycles

Prop. (1) If $t/2 \geq M$, the morphisms $\pi_{V,t}$ give rise to some $W_{\mathbb{Q}_p} \times T_\alpha \times GL_n(\mathbb{Q}_p)$-equivariant isomorphisms over $\bar{V} = V \times \bar{\mathbb{F}}_p$

$$\pi^* R\Psi(f_{M*}\mathbb{Z}/l^r\mathbb{Z}_{/X_M})_{|\bar{V}} \simeq p_2^* R\Psi(g_{M*}\mathbb{Z}/l^r\mathbb{Z}_{/\mathcal{M}_{\alpha,M}})_{|\bar{V}}.$$

(2) $\forall M \, \exists t_0$ such that the above isomorphism piece together, for $t \geq t_0$.

Rmk. As M varies, the above sheaves form a system with an action of $\langle GL_n(\mathbb{Z}_p), pI_n \rangle \subset GL_n(\mathbb{Q}_p)$. By introducing many more models one is able to recover the action of the whole group $GL_n(\mathbb{Q}_p)$.

The cohomology of the RZ spaces

Let Π be an admissible $\mathbb{Q}_l$-representation $T_\alpha \times W_{\mathbb{Q}_p}$ (e.g. $\Pi = H_c^q \cdot (J_{\alpha,U^p}, \mathbb{Q}_l)$, for some $q \geq 0$).

Thm. (1) All the representations appearing below are admissible.
(2)There is an equality of virtual $\mathbb{Z}/l^r\mathbb{Z}$-representaion of $GL_n(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$

$$\varinjlim_M Tor^{\bullet}_{T_\alpha}(H_c^{\bullet}(\bar{\mathcal{M}}_{\alpha,M}, R^{\bullet}\Psi(\mathbb{Z}/l^r\mathbb{Z})), \Pi) =$$

$$= \varinjlim_M Ext^{\bullet}_{T_\alpha}(H_c^{\bullet}(\mathcal{M}^{\mathrm{rig}}_{\alpha,M} \times \bar{E}_u, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi).$$