Achieving Mass Conservation in Adaptive Semi-Lagrangian Advection Schemes

Jörn Behrens

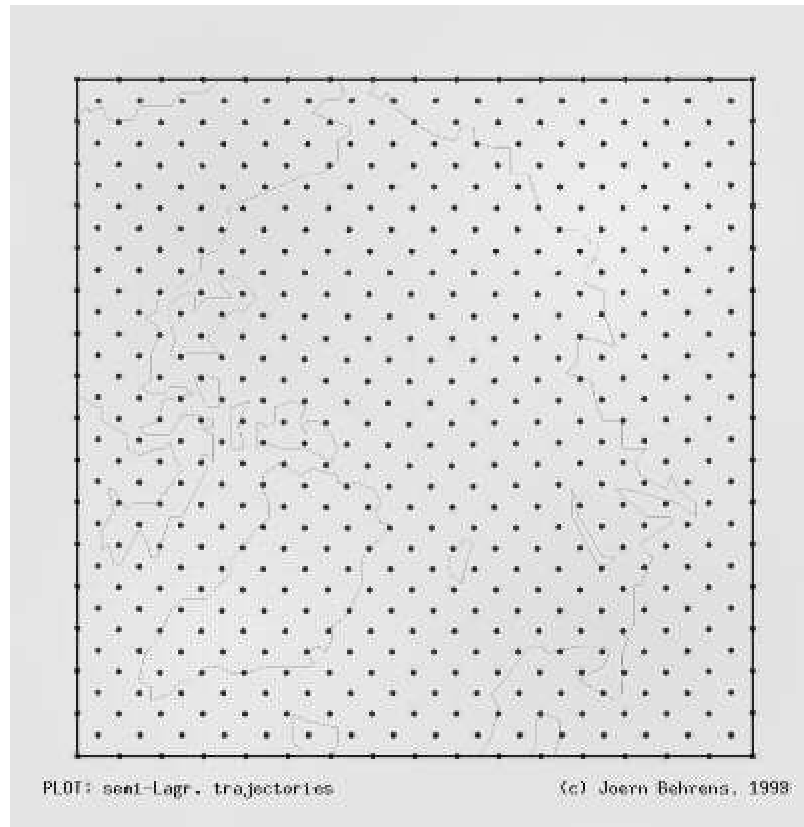
Technische Universität München
Munich, Germany

behrens@ma.tum.de
www-m3.ma.tum.de/m3/behrens

J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Adaptive Semi-Lagrangian Simulation I

Adaptive simulation of Trace Gas Transport in the Arctic Stratosphere

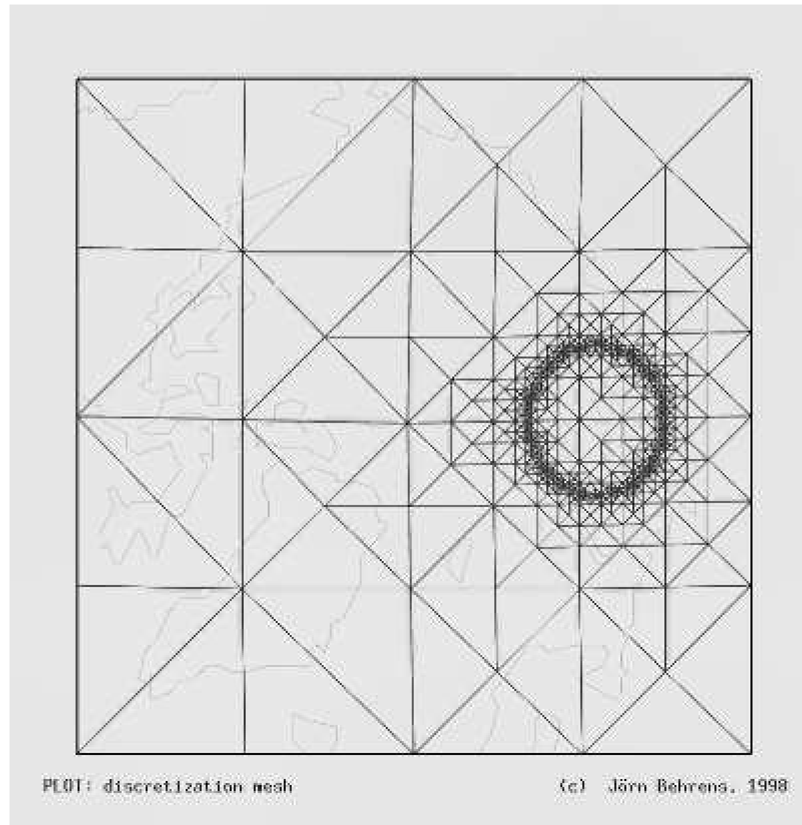


J. B., K. Dethloff, W. Hiller, A. Rinke (2000)

J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Adaptive Semi-Lagrangian Simulation II

Adaptive simulation of Trace Gas Transport in the Arctic Stratosphere

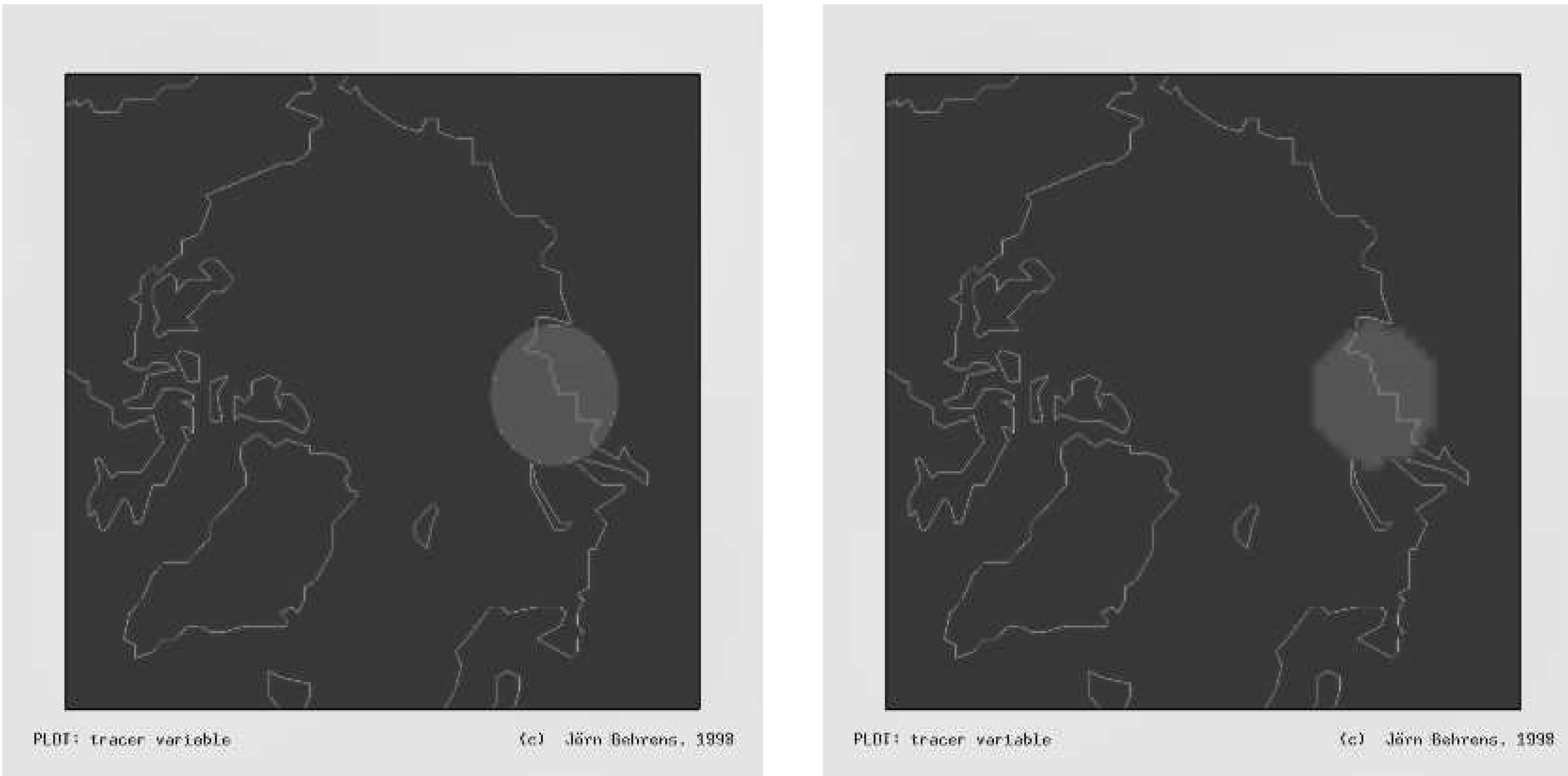


J. B., K. Dethloff, W. Hiller, A. Rinke (2000)

J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Adaptive Semi-Lagrangian Simulation III

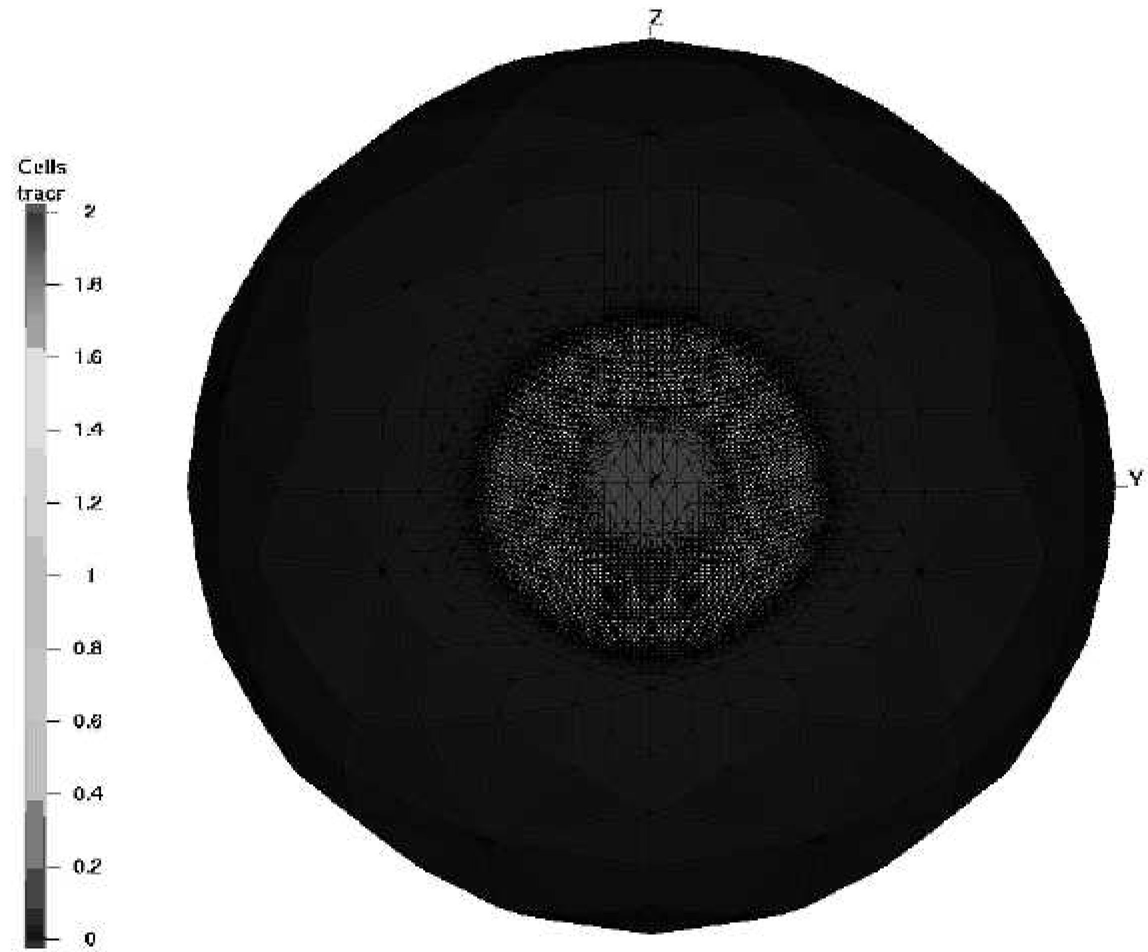
Adaptive simulation of Trace Gas Transport in the Arctic Stratosphere



J. B., K. Dethloff, W. Hiller, A. Rinke (2000)

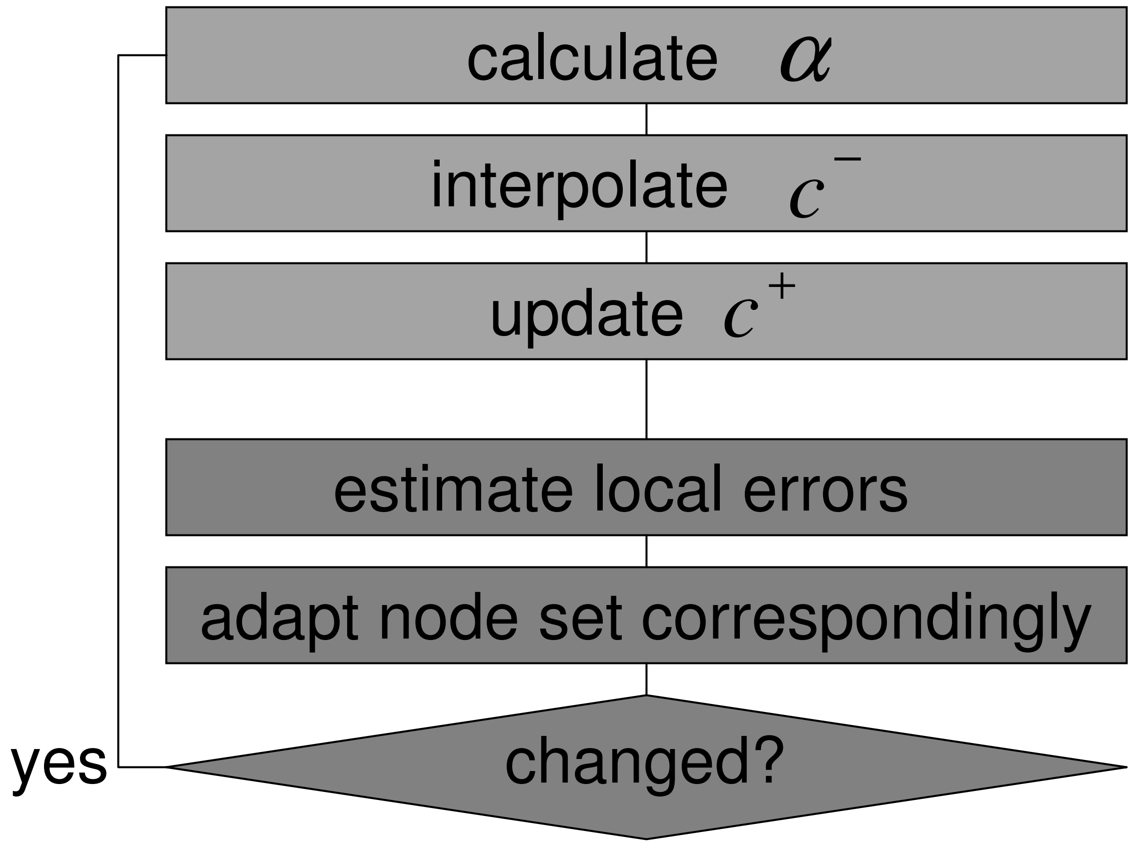
J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Adaptive Semi-Lagrangian Simulation IV



J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Adaptive Semi-Lagrangian Algorithm



J. B., (1996)/(1998)

J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Features of SLM

- Suitable for adaptivity
- Stable – no time step control for stability
- Parallelizable – each trajectory independent

Not conservative by construction

$\longrightarrow$ Objective: Construct Conservative Scheme

J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Existing Work

Priestley‘s Scheme

- Based on conserving mass globally
- Linear programming approach
- Exact conservation
- Possibly non-physical

Xiao/Yabe Scheme

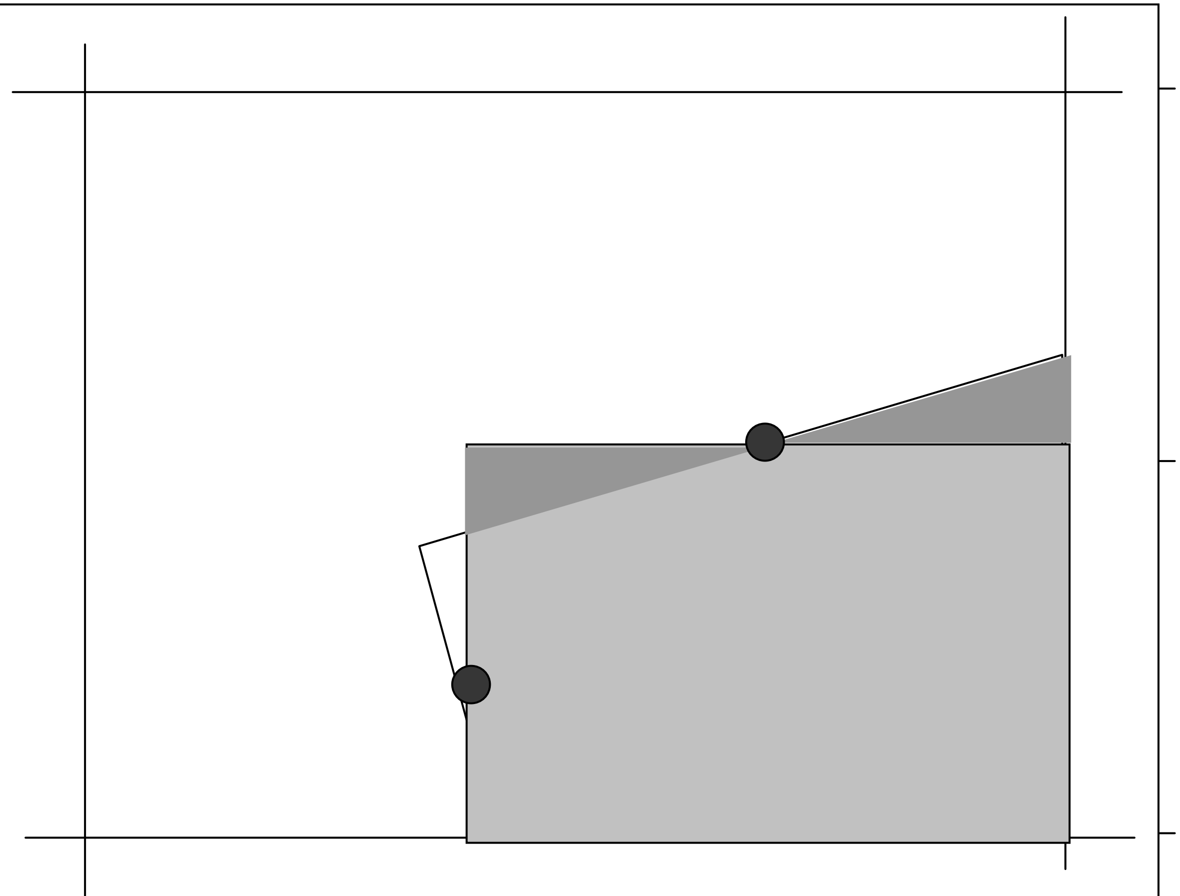
- Based on geometry
- Exact conservation
- Locally exact

A. Priestley (1993), Xiao, Yabe (2000)

J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Cell Integrated Scheme



$$c(\vec{x}, t+\Delta t) = c(\vec{x} - 2\vec{\alpha}, t - \Delta t)$$

$$c(\vec{x}, t+\Delta t) = Vol(\ \square\ ) \cdot c(\vec{x}\,^*, t - \Delta t)$$

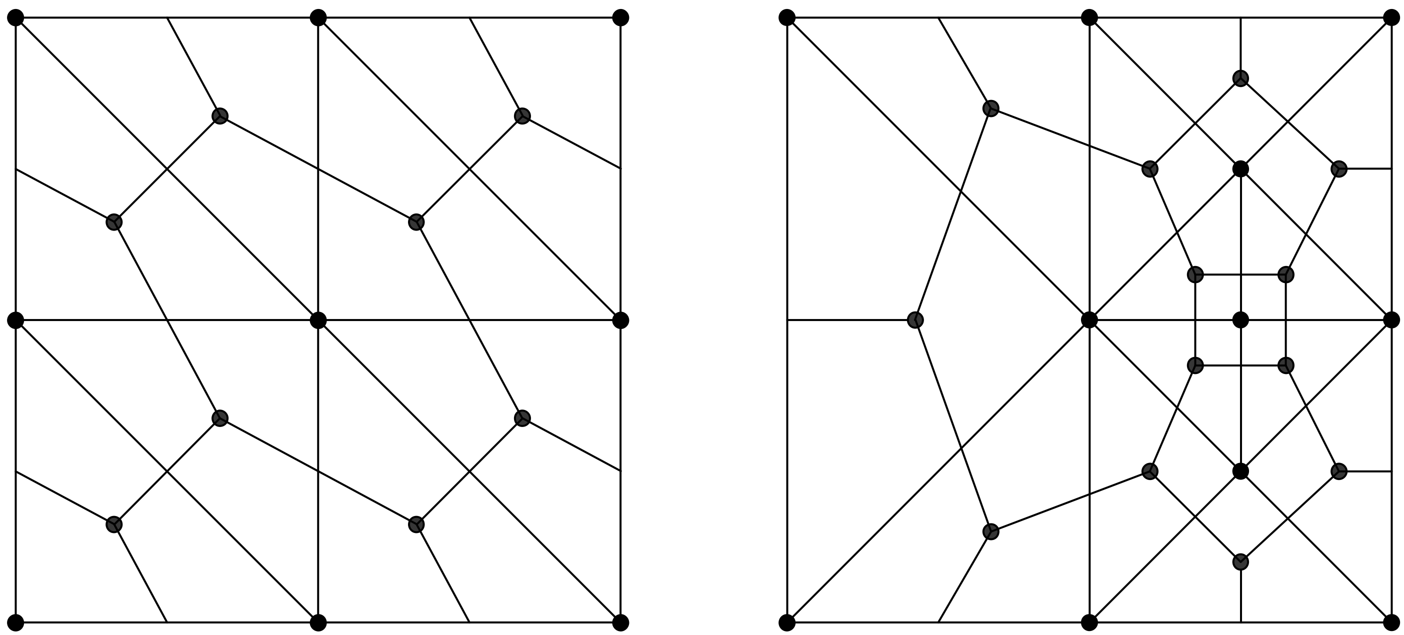
Scroggs, Semmazzi (1995), Machenhauer, Olk (1995)

Modification to Cell Integrated Scheme

Dual Mesh

- Dual grid point
  $\cong$
  Primary grid cell

- Primary grid point
  $\cong$
  Dual grid cell



J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Modification to Cell Integrated Scheme

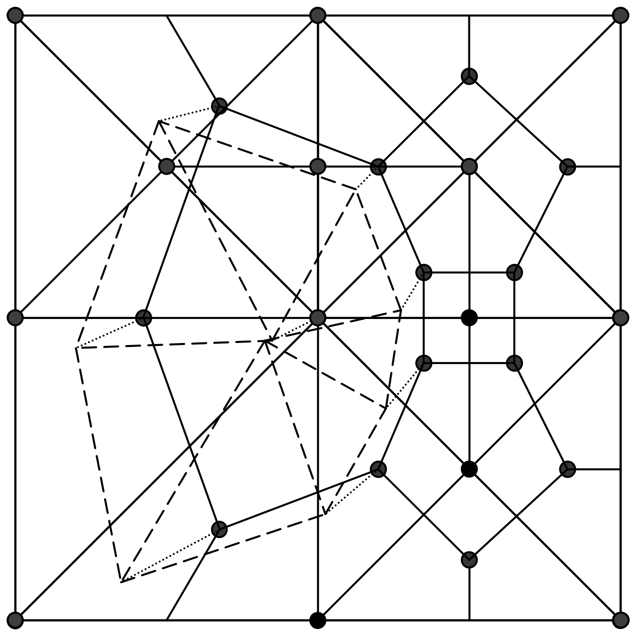
1. Look at **dual cell**, corresponding to **gridpoint**

2. **Advect** dual nodes

3. Create **upstream dual cell**

4. Create **local Triangulation**

5. Intersect with **old mesh**



J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

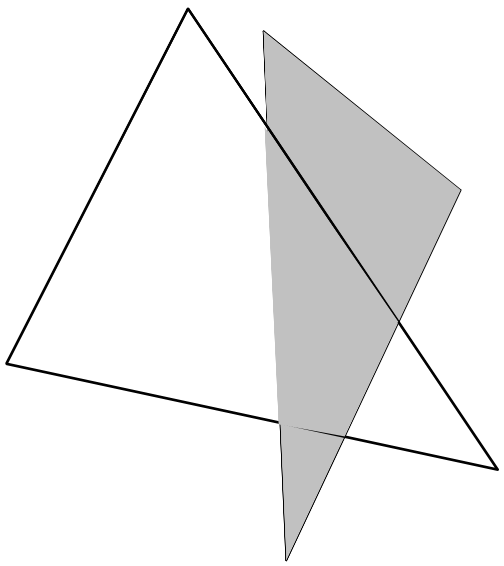
Polygon/Triangle Intersection

Algorithm:

`INPUT`: two oriented
AND convex polygons

`For` all edges:
 clip what's to the right
`End`

`OUTPUT`: Intersection polygon



$O(3(3\cdot 2+3))=O(27)$ Crossproducts

Angell (~1989)

J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

1st Order Cell Integrated Scheme

Let λ the dual cell corresponding to $\vec{x}$

$$c(\vec{x}, t + \Delta t) = \sum_{\substack{\tau \in T \\ \tau \cap \lambda \neq \emptyset}} Vol(\tau \cap \lambda) \cdot \overline{c}(\tau)$$

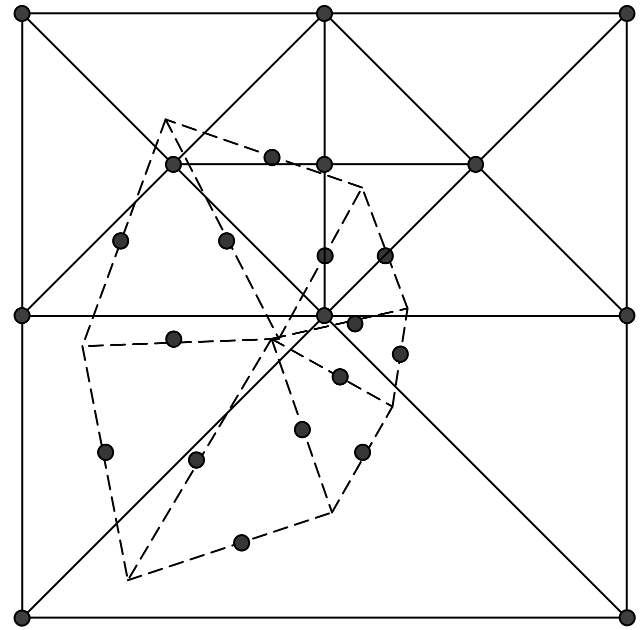
For higher order:

$$c(\vec{x}, t + \Delta t) = \sum_{\substack{\tau \in T \\ \tau \cap \lambda \neq \emptyset}} Vol(\tau \cap \lambda) \cdot I(\tau \cap \lambda)$$

$$I(\tau \cap \lambda) = \int_{\tau \cap \lambda} c(\xi, t - \Delta t) d\xi$$

J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Quadrature Based Scheme



Create **upstream dual cell**

Interpolate at **quadrature points**

$$c(\vec{x}, t+\Delta t) = \sum_{\tau \in \mathrm{P}} Vol(\tau) \sum_{\xi \in \tau} c(\xi, t-\Delta t)$$

Giraldo (1995)

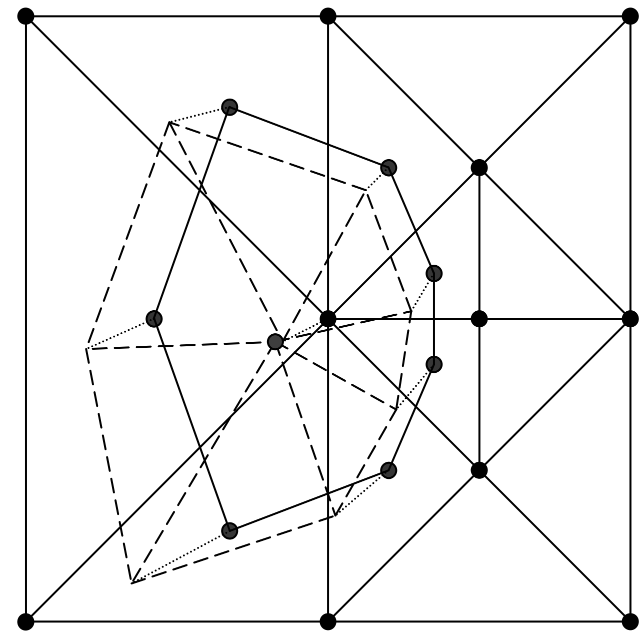
J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Dual Mesh Correction

Create **dual cell**

Create **upstream dual cell**

Interpolate **upstream value**

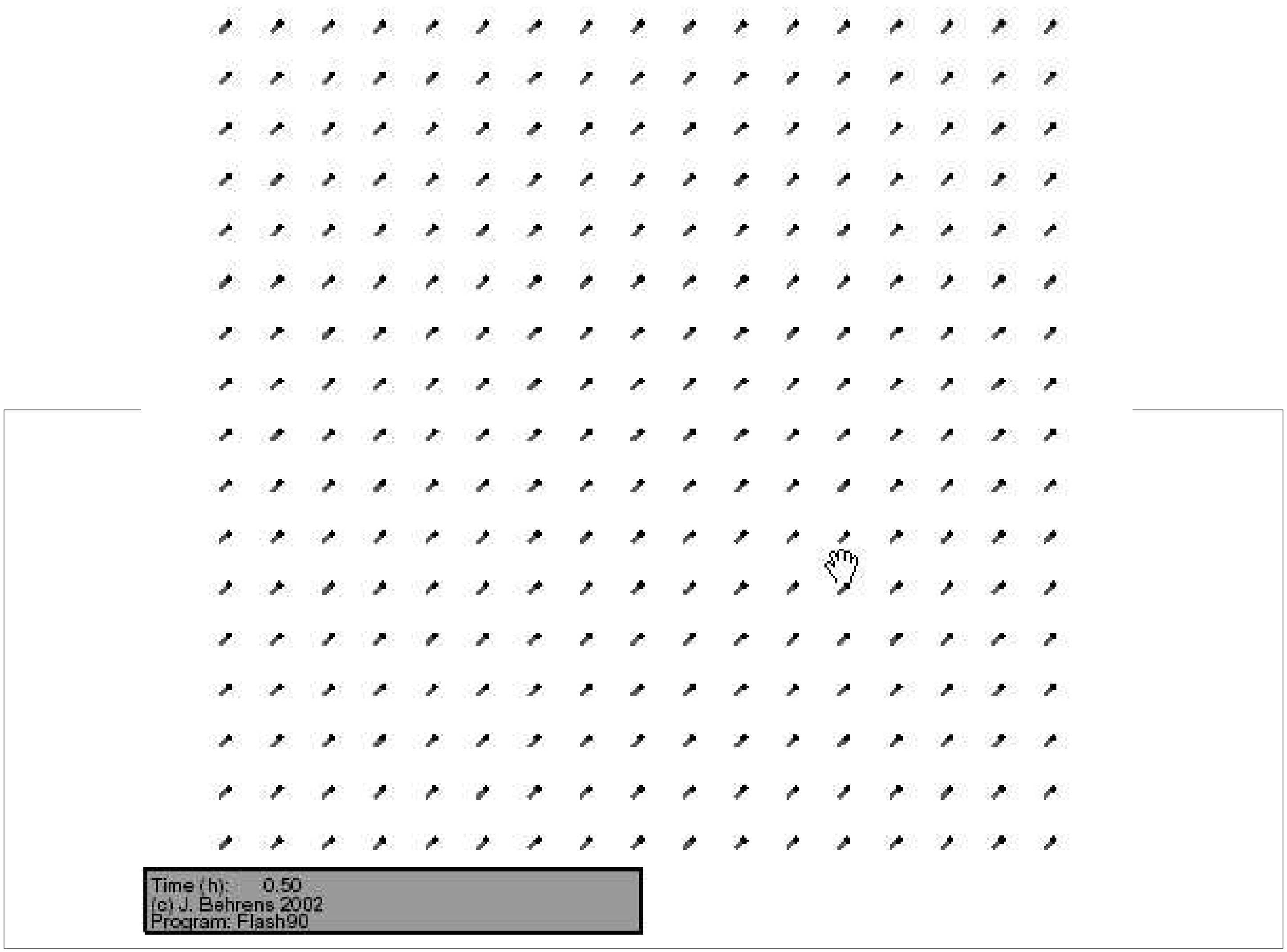


Correct value by:

$$c(\vec{x},t+\Delta t)=c(\vec{x}-\vec{\alpha},t-\Delta t)\cdot Vol(\;)/Vol(\;)$$

J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

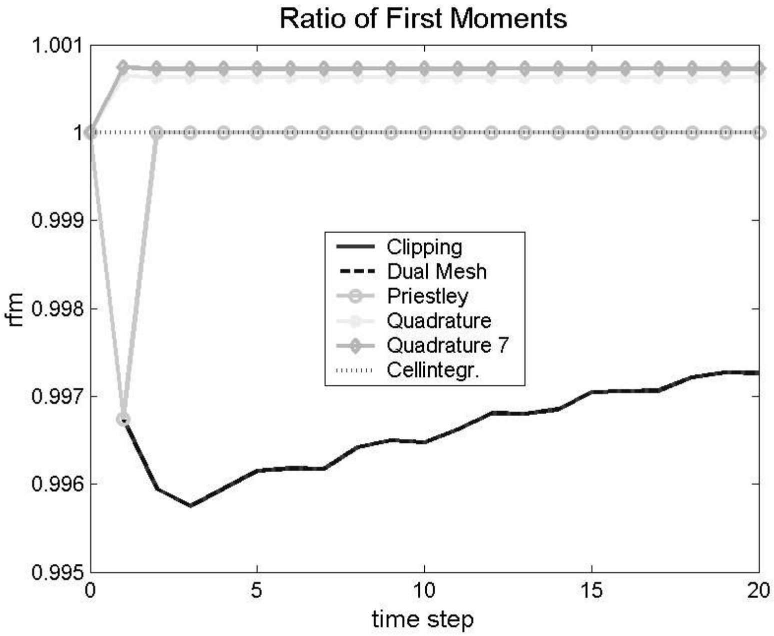
Testcases: Monotone Wind I

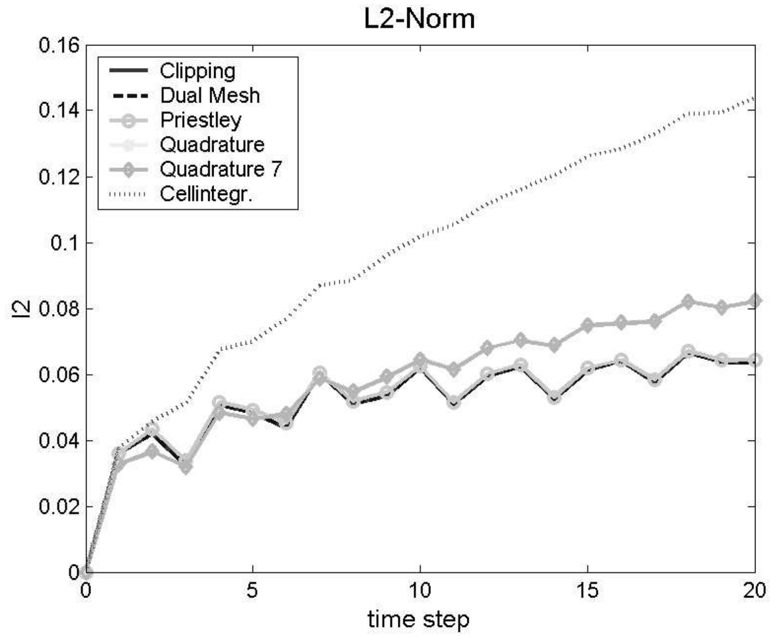


J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Testcases: Monotone Wind II

Non-Adaptive Case

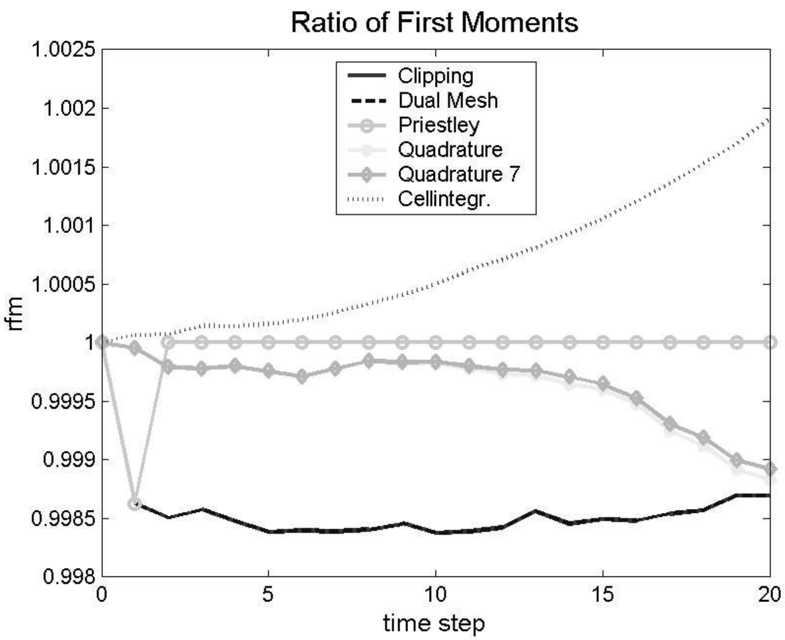


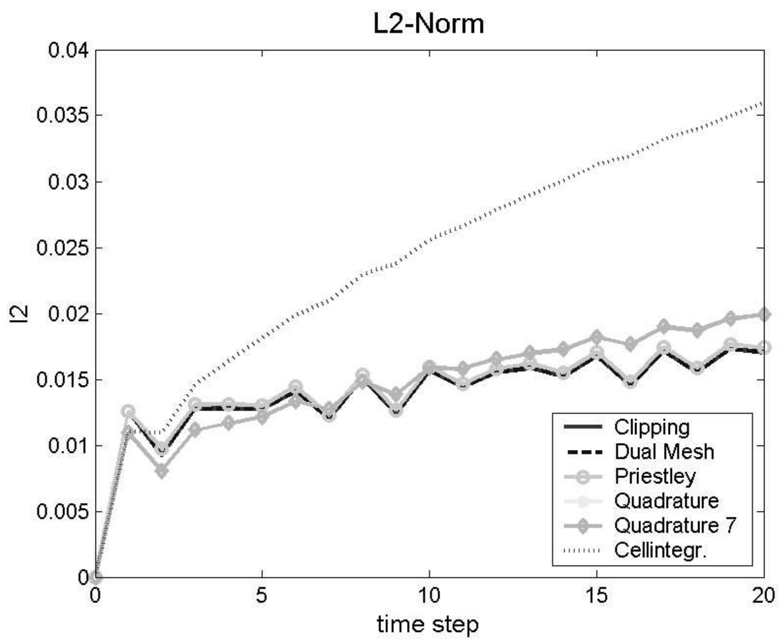


J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Testcases: Monotone Wind II

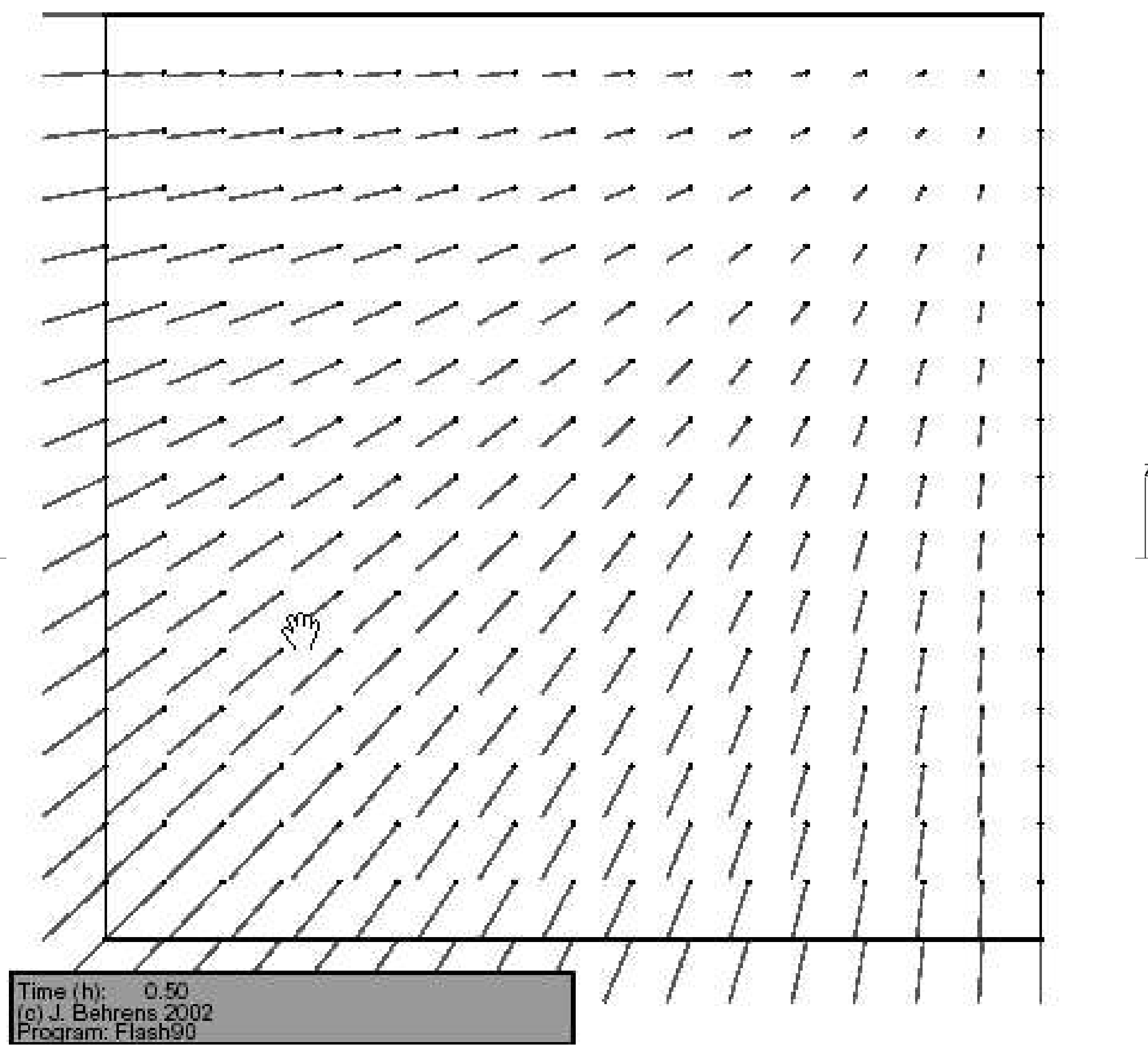
Adaptive Case





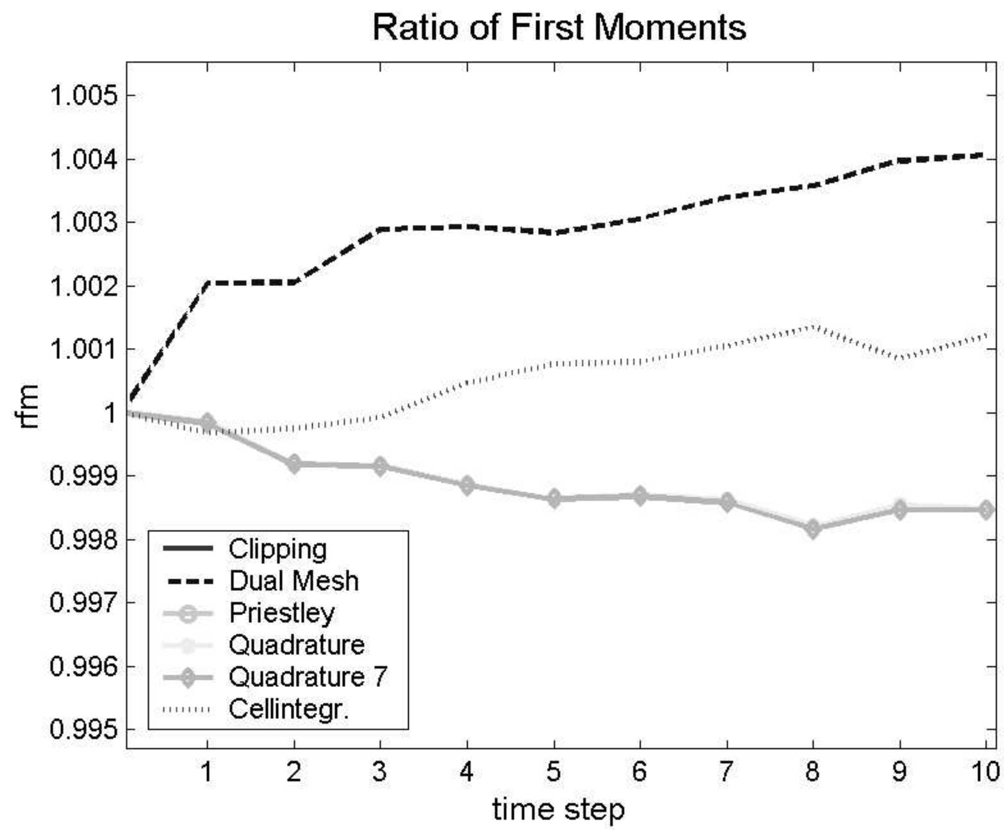
J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Testcases: Converging Wind I



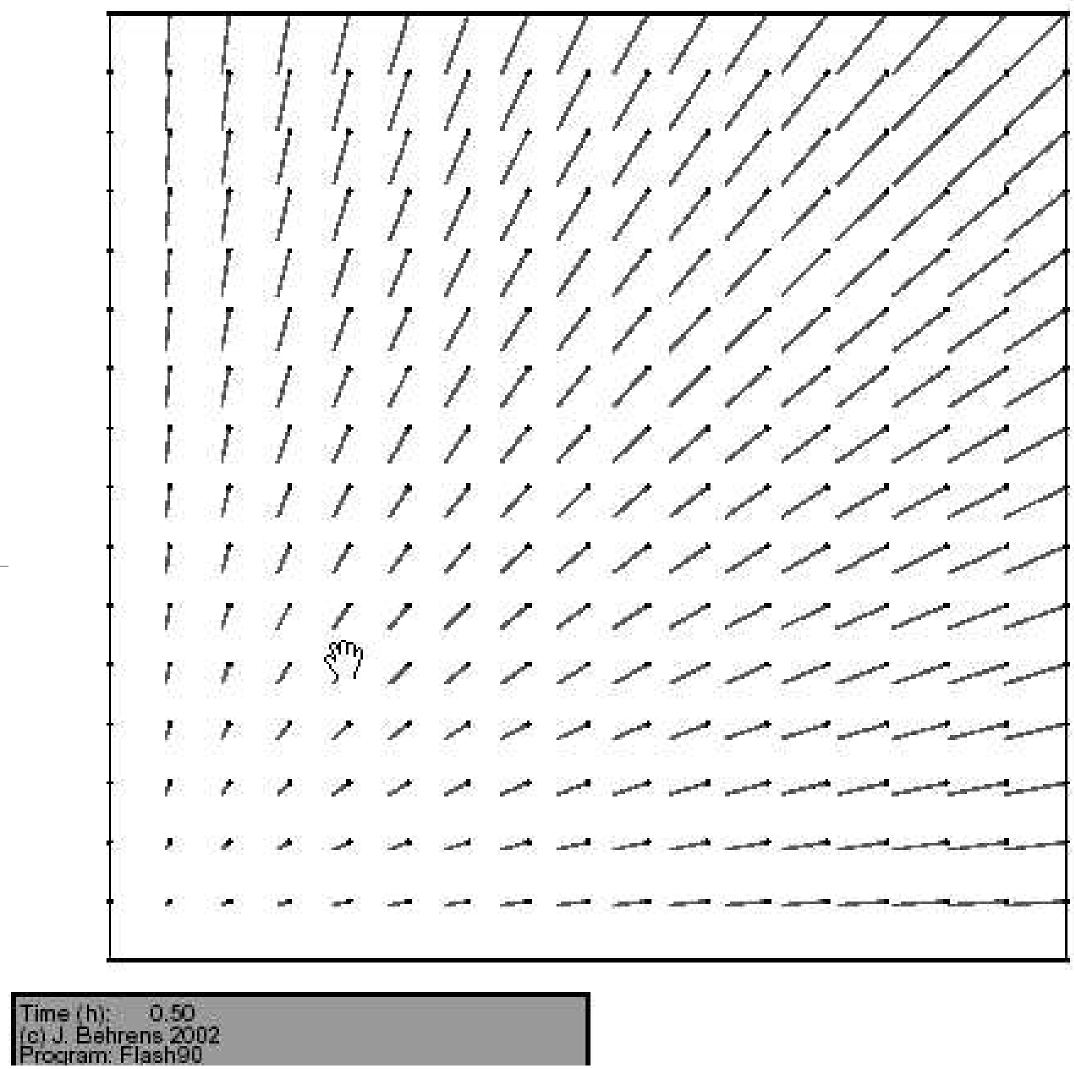
J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Testcases: Converging Wind II



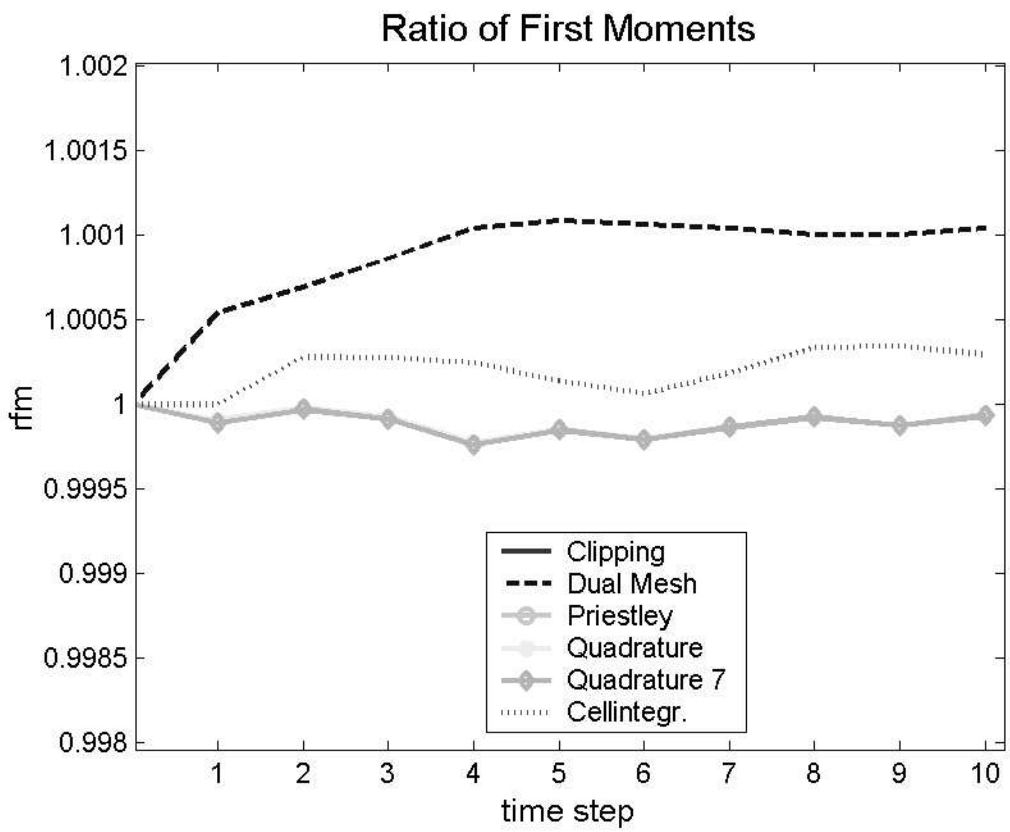
J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Testcases: Diverging Wind I



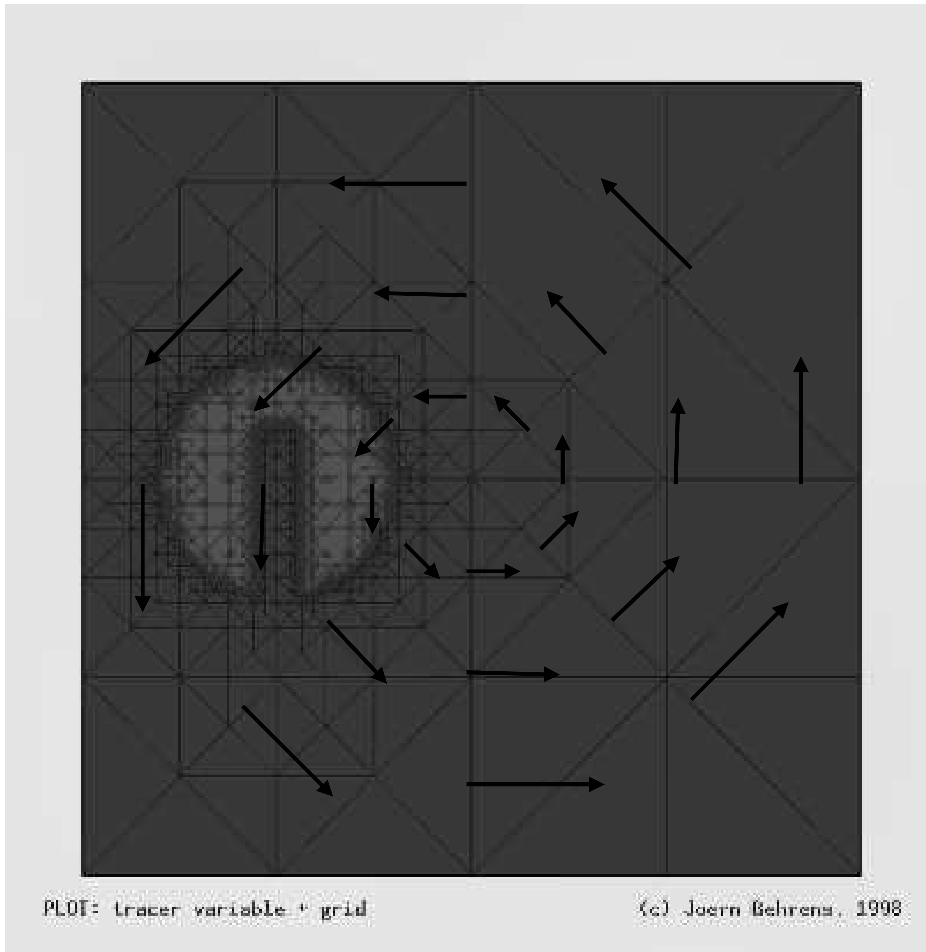
J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Testcases: Diverging Wind II



J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

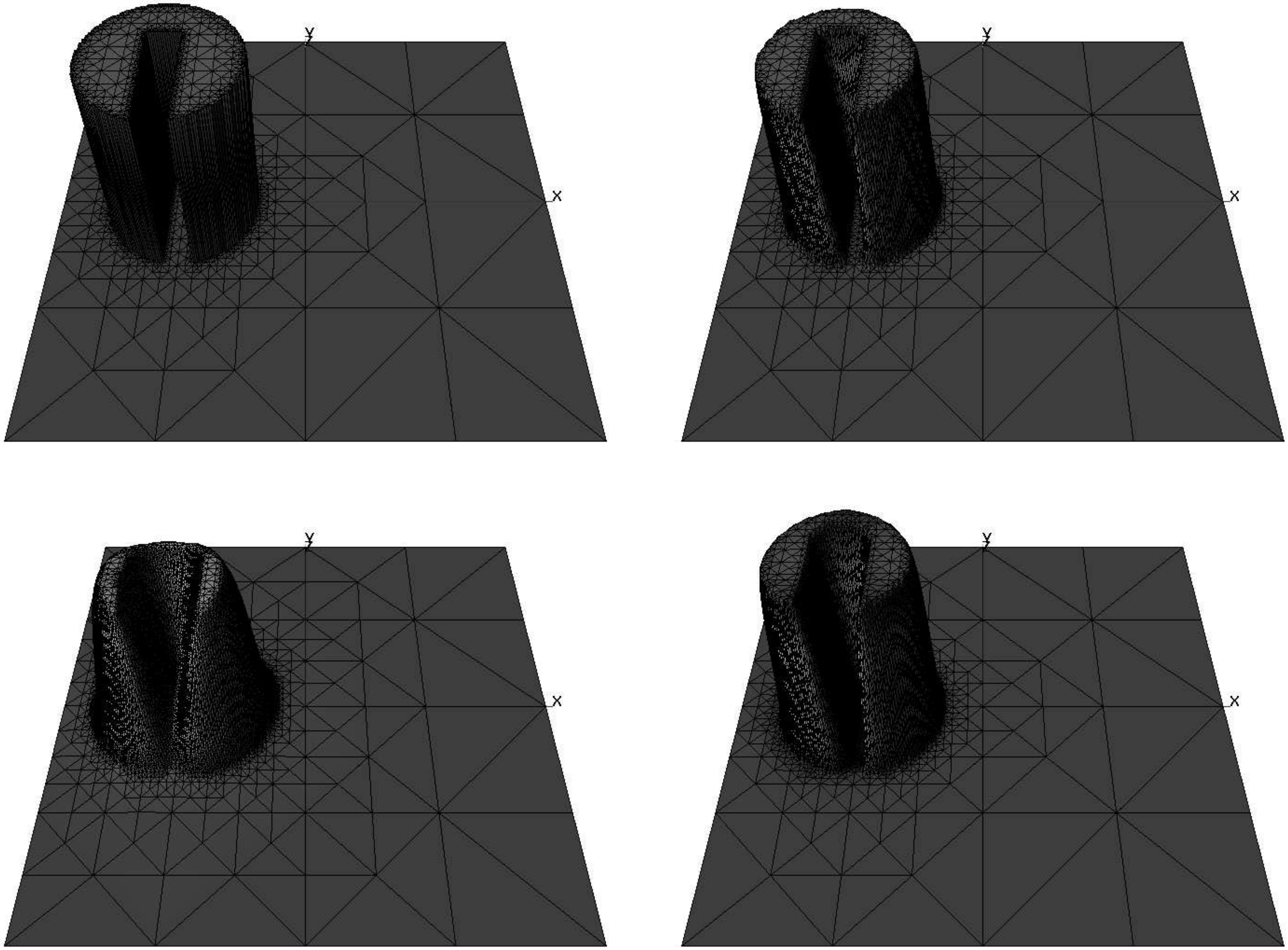
Slotted Cylinder



Zalesak (1979)

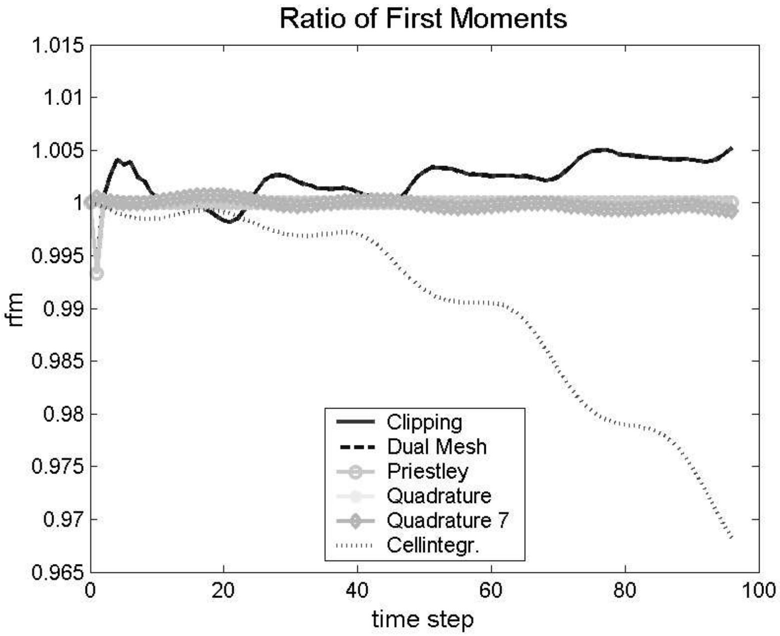
J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

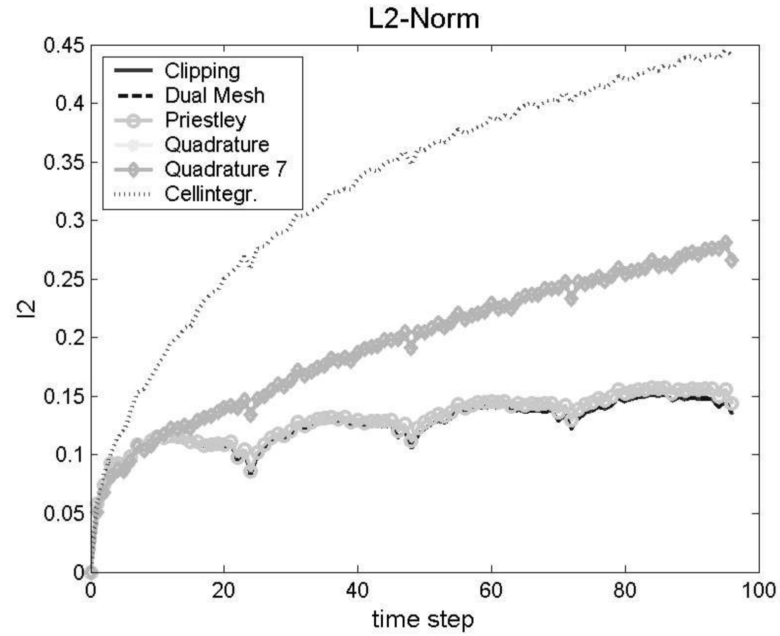
Testcases: Slotted Cylinder II



J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

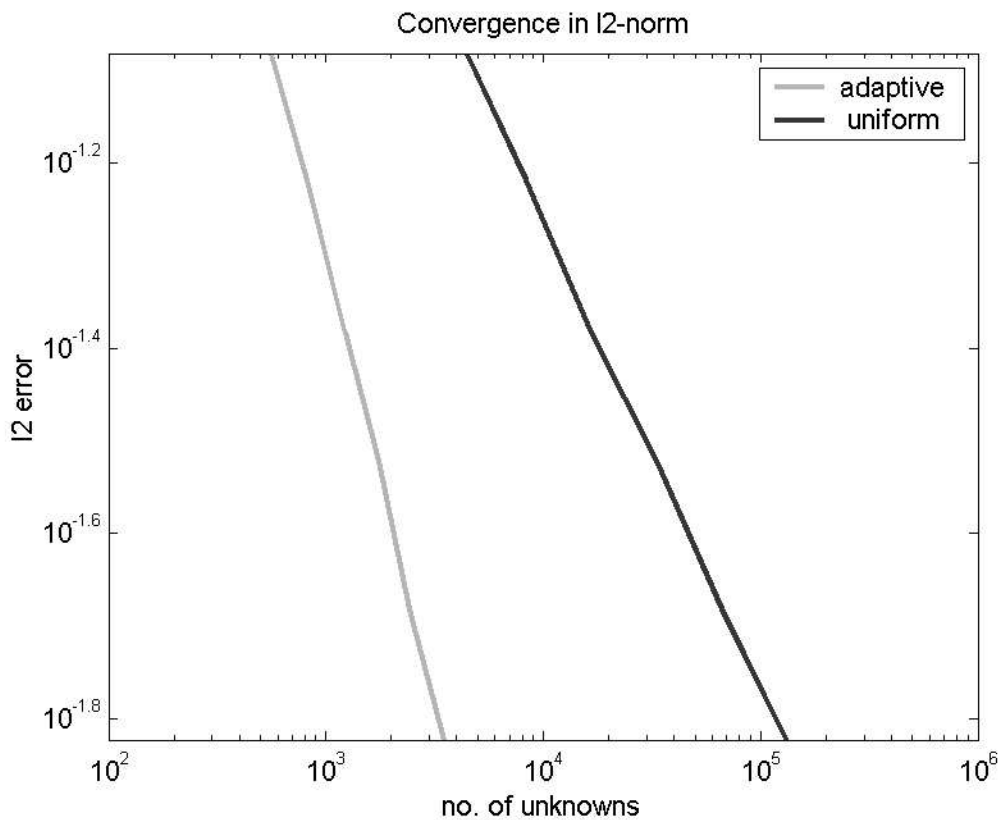
Testcases: Slotted Cylinder III





J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Testcases: Slotted Cylinder IV



J. Behrens
TU München
Scientific Computing
behrens@ma.tum.de

Conclusions

- Several (almost) mass conserving
  Semi-Lagrangian Schemes presented
- Computational geometry helps solve the problem
- Adaptivity helps to improve order of convergence
- Not complete yet: boundary, sphere, etc.

J. Behrens
TU München
Scientific Computing
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Scientific Computing
behrens@ma.tum.de

behrens@ma.tum.de
www-m3.ma.tum.de/m3/behrens/