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Fast Spherical Harmonic Transform of FLTSS and its evaluation

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Fast Legendre Transform - Contents

- FLTSS
  - Implementation and its performance
- Applications
  - Shallow-water equations
  - Turbulent flow

FLTSS

- FLTSS --- Fast Legendre Transform with Stable Sampling
  - Fast Legendre transform algorithm based on the FMM (Fast Multipole Method)
    - Approximate transform
    - Complexity $O(T^2 \log T \log(1/\varepsilon))$
    - Numerically stable
  - Error control in weighted 2-norm (damping)

The idea of fast transform (1)

- Legendre functions can be interpolated as polynomials

$$P_n^m(y_j) = P_m^m(y_j)\omega(y_j)\sum_i \frac{1}{y_j - x_i} \frac{P_n^m(x_i)}{P_m^m(x_i)\omega_i(x_i)}$$

$$\omega(y) = \prod y - x_j$$

$$\omega_i(y) = \omega(y)/(y - x_i)$$

FMM computes the summation in ***linear time approximately***

The idea of fast transform (2)

Evaluation of an N-term transform on K points

$O(KN)$ by direct computation

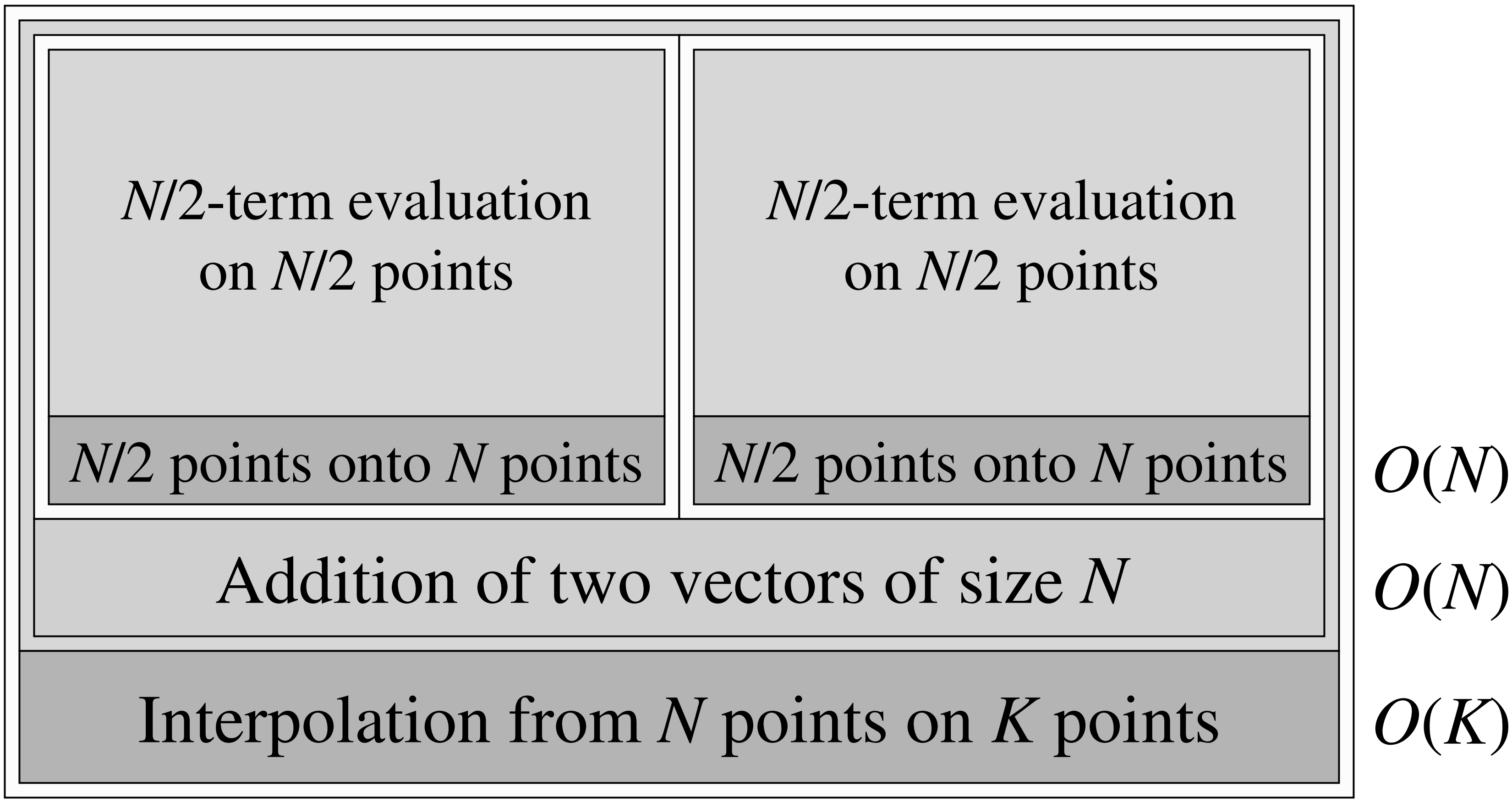
The idea of fast transform (2)

N-term evaluation on N points

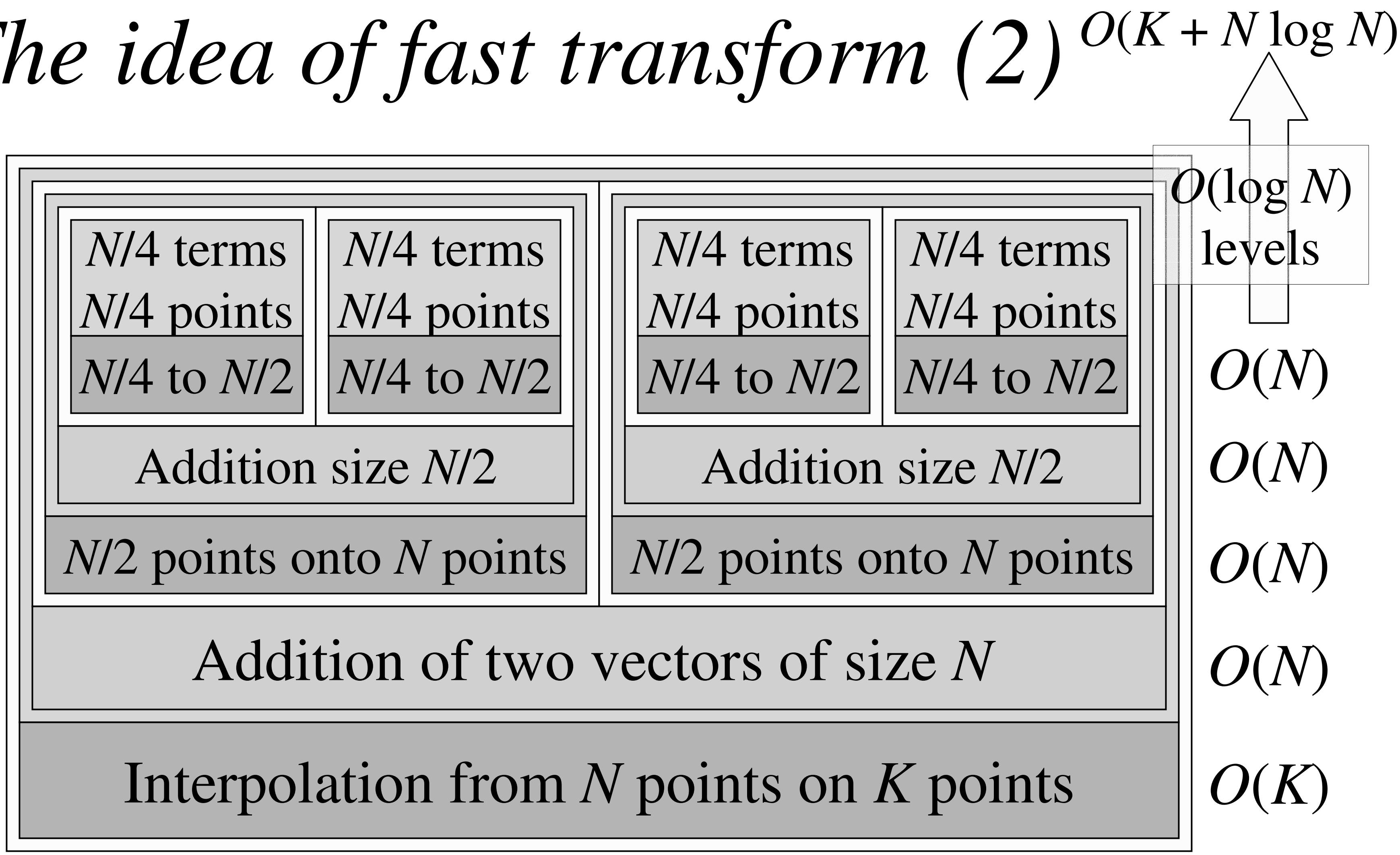
$O(N^2)$ in direct computation

Interpolation from N points on K points $O(K)$

The idea of fast transform (2)



The idea of fast transform (2)



Speedup rates in flop counts

T	$\varepsilon = 10^{-6}$		$\varepsilon = 10^{-10}$	
	error	speedup	error	speedup
127	1.00e-6	1.456	1.05e-10	1.281
170	1.16e-6	1.597	1.13e-10	1.376
255	1.24e-6	1.825	1.32e-10	1.536
511	1.52e-6	2.281	1.37e-10	1.903
1365	2.02e-6	3.983	2.24e-10	2.720
4095	2.62e-6	9.114	2.85e-10	5.415

Implementation of FLTSS

- Transform code written in fortran90
  - Preprocessing code in C
    - Complexity $O(T^3 \log T)$ or $O(T^4)$
    - Error control and stability optimization
    - Preprocessed data stored in disk
- Evaluation and expansion routines
  - Legendre functions and differentiated ones
  - real and complex

First implementation (ver0.10)

- Straightforward implementation of the evaluation algorithm for real
  - Using *recursive* feature of fortran90
- Timing results --- miserable!

T170	original	fast	
		$\varepsilon=10^{-6}$	$\varepsilon=10^{-10}$
time	39.26	63.50	48.26

chammp case-1 12hours: alpha 21264 666MHz

Improved implementation (ver0.20)

- Unfold all computations
  - Flat structure of multiply-and-add's ... similar to sparse matrix-vector product
- Timing results
  - Much better, but yet not satisfactory

T=170	original	fast	
		$e=10^{-6}$	$e=10^{-10}$
time	39.26	31.58	33.37

chammp case-1 12hours: alpha 21264 666MHz

Performance improvements?

- On RISC/vector processors
  - Unrolling the outer loop
  - Multiple-vector transform
    - High perf in single-vector transform hopeless?
- On parallel processors
  - Trivial parallelism in m-dimension
    - Load balancing is not trivial
  - Further parallelization of finer grain?

Application of FLTSS

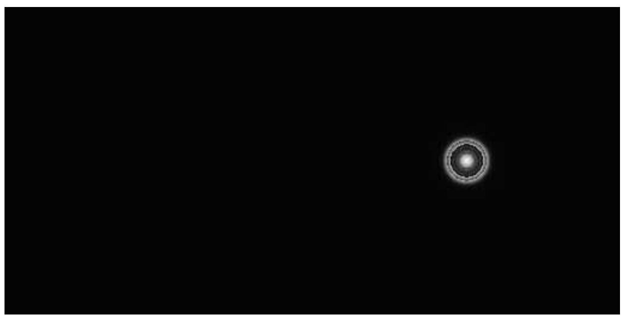
Applications of FLTSS

- Replacing Legendre transforms in existing codes: T170, $\varepsilon = 10^{-10}/10^{-6}$
- Shallow-water test sets on a sphere
  - chammp (stswm)
  - Test cases 1, 3 and 6
- Turbulent flow on a sphere
  - Developed by Yoden and Ishioka

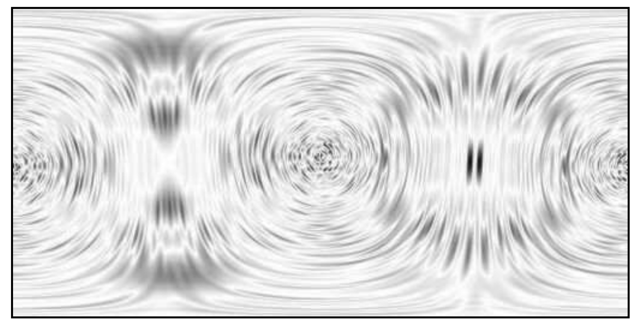
Case 1: error and difference

$\alpha = \pi / 2$

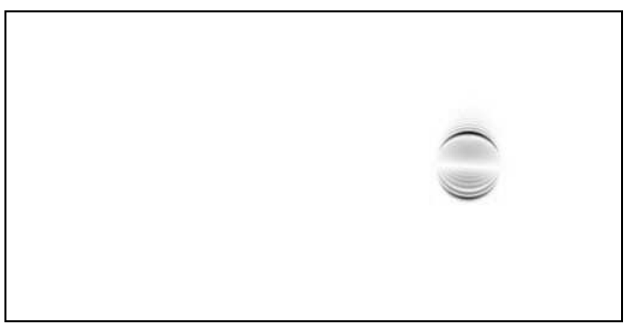
Height 12 days: max 1000m



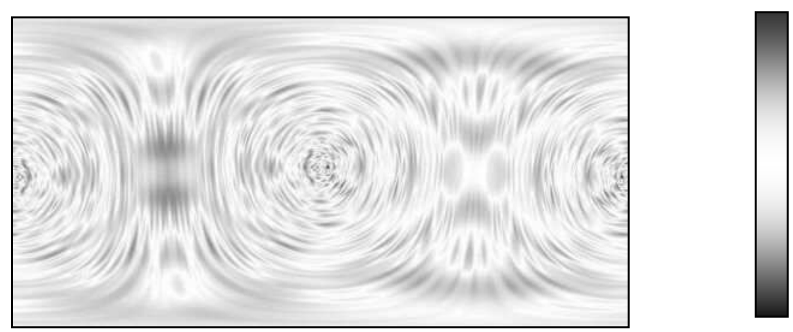
Diff 10^{-10}: max 10^{-6} m



Error: max 5m



Diff 10^{-6}: max 10^{-2} m

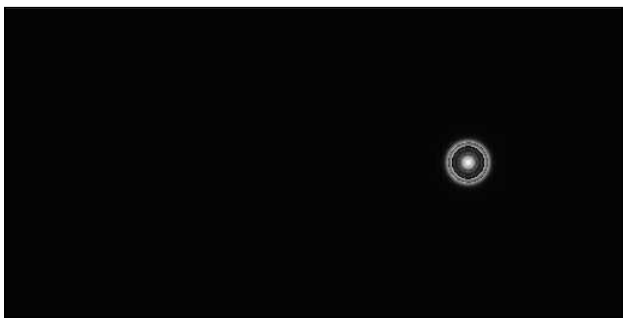


Difference from the original code is spread over the sphere

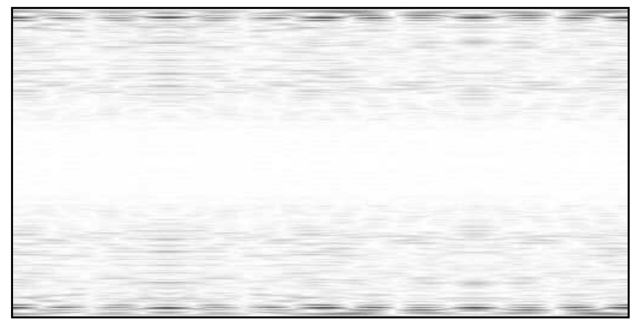
Case 1: error and difference

$\alpha = 0$

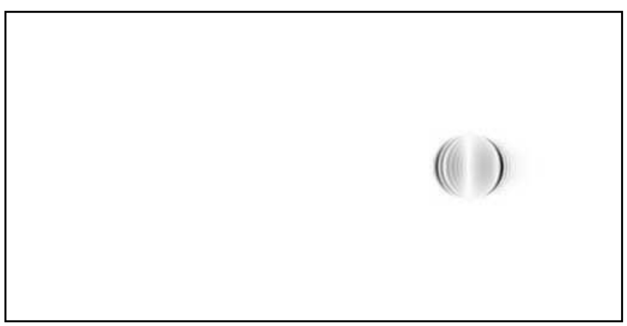
Height: max 1000m



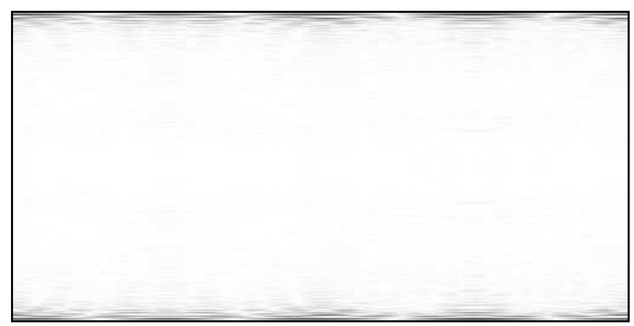
Diff 10^{-10}: max 10^{-5} m



Error: max 5m



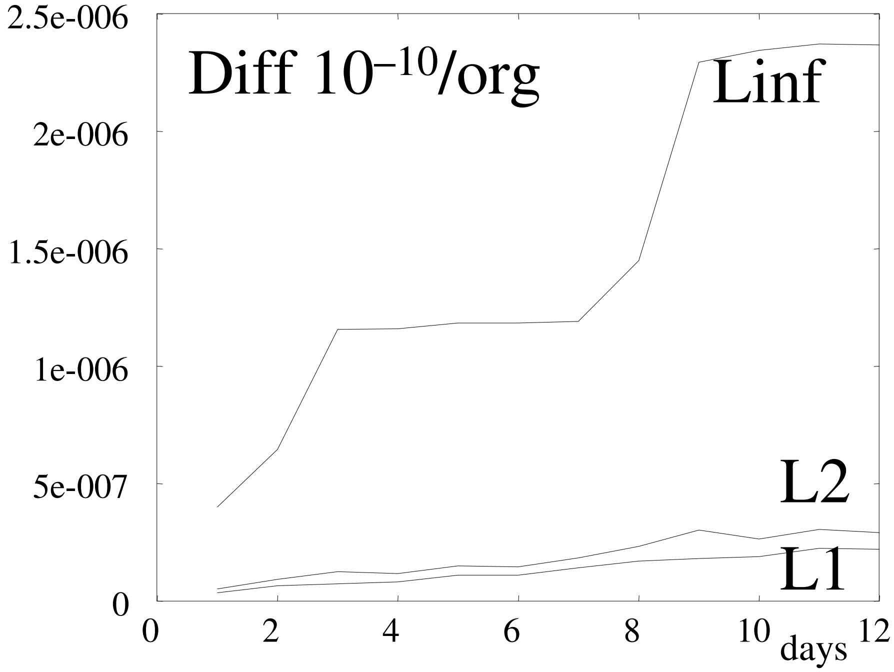
Diff 10^{-6}: max 0.5 m



Difference larger and localized about the poles

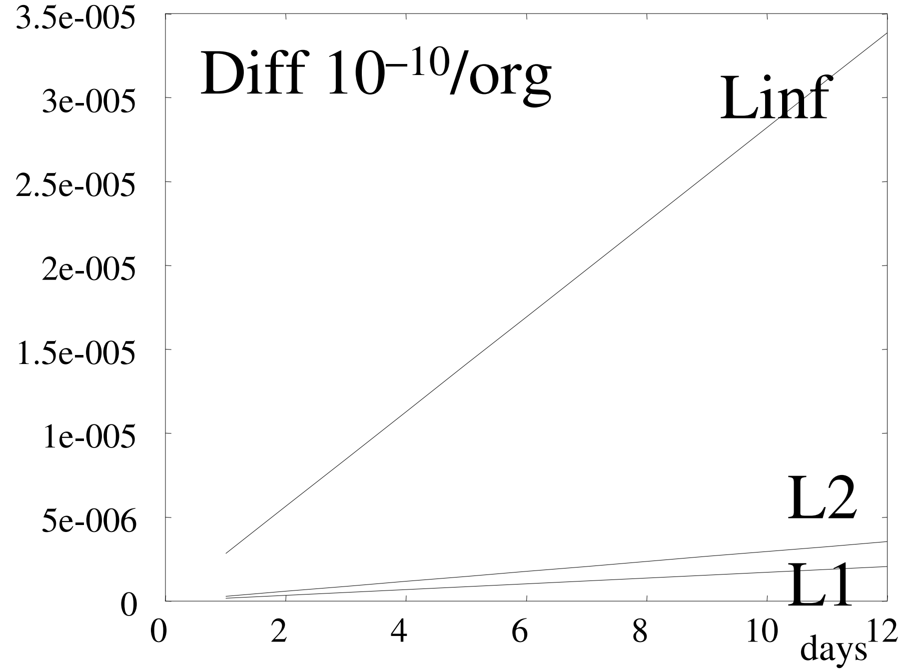
Case 1: devlp of difference

$\alpha = \pi / 2$



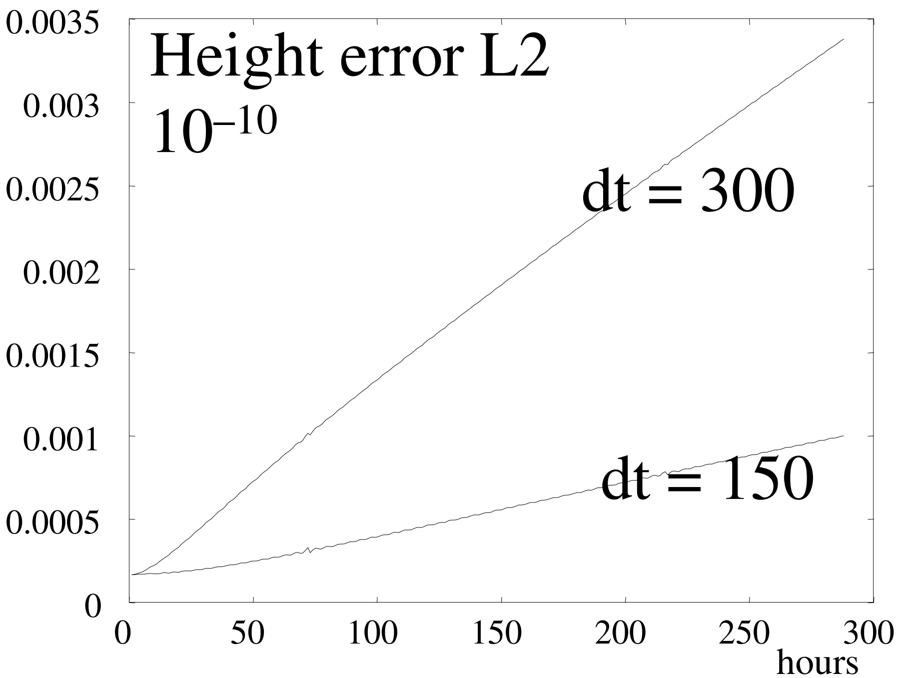
Abrupt increase at days 3 and 9

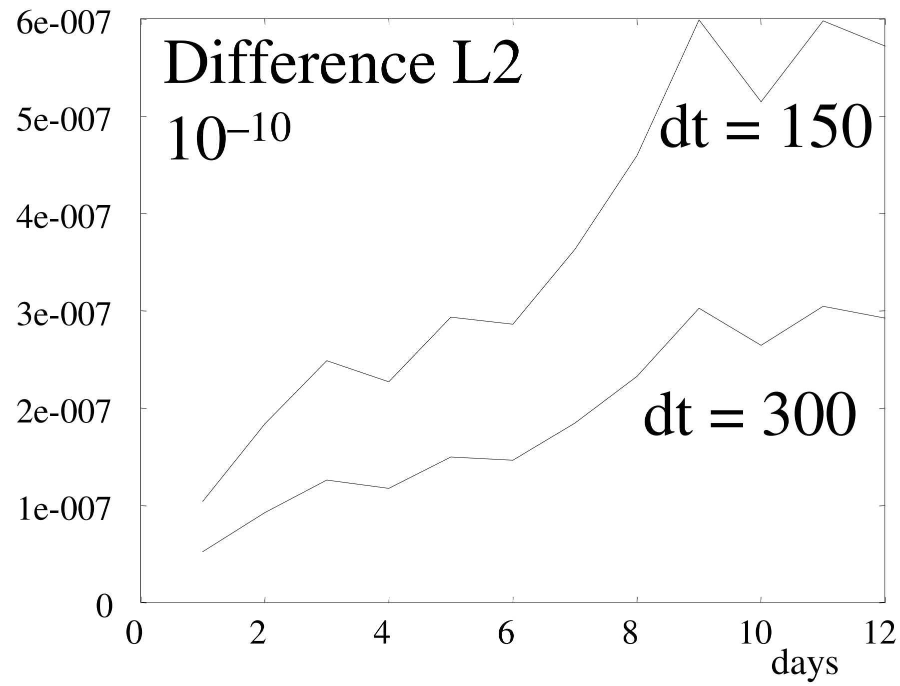
$\alpha = 0$



Linear but 10 times larger

Case 1: effects of time step





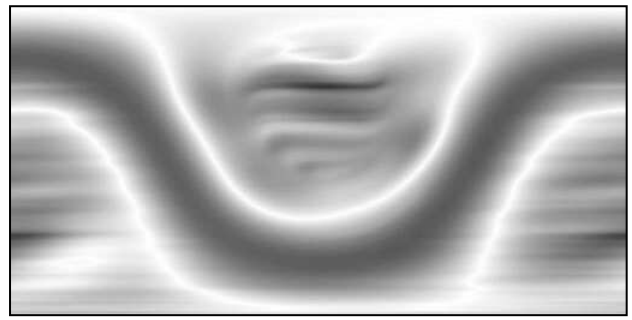
Difference is proportional to the *number* of transforms

Case 3: error and difference

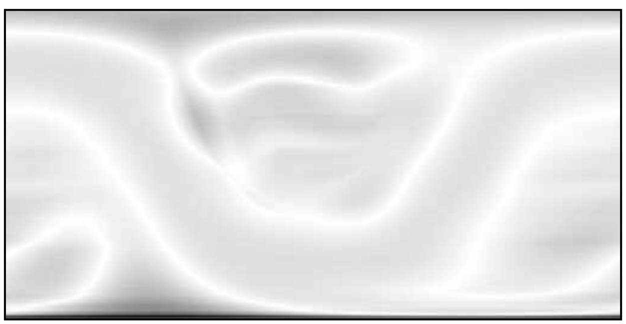
$\alpha = \pi / 3$

Height: max 3000m

Diff 10^{-10}: max 10^{-5} m



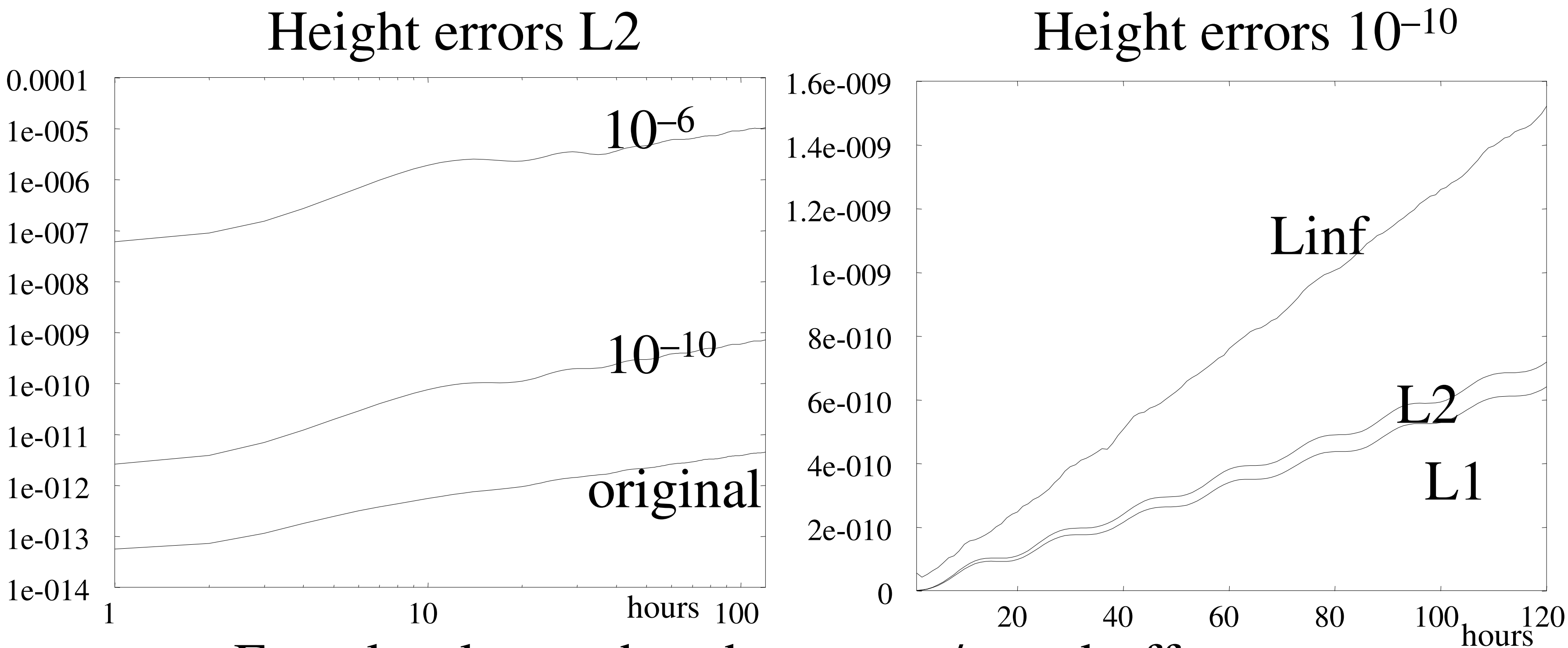
Error org: max 10^{-7} m



Diff 10^{-6}: max 0.1 m

Difference larger than original error

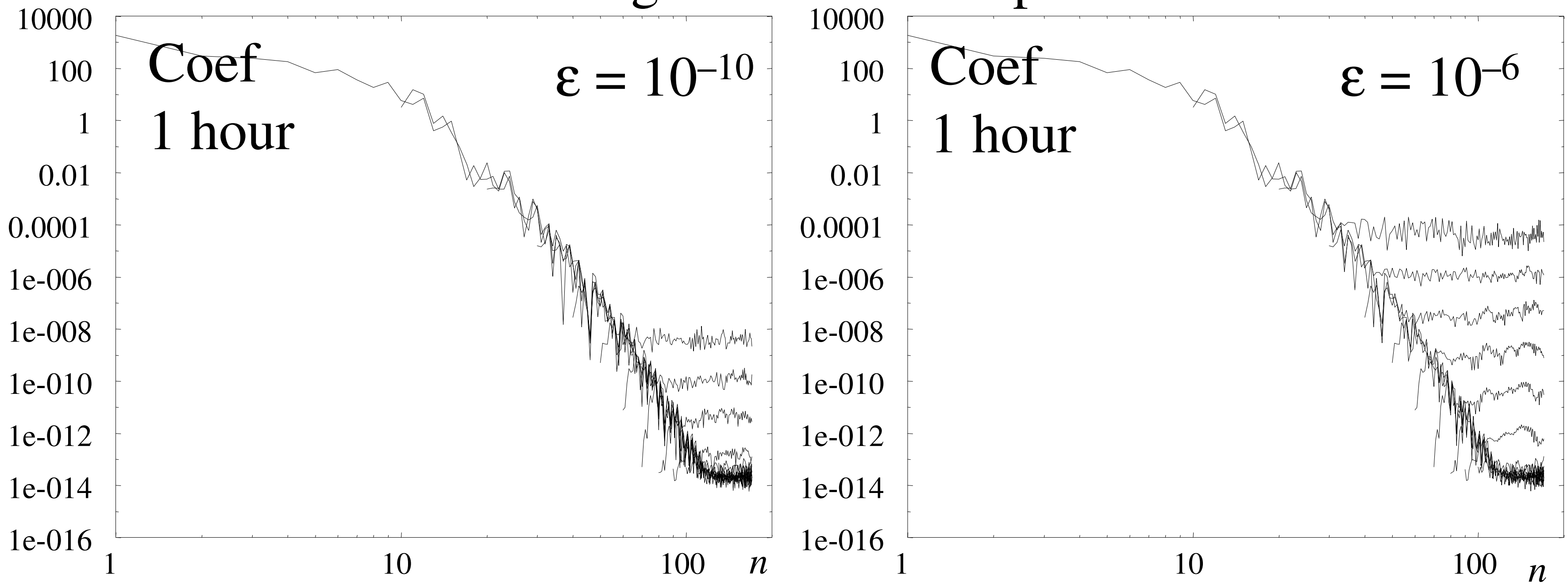
Case 3: height errors ≈ differences



Error levels matches the approx/round-off errors
Nearly linear increase with time

Case 3: expansion coefficients

the maximum magnitude of the expansion coefs

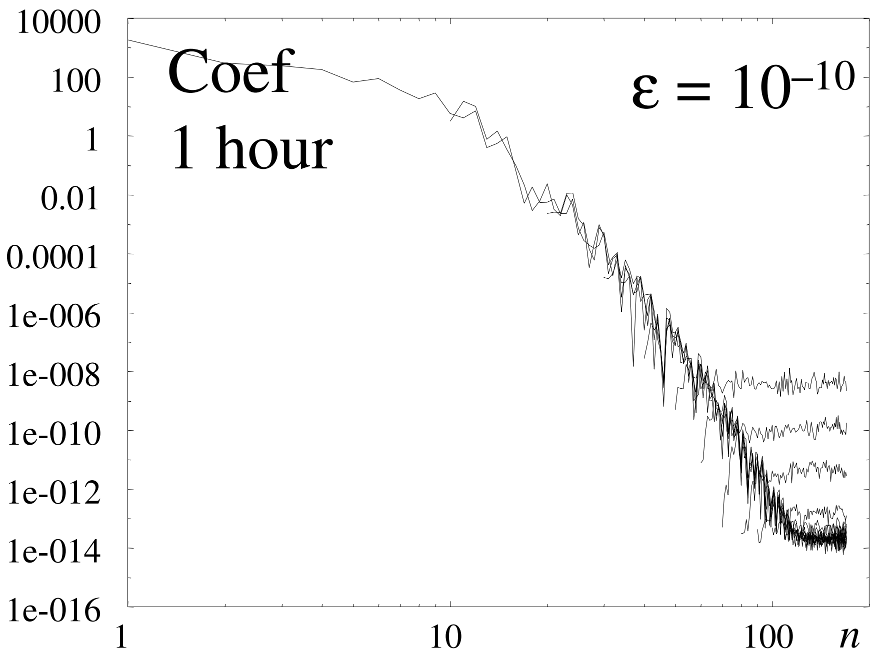


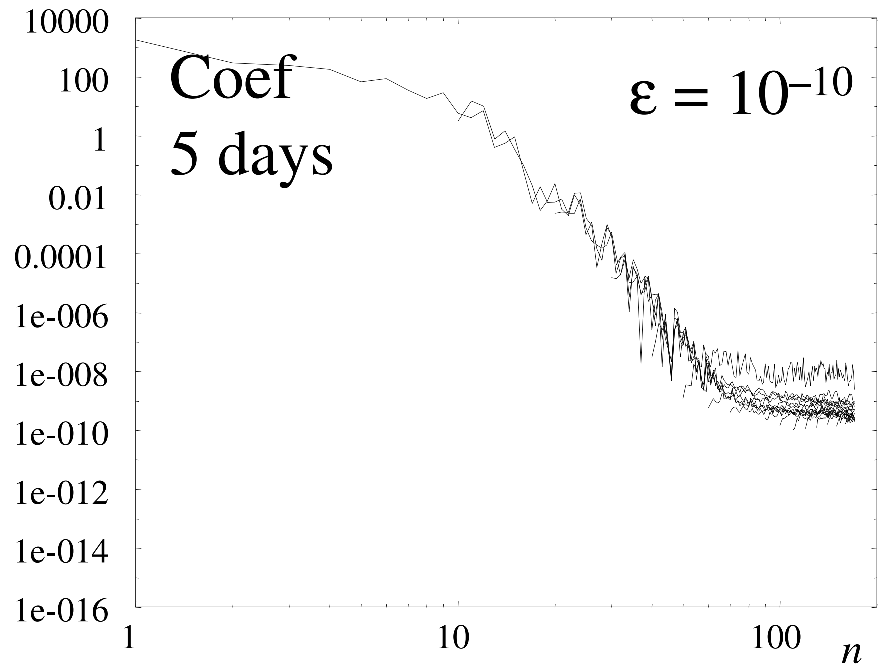
Rapid decrease; the round-off level at $n = 100$

Filled by errors of approx error level for each m

Case 3: expansion coefficients

the maximum magnitude of the expansion coefs

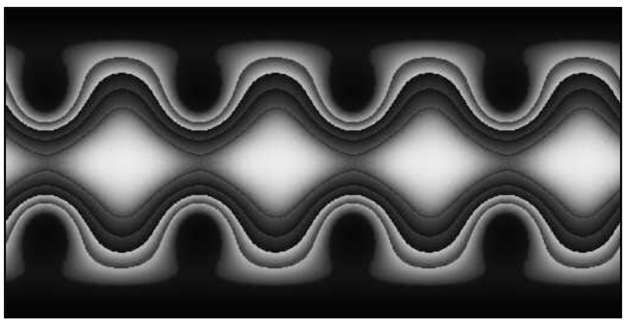




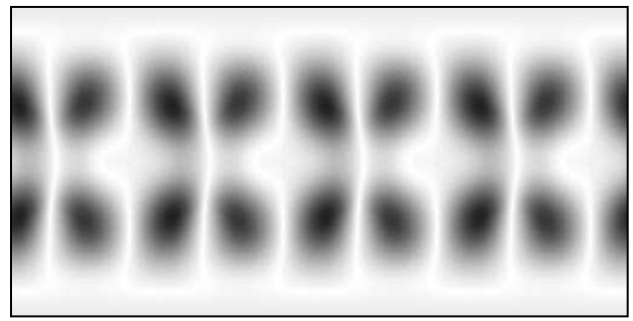
Filled by errors of the same level after long time integration

Case 6: result and difference

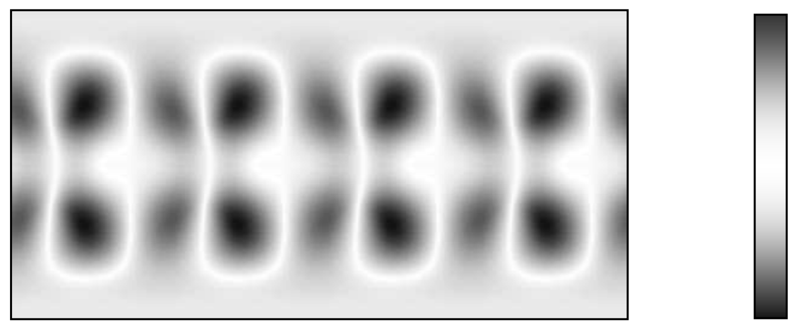
Height 14days: max 6000 m



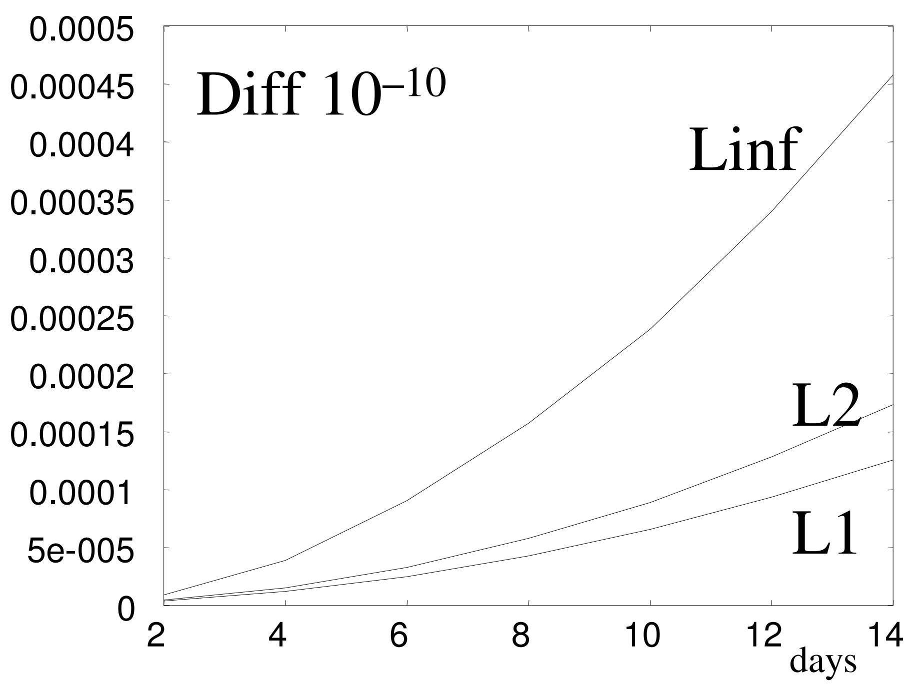
Diff 10^{-10}: max 10^{-3} m

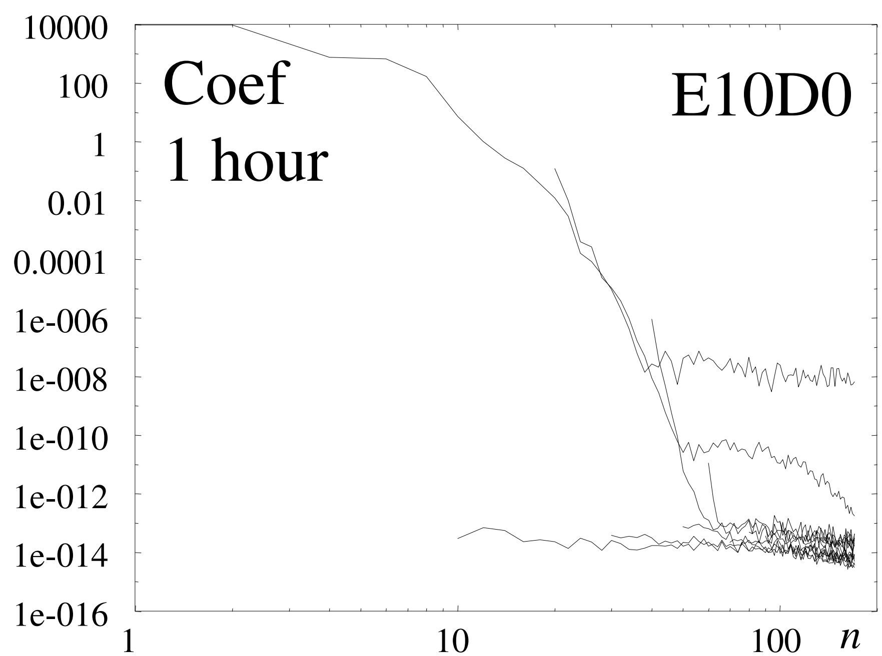


Diff 10^{-6}: max 16 m



Case 6: devlp of difference





Case 6: conservation analysis

t=14days	org	10^{-10}	10^{-6}
mass	0.0	-1.05e-10	-3.61e-7
energy	-1.19e-4	-1.19e-4	-6.43e-5
enstrophy	5.52e-13	5.52e-13	5.52e-13
divergence	6.03e-20	8.00e-18	-6.65e-13
vorticity	1.03e-22	2.74e-22	1.67e-22
height	1.81e-2	1.81e-2	1.86e-2

Turbulent flow: the equations

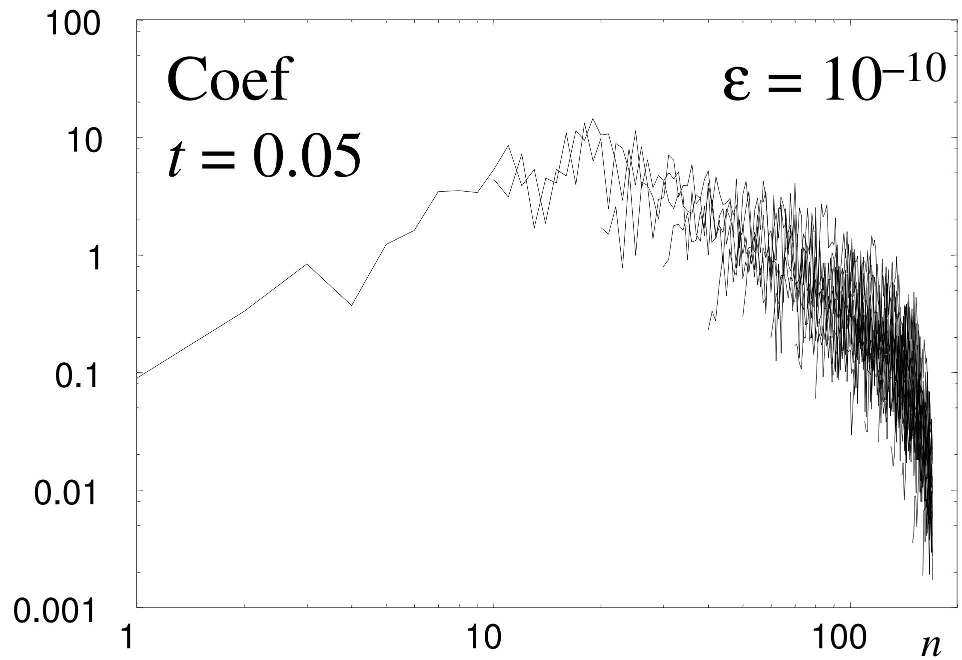
- Incompressible flow in vorticity eq

$$\frac{\partial \omega}{\partial t} = -\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \lambda}\frac{\partial \omega}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial \omega}{\partial \lambda}\frac{\partial \psi}{\partial \theta} - 2\Omega\frac{\partial \psi}{\partial \lambda}$$

$$\omega = \nabla^2 \psi$$

- Hyperviscocity $-\nu\nabla^8\omega$ with $\nu = 10^{-38}$
- Parameters $\Omega = 50$ (earth), $t = 1$ (a week)

Turbulent flow: expansion coefs

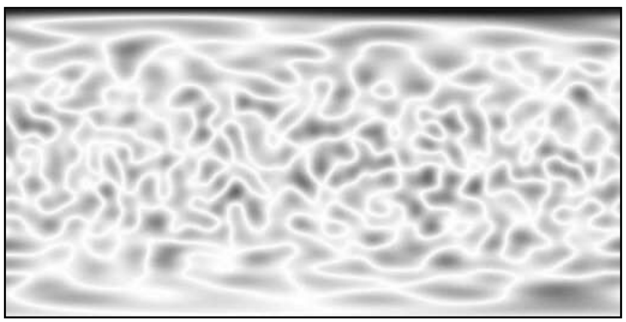


Random field at $t = 0$

Very slow decrease: 10^{-2} at $n = 170$

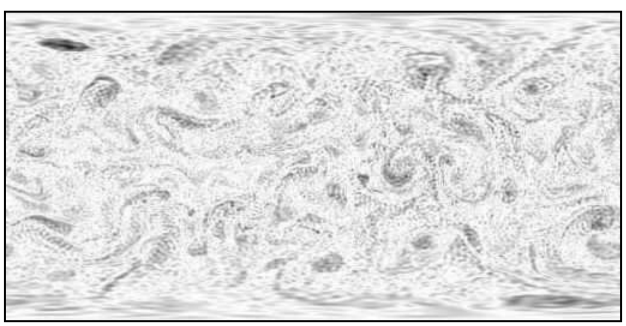
Turbulent flow: vorticity field

t = 0: max = 90



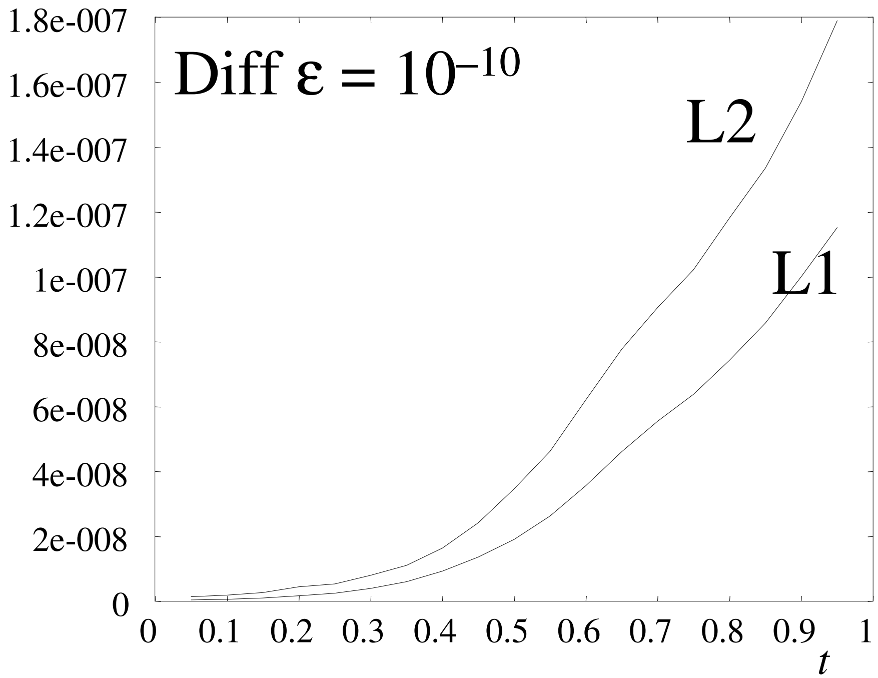
Diff 10^{-10}: max = 4e-6

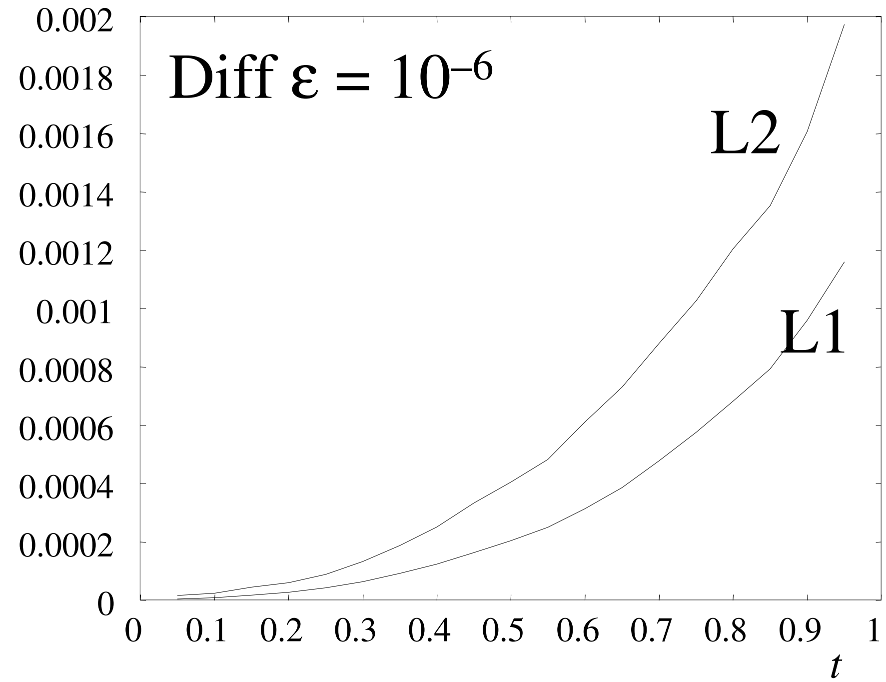
t = 0.9: max = 105



Diff 10^{-6}: max = 4e-2

Turbulent flow: devlp of difference





Turbulent flow: conservation

t = 0.95	org	10^{-10}	10^{-6}
energy	1.46e-6	1.46e-6	1.78e-6
enstrophy	1.98e-4	1.98e-4	1.98e-4

Approximation effects less than the hyperviscosity

Summary at this point

- The effects of the approximation error is easy to understand (in most cases)
- Easy to user, hard to develop
  - Behavior is easy to understand
  - Many things to do for higher performance

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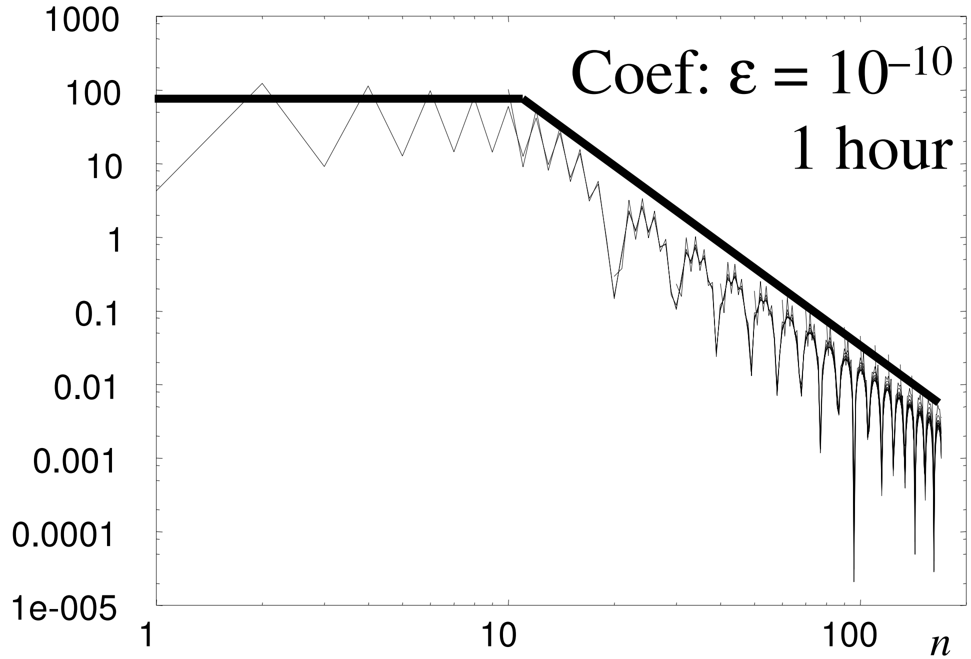
damping (weighted error control)

Case 1: expansion coefficients

the maximum magnitude of the expansion coefs



Damping: weighted error control

- Transform (evaluation)

$$g_m(\mu) = \sum g_n^m P_n^m(\mu)$$

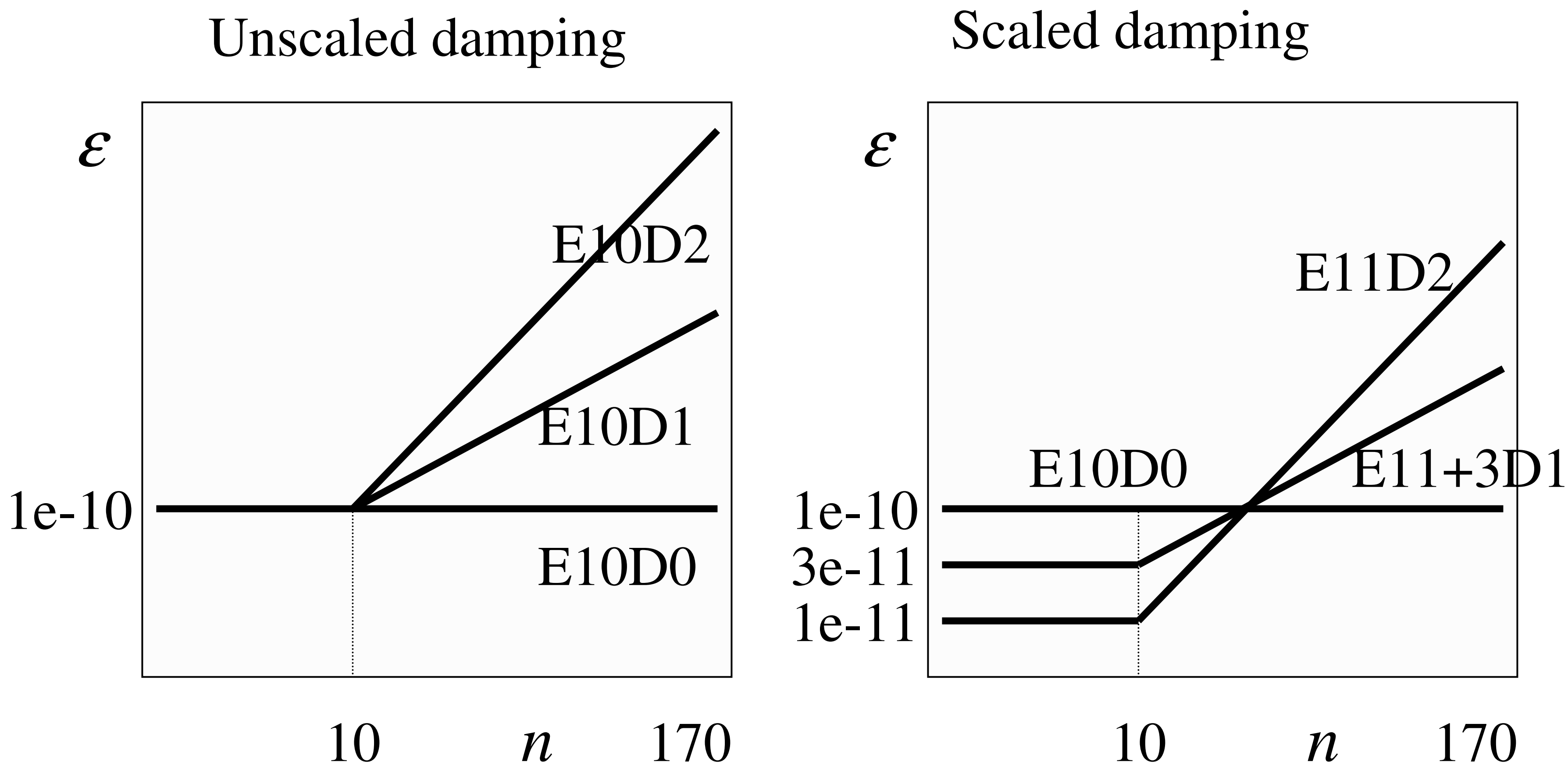
with approximation

$$\left|\tilde{g}_m(\mu) - g_m(\mu)\right| \le \sum \underbrace{\left|g_n^m\right|}_{\text{smaller coefficient}} \underbrace{\left|\tilde{P}_n^m(\mu) - P_n^m(\mu)\right|}_{\text{larger approximation error}}$$

smaller coefficient

larger approximation error

Damping and scaling



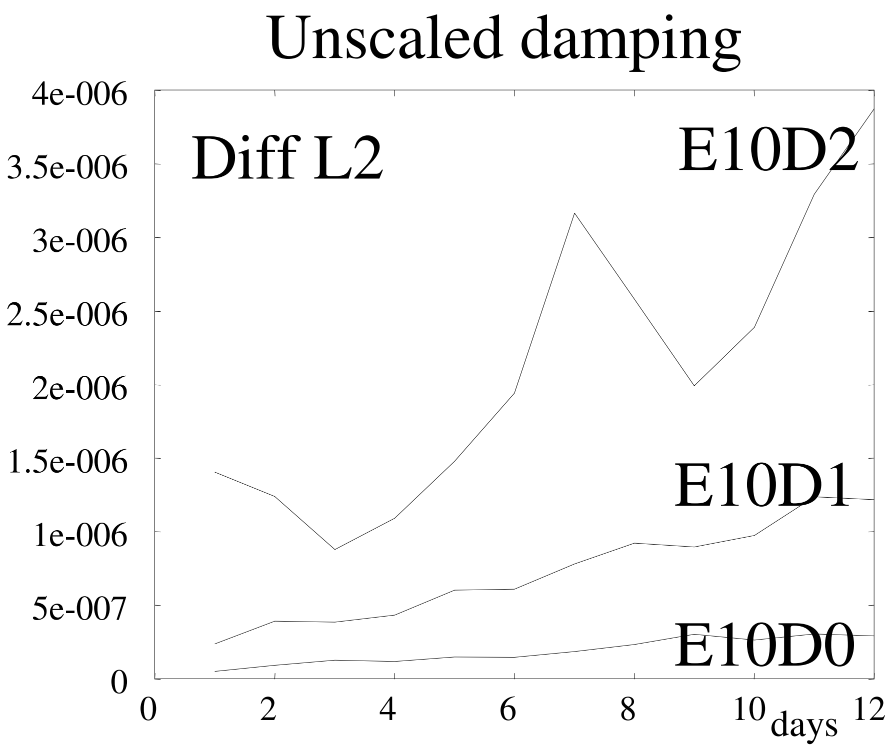
Speedup of scaled damping

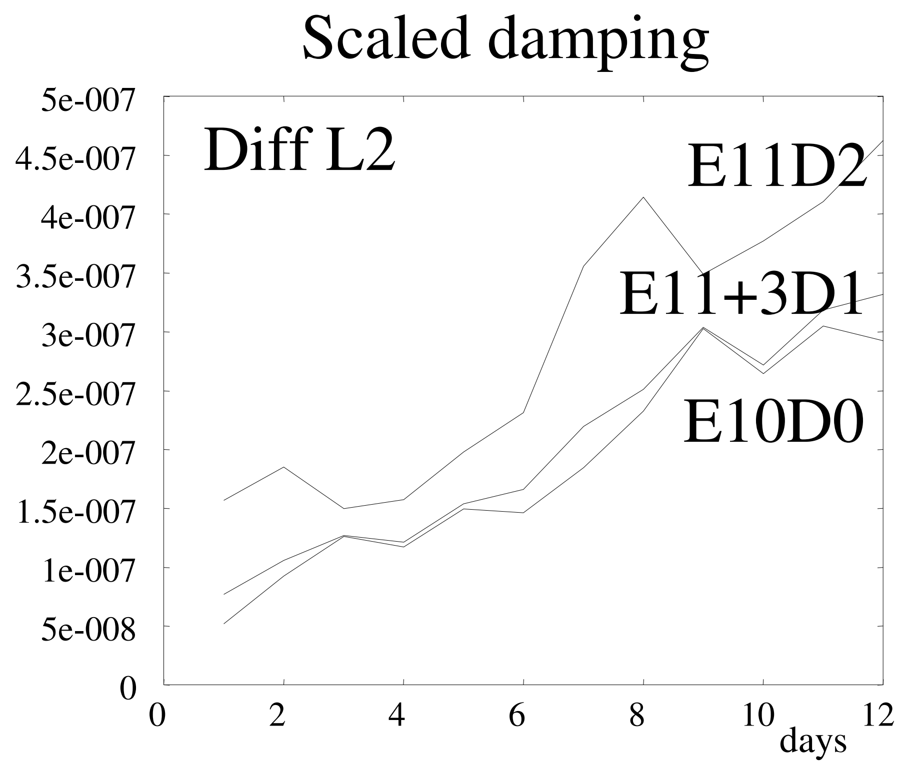
Speedup against direct computation of Legendre transform

	170	341	682	1365
E10D0	1.316	1.565	1.878	2.278
E11+3D1	1.319	1.578	1.904	2.358
E11D2	1.314	1.577	1.907	2.409

Speedup with scaled damping --- only in larger transforms

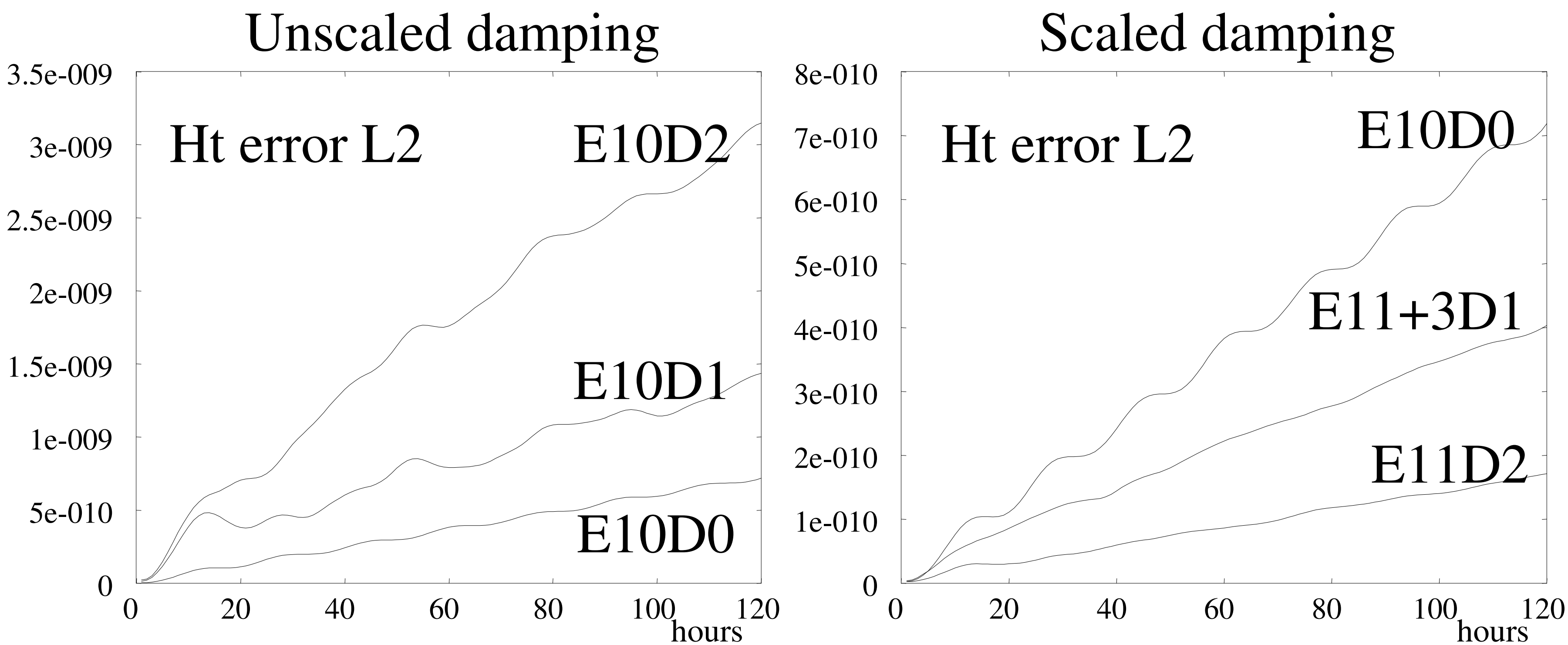
Case 1: damping effects on diff





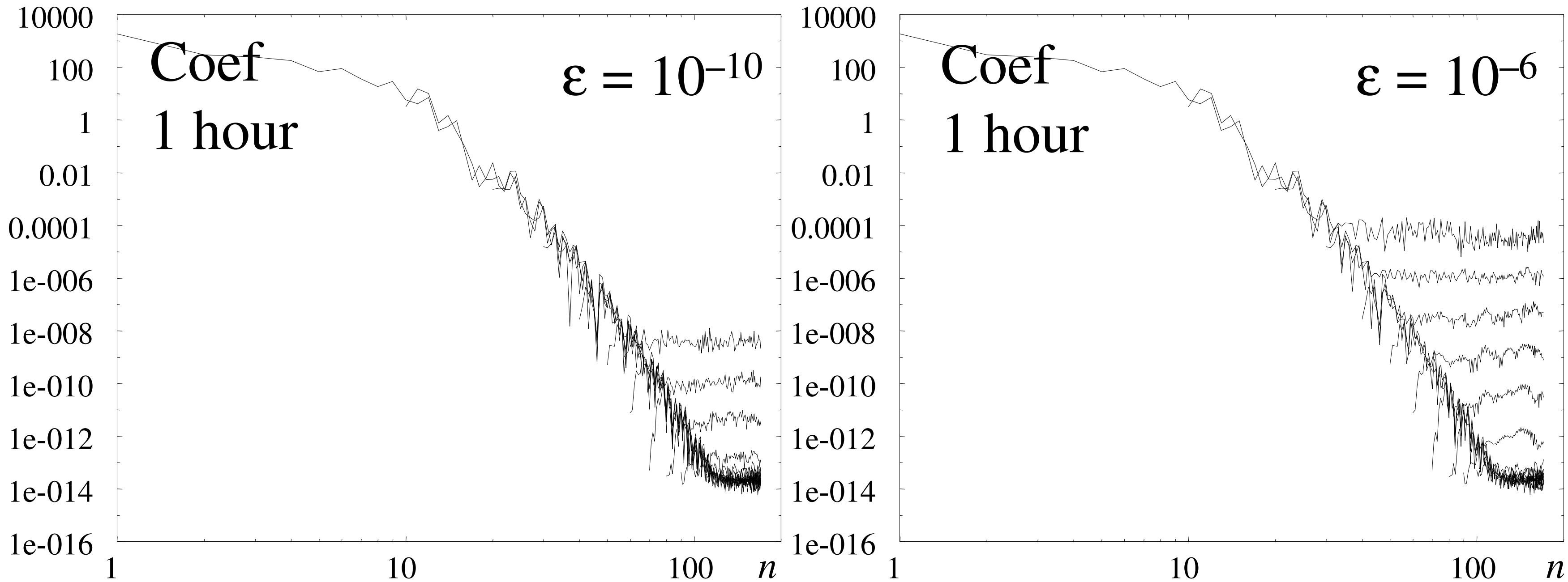
Scaling cancels the increase of the error of the damping

Case 3: height errors with damping



Scaling effects much; because of the rapid decrease of coef?

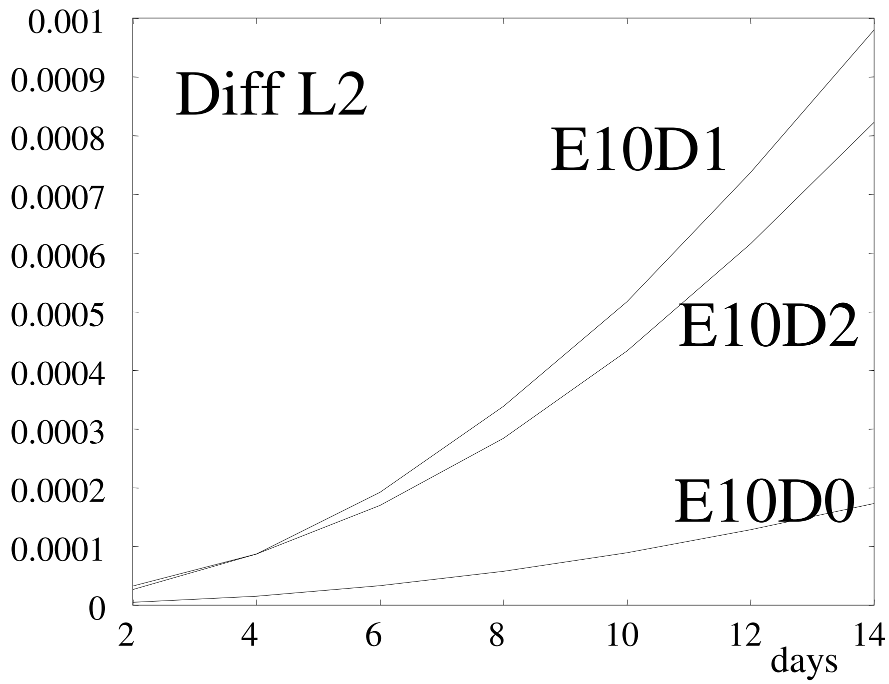
Case 3: expansion coefficients

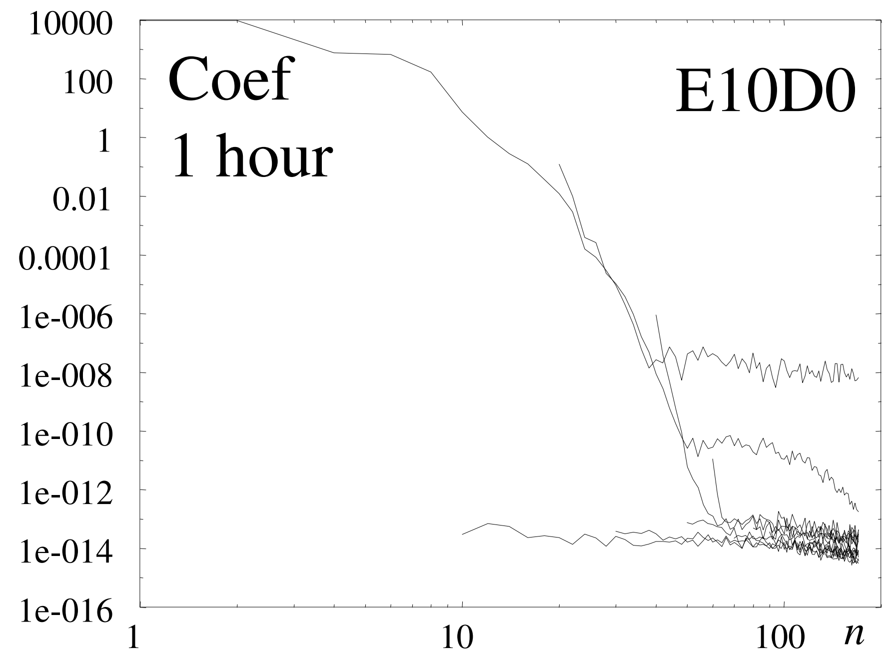


Rapid decrease; the round-off level at $n = 100$

Filled by errors of approx error level for each m

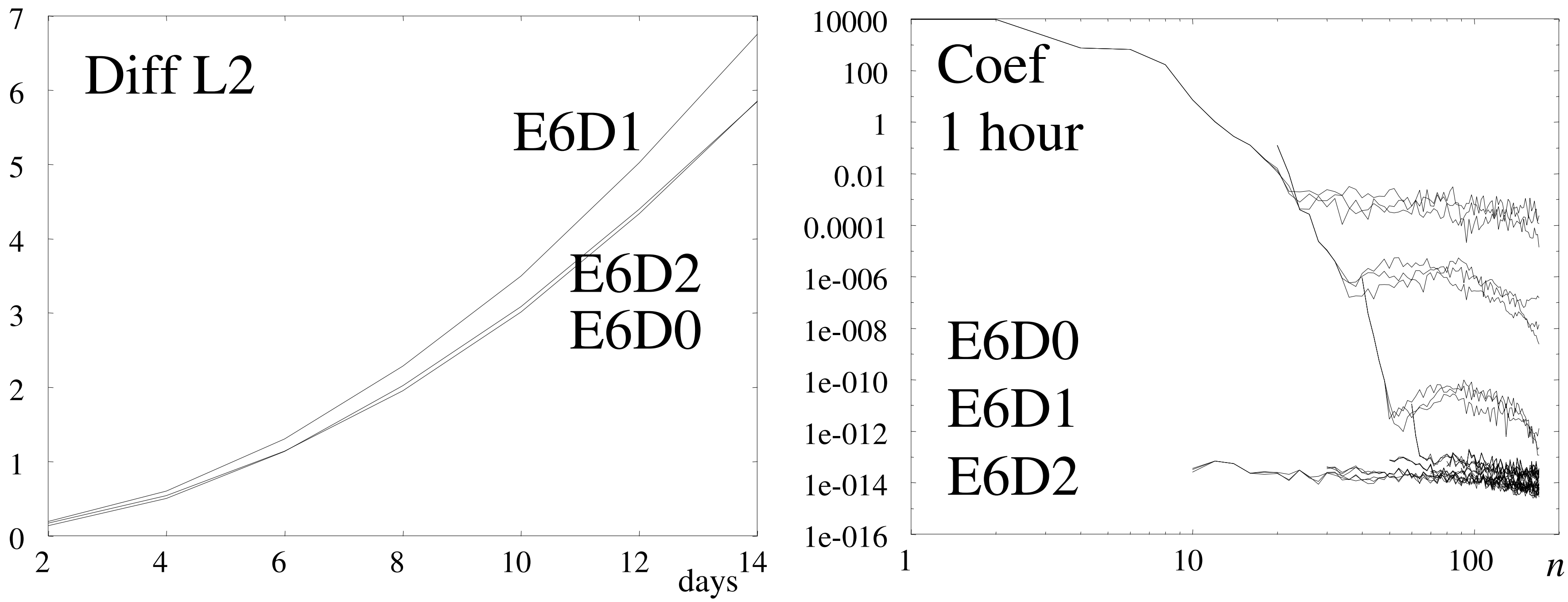
Case 6: difference with damping





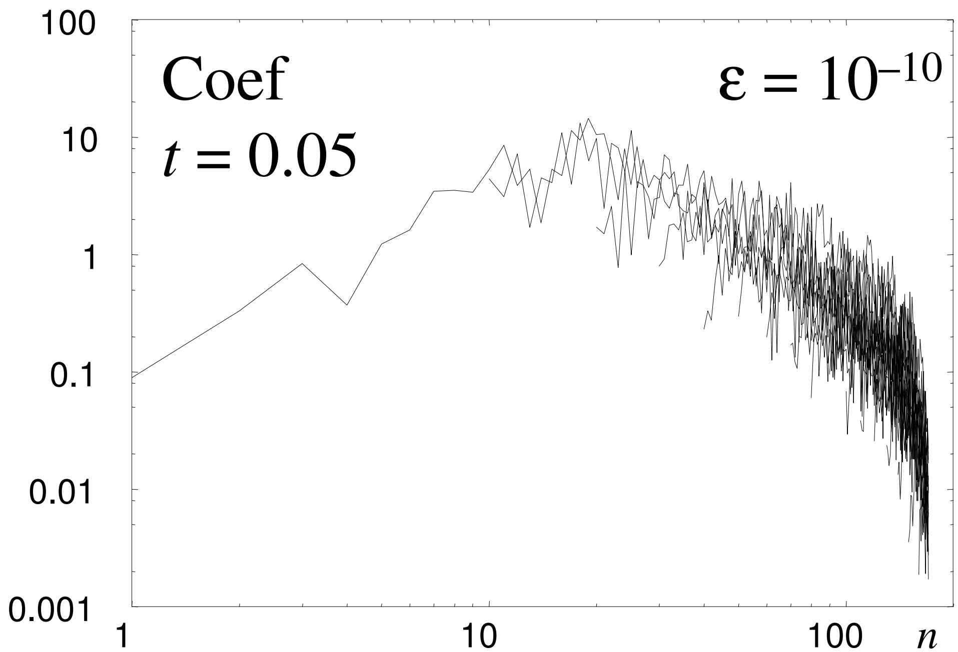
damping effects less ... from rapid decrease of coef?

Case 6: expansion coefficients



damping effects less ... from rapid decrease of coef?

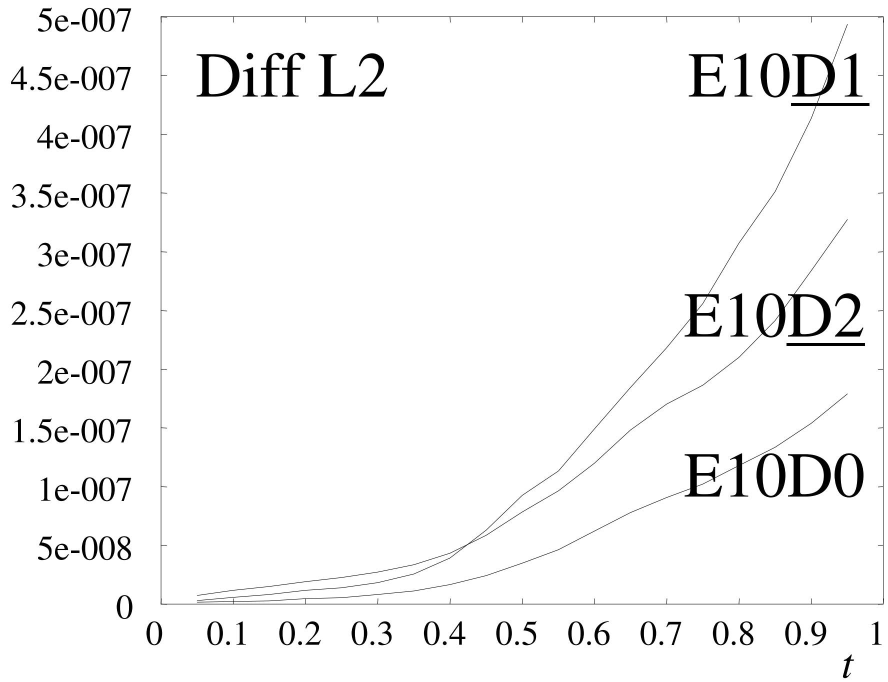
Turbulent flow: expansion coefs

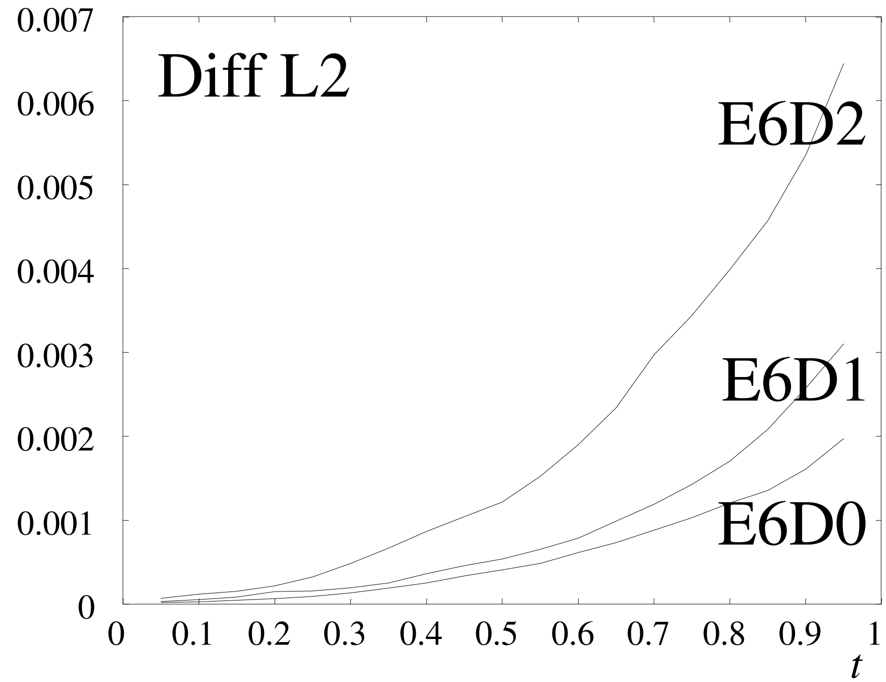


Random field at $t = 0$

Very slow decrease: 10^{-2} at $n = 170$

Turbulent flow: devlp difference





Regular increase by damping with exceptional E10D1(?)

Summary

- Easy to use, hard to develop
  - Behavior is easy to understand
  - Many things to do for higher performance
- Damping seems to work
  - Needs more research to predict the effects