Building Solutions to Nonlinear Elliptic and Parabolic Partial Differential Equations

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Early History of PDEs

Early PDEs

- Wave equation, d'Alembert 1752, model for vibrating string
- Laplace equation, 1790, model for gravitational potential
- Heat equation, Fourier, 1810-1822
- Euler equation for incompressible fluids, 1755
- Minimal surface equation, Lagrange, 1760
- Monge-Ampère equation by Monge, 1775
- Laplace and Poisson, applied to electric and magnetic problems:

Poisson 1813, Green 1828, Gauss, 1839

Solution *methods* were introduced

- separation of variables,
- Green's functions,
- Power Series,
- Dirichlet's principle.

H. Poincarré

An influential paper by H. Poincaré in 1890, remarked that a wide variety of problems of physics:

- electricity,
- hydrodynamics,
- heat,
- magnetism,
- optics,
- elasticity, etc. . .

have“un air de famille” and should be treated by common methods.

Stressed the importance of rigour despite the fact that the models are only an approximation of physical reality. Justified rigour

- For intrinsic mathematical reasons
- Because PDEs may be applied to other areas of math.

Nonlinear PDE and fixed point methods

Picard and his school, beginning in the early 1880's, applied the method of successive approximation to obtain solutions of nonlinear problems which were mild perturbations of uniquely solvable linear problems.

S. Banach 1922, fixed point theorem:
In a complete metric space X, a mapping $\mathcal{S} : X \to X$ which satisfies

$$\|S(x) - S(y)\| < K\|x - y\|, \qquad \text{for all } x, y \in X, \text{ and for } K < 1,$$

has a unique fixed point.

Modern theory: non-constructive

Prior to 1920: classical solutions, constructive solution methods.

The development around 1920s of
1. Direct methods in calculus of variations.
   (Classical spaces not closed: weak solutions lie in the completion.)
2. Approximation procedure used to construct a solution.
   (Approximate solutions no longer classical.)

Led to notion of weak solution.

New methodology, separated issues of
   i. Existence of weak solution
   ii. Uniqueness of weak solution
   iii. Regularity of weak solution

but no longer had
   iv. Explicit construction of solutions

R. Courant, K. Freidrichs, H. Lewy 1928

Seminal paper in numerical analysis, predated computers.

Constructive solution methods for classical linear PDEs of math physics:

- elliptic boundary value and eigenvalue,
- hyperbolic initial value,
- parabolic initial value.

The finite difference method:

- *replace differentials by difference quotients* on a mesh.
- Obtain algebraic equations, construct solutions to these equations.
- Prove convergence (in L^2 norm).

Elliptic PDE: *implicit* scheme.

Hyperbolic/Parabolic PDE: *explicit* scheme
but with restriction on the time step, (the *CFL condition*.)

Finite Differences for Laplacian and Heat Equation

Centered difference scheme for $-u_{xx}$.

$$\mathcal{F}^i(\mathbf{u}) = \frac{1}{dx}\left(\frac{u_i - u_{i-1}}{dx} + \frac{u_i - u_{i+1}}{dx}\right)$$

Implicit and *explicit* Euler scheme for $u_t = u_{xx}$

$$\left(\frac{u_i^{n+1} - u_i^n}{dt}\right) + \mathcal{F}^i(\mathbf{u}^{n+1}) = 0, \qquad \left(\frac{u_i^{n+1} - u_i^n}{dt}\right) + \mathcal{F}^i(\mathbf{u}^{n}) = 0.$$

Explicit scheme gives a map

$$u^{n+1} = \mathcal{S}_{dt}(u^n) = u_i^n - dt\,\mathcal{F}^i(\mathbf{u}^n).$$

For explicit scheme, require

$$dt \leq \frac{1}{2}dx^2 \tag{CFL}$$

for stability in L^2.

Convergence of Approximation methods

Lax-Richtmeyer 1959, stability necessary for convergence of linear difference schemes in L^2.

Lax Equivalence theorem a "Meta-theorem" of Numerical Analysis:

Consistent, stable schemes are convergent.

Need to make these notions precise to get a theorem,
in particular, need to assign a norm for stability.

For nonlinear or degenerate PDE, the solutions may not be smooth.

It is *essential* for convergence
that the norm used in the existence and uniqueness theory
be the norm used for stability of the approximation.

Stability in ℓ^∞ and in ℓ^2

Let M be linear map $M : \mathbb{R}^n \to \mathbb{R}^n$.

$$\|Mx\|_2 \leq \|x\|_2 \text{ for all } x \quad \text{iff} \quad \text{all eigenvalues of } MM^T \text{ in unit ball}$$

$$\|Mx\|_\infty \leq \|x\|_\infty \text{ for all } x \quad \text{iff} \quad \sum_{j=1}^{n} |M_{ij}| \leq 1, \quad i = 1, \ldots n.$$

Explicit Euler for heat equation: stability conds. in ℓ^2 and ℓ^∞ coincide.

In general these notions do not coincide.
For linear maps, stability in ℓ^∞ is stronger than stability in ℓ^2.

Note:

- Stability in ℓ^∞: examine coefficients.
- Stability in ℓ^2: check a spectrum.

T.S. Motzkin and W. Wasow 1953

Finite difference schemes for linear elliptic equations

$$A_{dx}u = -\sum_j a_j(dx)u(x - j\,dx), \qquad \text{in } \mathbb{R}^n.$$

Scheme is of "positive type" if $a_j \geq 0$ for $j \neq 0$ and $a_0 < 0$.

Prove discrete maximum principle by "walking to the boundary," prove convergence (now using L^∞ norm) as $dx \to 0$.

Rewrite $A_{dx}u$ as

$$A_{dx}u = \frac{1}{dx^2}\sum_{i\neq 0} p_i(u(x) - u(x - ih)) + p_0 u(x),$$

where now $p_i \geq 0, i \neq 0$.

Random Walk:

p_i probability of jump from x to $x - ih$, p_0 prob of decay.

The Comparison Principle

Viscosity Solutions, Monotone schemes

The comparison principle

Schematic:

$$data \to PDE \to solution.$$

Comparison principle:

$$\text{If } data_1 \leq data_2 \text{ then } solution_1 \leq solution_2.$$

E.g. data corresponds to:
boundary conditions for elliptic equations,
initial conditions For parabolic equations.

Solutions are functions on the domain.

Monotonicity for schemes:

The discrete comparison principle.

Schematic:

$$data \rightarrow \text{ numerical scheme } \rightarrow solution.$$

Monotonicity:

$$\text{If } data_1 \leq data_2 \text{ then } solution_1 \leq solution_2.$$

Data: a finite number of function values at points on the boundary of the computational domain:
boundary conditions for elliptic equations,
initial conditions For parabolic equations.

Solutions are finite number of function values at grid points (nodes) in the entire domain.

Local structure conditions

Local structure conditions on the PDE (degnerate ellipticity) ensures that the comparison principle holds.

We find (A.O.) local structure conditions on the numerical schemes which ensures that monotonicity holds. Furthermore, this structure condition leads to

- self-consistent existence and uniqueness proofs for solutions of the scheme,
- an explicit iteration scheme which can be used to find solutions.

Elliptic equations lead to implicit schemes, whereas *explicit, monotone* schemes for parabolic equations can be built from the scheme for the underlying elliptic equation.

Viscosity Solutions

Weak notion of solution for PDEs where the comparison principle holds.

$$F(x, u, u_x, u_{xx}) = 0, \text{ in one space dimension,}$$
$$F(x, u, Du, D^2u) = 0, \text{ in higher dimensions,}$$

$F(x, r, p, M) \to \mathbb{R}$. $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$, where S^n space of symmetric $n \times n$ matrices.

Definition: the function F is *degenerate elliptic*, if it is non-increasing in M and non-decreasing in r.
Degenerate ellipticity is a local structure condition on the function F which yields the global comparison principle.

Examples:

$$\min\{u_t - u_{xx}, u - g(x)\} = 0 \qquad \text{parabolic obstacle problem}$$
$$u_t - |u_x| = 0 \qquad \text{front propagation}$$

Note: "degenerate elliptic" includes parabolic: degenerate in t var.

Viscosity Solutions - Definition

The bounded, uniformly continuous function u is a viscosity solution of the degenerate elliptic equation

$$F(x, u, Du, D^2u) = 0 \text{ in } \Omega$$

if and only if for all $\phi \in C^2(\Omega)$, if $x_0 \in \Omega$ is a nonnegative local maximum point of $u - \phi$, one has

$$F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0,$$

and for all $\phi \in C^2(\Omega)$, if $x_0 \in \Omega$ is a nonpositive local minimum point of $u - \phi$, one has

$$F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0.$$

Monotonicity is a *global* condition.

Existence and Uniqueness of Viscosity Solutions

M. Crandall, P.L. Lions, G. Barles, L.C. Evans, H. Ishii, P.E. Souganidis

Theorem. *For a wide class of* degenerate elliptic equations *there exist unique viscosity solutions.*

Viscosity solutions are the correct framework for proving existence and uniqueness results for PDE for which the *Comparison Principle* holds.

Convergence of Approximation Schemes

G. Barles and P.E. Souganidis (1991)

Theorem. *The solutions of a* stable, consistent, monotone *scheme converge to the unique viscosity solution of the PDE.*

Q: Does it really matter if the schemes are not monotone?
Q: How do we find monotone schemes?

End of introduction

To follow:
definitions, and theorems regarding: building monotone schemes.

Results for

- Math Finance, HJ equations
- Nonconvergent methods
- Convergent schemes for motion by mean curvature, infinity laplacian

Heuristic: norms for convergence

Correct norms reflect underlying physical and analytical properties,

- Conservation of Energy
- Conservation of Mass
- The Comparison Principle

For heat equation, $u_t = u_{xx}$, use L^2 norm

$$\frac{d}{dt}\int \frac{u^2}{2}\,dx = \int u u_t\,dx = \int u u_{xx}\,dx = -\int u_x^2\,dx \leq 0.$$

For conservation law $u_t = -(u^2)_x,$, use L^1 norm,

$$\frac{d}{dt}\int u\,dx = \int u_t\,dx = -\int u_x^2\,dx \leq 0.$$

For nonlinear, degenerate elliptic, $u_t = F(u_{xx})$ with F nondecreasing, use L^∞, or oscillation norm,

$$\frac{d}{dt}(\max u - \min u) = F(u_{xx})|_{\max} - F(u_{xx})|_{\min} \leq 0.$$

Numerical methods reflect the heuristic

Divergence structure elliptic:

Finite element method or Energy method for variational problems.

L^2 norms.

Conservation Laws:

finite differences, (node values: cell averages), "finite volume"

L^1 norms.

Fully nonlinear degenerate elliptic:

monotone finite difference methods. (node values: function values)

L^∞ norms

Conservation Laws and Hamilton-Jacobi Equations

PDE theory for conservation laws preceded theory of viscosity solutions. The connection between conservation laws and Hamilton-Jacobi equations in one dimension is given by differentiating,

$$u_t + \frac{1}{2}u_x^2 = 0 \qquad \text{(HJ)}$$

$$v_t + \frac{1}{2}\left(v^2\right)_x = 0, \qquad \text{where } v = u_x \qquad \text{(Cons Law)}$$

Numerics for cons. laws relies on an entropy preserving flux function.

The flux functions lead to *monotone schemes* for HJ equations in one spatial dimension. Extended to higher dimensions.

Monotone schemes suffer from low accuracy.
ENO (Essentially Non-Oscillatory) and WENO (Weighted ENO) schemes selectively use high order interpolation in smooth regions, monotone flux function in nonsmooth regions to get better performance.

Results for HJ equations

Improving methods for HJ requires either

- higher order interpolation, or
- better flux functions.

Challenge often

- finding a monotone flux, and
- checking monotonicity of the flux.

Theorems to follow (A.O.) provide:

- simple local structure condition which guarantees monotonicity,
- methods for building monotone schemes.

Remarks on Explicit, Monotone schemes

1. Because for HJ equations, monotone schemes were supplanted by ENO and WENO, there is a misconception that monotone schemes are not practical. *This is not the case*. For second order equations, not only are they practical, they may be *the only convergent methods available*.

2. For linear equations, the time step restriction imposed for the CFL condition may be undesirable. Since it is quite inexpensive to solve a linear system of equations, implicit methods are often preferred.

However, for *nonlinear equations*, due to the iterative methods which must be used to solve nonlinear equations, one implicit time step may be more costly than thousands of explicit steps. So explicit methods are preferred.

Math Finance

While valuation models (American options in complete markets) lead to obstacle problems for linear PDEs, more general valuation problems lead to *linear, but degenerate elliptic* PDEs.

Portfolio optimization problems (stochastic control) lead to fully non-linear Hamilton-Jacobi-Bellman equations

$$\sup_i \{A_i u - f_i\}$$

where A_i family of linear elliptic operators.

For these types of equations: only PDE theory available is viscosity solutions.

Numerics for Math Finance

Most methods in use, e.g. Finite Element methods, are convergent for the simplest problems, but are not monotone for the more general nonlinear or degenerate problems.

So in these more general cases the methods do not converge.

Nevertheless, many practitioners use these methods.

In addition, it is desirable to build a comprehensive class of schemes which can solve the large number of models.

Research program (A.O. and T. Zariphopoulou).
Build a *framework* of monotone schemes for nonlinear PDEs which arise in valuation and optimal portfolio problems in math finance.

Motivating Example

Nonmonotone schemes may diverge, even if they are stable in L^2.

Toy example of a linear, degenerate elliptic equation.

For this degenerate equation, no convergence in L^2.

Require monotonicity for convergence.

Monotonicity for linear maps

This is simpler than monotonicity in general.

Defn: For vectors x, y in $\mathbb{R}^n$, $x \leq y$ means $x_i \leq y_i$ for $i = 1, \ldots, n$

Defn: The linear map $M : \mathbb{R}^n \to \mathbb{R}^n$ is *monotone* if

$$x \leq y \text{ implies } Mx \leq My \text{ for all } x, y$$

Monotonicity condition (for linear maps):

$$M \text{ is monotone iff } M_{ij} \geq 0 \text{ for } i, j = 1 \ldots, n$$

Monotonicity for explicit nonlinear schemes: The solution at the grid point u_0 must be a non-decreasing function of its neighbors.

Monotonicity in general: to follow.

Schemes for the degenerate equation

Consider the degenerate elliptic equation in $\mathbb{R}^2$

$$u_t = (u_{xx} + 2u_{xy} + u_{yy}).$$

Use centered difference for u_{xx}, u_{yy}. Obtain three different explicit schemes, distinguished by the u_{xy} discretization,

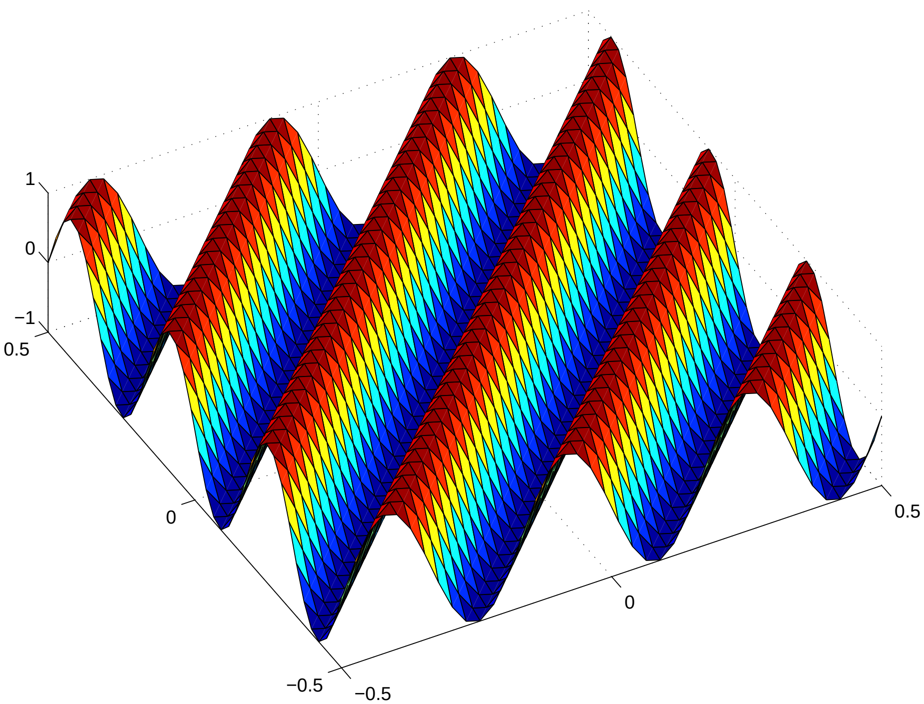
$$\mathcal{S}^{\text{Diag}} = \begin{bmatrix} & & 1/2 \\ & 0 & \\ 1/2 & & \end{bmatrix}, \qquad \text{monotone, } \ell^2\text{-stable} \to \text{converges.}$$

$$\mathcal{S}^{\text{Centered}} = \begin{bmatrix} -1/8 & 1/4 & 1/8 \\ 1/4 & 0 & 1/4 \\ 1/8 & 1/4 & -1/8 \end{bmatrix}, \text{ nonmonotone, } \ell^2\text{-stable in} \to \text{?}$$

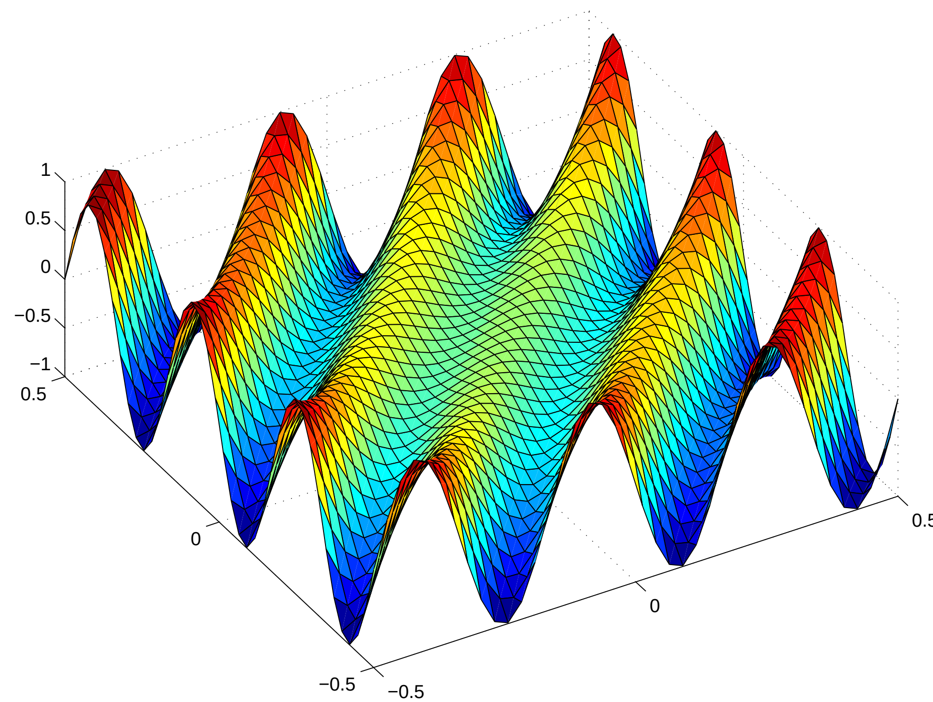
$$\mathcal{S}^{\text{Anti-Diag}} = \begin{bmatrix} -1/6 & 1/3 & \\ 1/3 & 0 & 1/3 \\ & 1/3 & -1/6 \end{bmatrix}, \text{ nonmonotone, } \ell^2\text{-unstable} \to \text{blows up .}$$

Numerical experiments

Diagonal scheme:



Centered scheme:



Solution of the centered scheme differs by 1 from the exact solution.

Conclusion:

Consistency and stability (in the ℓ^2 norm) does not imply convergence for this *linear, degenerate* PDE. Require *monotonicity*.

Develop a self-consistent, rigorous framework

for monotone difference schemes

Q: What is a finite difference scheme? Can we find a good definition?

Q: For nonlinear schemes, under what conditions can we prove monotonicity (the comparison principle)?

Q: Can we also prove in a self-consistent way, existence and uniqueness of solutions for the *schemes themselves*.

The methods should reflect, at the discrete level, the methods used for the PDEs.

What is a finite difference scheme?

Structure conditions:

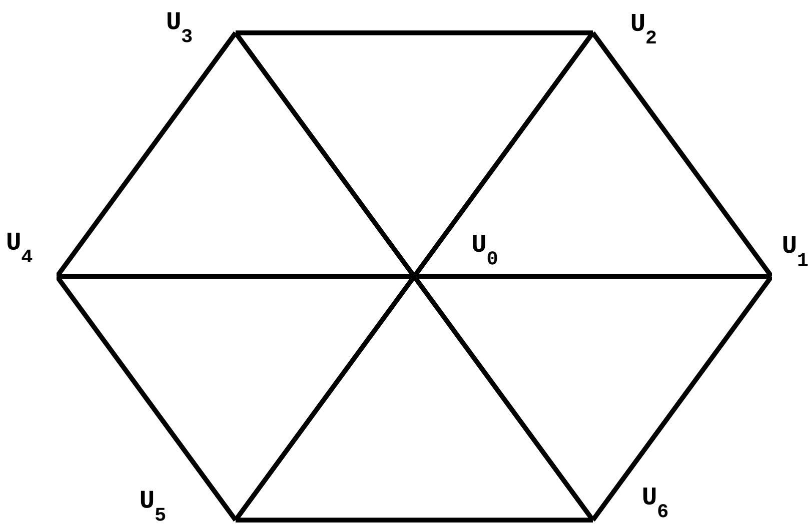
Scheme at x_i should depend only on u_i and the *differences* $u_i - u_j$.

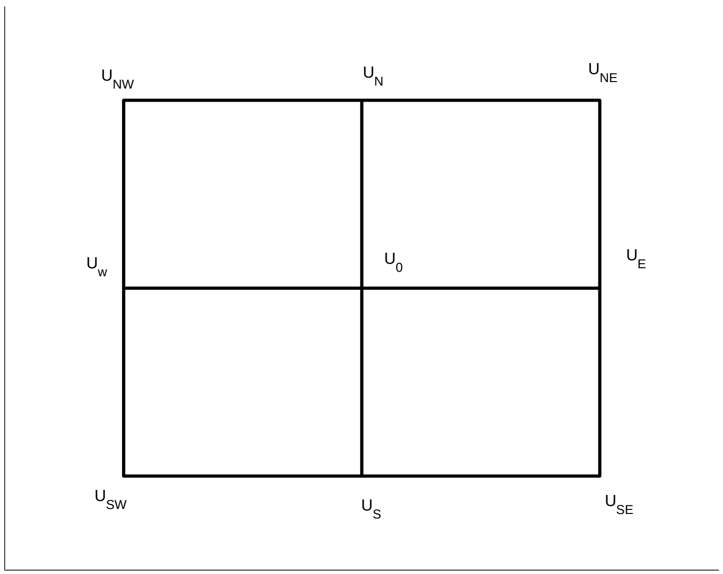
Definition:

A function $\mathcal{F} : \mathbb{R}^N \to \mathbb{R}^N$, is a *finite difference scheme* if

$$\mathcal{F}(\mathbf{u})^i = \mathcal{F}^i\left(u_i, u_i - u_{i_1}, \ldots, u_i - u_{i_{n_i}}\right) \qquad (i = 1, \ldots, N)$$

for some functions $\mathcal{F}^i(x_0, x_1, \ldots, x_{n_i})$.





Discrete Ellipticity

Q: Can we find a structure condition on *nonlinear* difference schemes which implies monotonicity (the discrete comparison principle).

A: Yes. (A.O.)

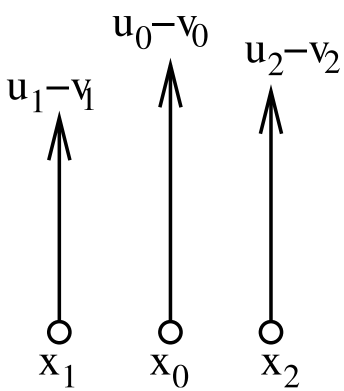
$\mathcal{F} : \mathbb{R}^N \to \mathbb{R}^N$, is a *discretely elliptic finite difference scheme* if

$$\mathcal{F}(\mathbf{u})^i = \mathcal{F}^i\left(u_i, u_i - u_{i_1}, \ldots, u_i - u_{i_{n_i}}\right) \qquad (i = 1, \ldots, N)$$

for some nondecreasing functions $\mathcal{F}^i(x_0, x_1, \ldots, x_{n_i})$.

Discrete ellipticity:
local structure condition for the nonlinear difference scheme which implies the global comparison principle.

Theorem (Monotonicity for schemes (A.O.)). *Let $\mathcal{F}$ be a strictly proper, discretely elliptic finite difference scheme. If $\mathcal{F}(\mathbf{u}) \le \mathcal{F}(\mathbf{v})$, then $\mathbf{u} \le \mathbf{v}$.*



Proof Suppose $\mathbf{u} \not\le \mathbf{v}$ and let i be an index for which $u_i - v_i = \max_{j=1,\dots,N}\{u_j - v_j\} > 0$, so that

$$u_i - u_j \ge v_i - v_j, \quad j = 1, \dots, N.$$

$$\begin{aligned} \mathcal{F}(\mathbf{u})^i &= \mathcal{F}^i(u_i, u_i - u') \\ &\ge \mathcal{F}^i(u_i, v_i - v'), && \text{by discretely elliptic} \\ &> \mathcal{F}^i(v_i, v_i - v') = \mathcal{F}(\mathbf{v})^i, && \text{by strictly proper} \end{aligned}$$

which is a contradiction.

Iterations and Convegence

Definition (Explicit Euler map). Define $\mathcal{S}_\rho : \mathbb{R}^N \to \mathbb{R}^N$ or $\rho > 0$, by

$$\mathcal{S}_\rho(\mathbf{u}) = \mathbf{u} - \rho \mathcal{F}(\mathbf{u}).$$

It is the explicit Euler discretization, with time step ρ, of the ODE

$$\frac{d\mathbf{u}}{dt} + \mathcal{F}(\mathbf{u}) = 0.$$

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$, define $\mathbf{u} \leq \mathbf{v}$ if and only if $u_i \leq v_i$, for $i = 1, \ldots, N$.
Definition (Monotonicity). The map $\mathcal{S} : \mathbb{R}^N \to \mathbb{R}^N$ is *monotone*, if

$$\mathbf{u} \leq \mathbf{v} \text{ implies that } \mathcal{S}(\mathbf{u}) \leq \mathcal{S}(\mathbf{v}).$$

Definition (Nonlinear CFL condition). Let $\mathcal{F}$ be a Lipschitz continuous, discretely elliptic scheme. The nonlinear Courant-Freidrichs-Lax condition for the Euler map $\mathcal{S}_\rho$ is

$$\rho \leq \frac{1}{K}, \tag{CFL}$$

where K is a Lipschitz constant for the scheme.

Constructive Existence Theorems for schemes

Theorem (Contractivity of the Euler map (AO)). *Let $\mathcal{F}$ be a Lipschitz continuous, discretely elliptic scheme. Then the Euler map is a contraction in $\mathbb{R}^N$ equipped with the maximum norm, provided* (CFL) *holds. If, in addition, $\mathcal{F}$ is uniformly proper, and strict inequality holds in* (CFL)*, then the Euler map is a strict contraction.*

Uniformly proper: mild technical condition, (add $dx^2 u_i$ to each component of the equation)

Building Schemes for parabolic equations

The following theorem gives a method for building *explicit* monotone schemes for parabolic equations from a discretely elliptic schemes for the spatial part of the equation.

The CFL condition (which determines a bound on the time step) is easily determined by calculating the Lipschitz constant of the scheme.

Theorem (Monotonicity of the Euler map (AO)). *Let $\mathcal{F}$ be a Lipschitz continuous, discretely elliptic scheme. Then the Euler map is monotone provided* (CFL) *holds.*

Method for building schemes

Let $F_i : (x, r, p, M) \to \mathbb{R} \quad i = 1, 2$
$\mathcal{F}_i$: grid functions $\to$ grid functions, $\quad i = 1, 2$
Let $g(x, y)$ be a non-decreasing function, e.g. max or min.

Observation 1 (Crandall-Ishii-Lions) If F_1, F_2 are degenerate elliptic, then so is

$$F = g(F_1, F_2).$$

Observation 2 (AO) If $\mathcal{F}_1, \mathcal{F}_2$ are discretely elliptic, then so is

$$\mathcal{F} = g(\mathcal{F}_1, \mathcal{F}_2).$$

This gives a direct method for building schemes for complicated equations from simpler ones

Examples of Consistent, Monotone Schemes

Use standard finite differences, on uniform grid, explicit in time. (written for clarity with particular values of dt.)

Heat, upwind advection,

$$u_t - u_{xx} = 0 \qquad \mathcal{S}_A(U) = \frac{U_L + U_R}{2} \qquad \text{when } dt = dx^2/2$$

$$u_t - u_x = 0 \qquad \mathcal{S}_R(U) = U_R \qquad \text{when } dt = dx$$

$$u_t + u_x = 0 \qquad \mathcal{S}_L(U) = U_L \qquad \text{when } dt = dx$$

Applications

Front Propagation

$$F = u_t - |u_x| = \max\{u_t + u_x, u_t - u_x\}$$

$$\mathcal{S} = \max\{\mathcal{S}_R, \mathcal{S}_L\}$$

Convergent, monotone scheme:

$$\mathcal{S}(U) = \max\{U_L, U_R\} \qquad \text{when } dt = dx$$

Obstacle problems

Let

$$\mathcal{F}_1 \text{ be a discretely elliptic scheme for } F(x, u, Du, D^2u)$$

and let

$$\mathcal{F}_2(\mathbf{u}) = \mathbf{u} - \mathbf{g} \text{ be the constant scheme } u - g = 0$$

The obstacle problem

$$\min(F(x, u, Du, D^2u), u - g(x)) = 0$$

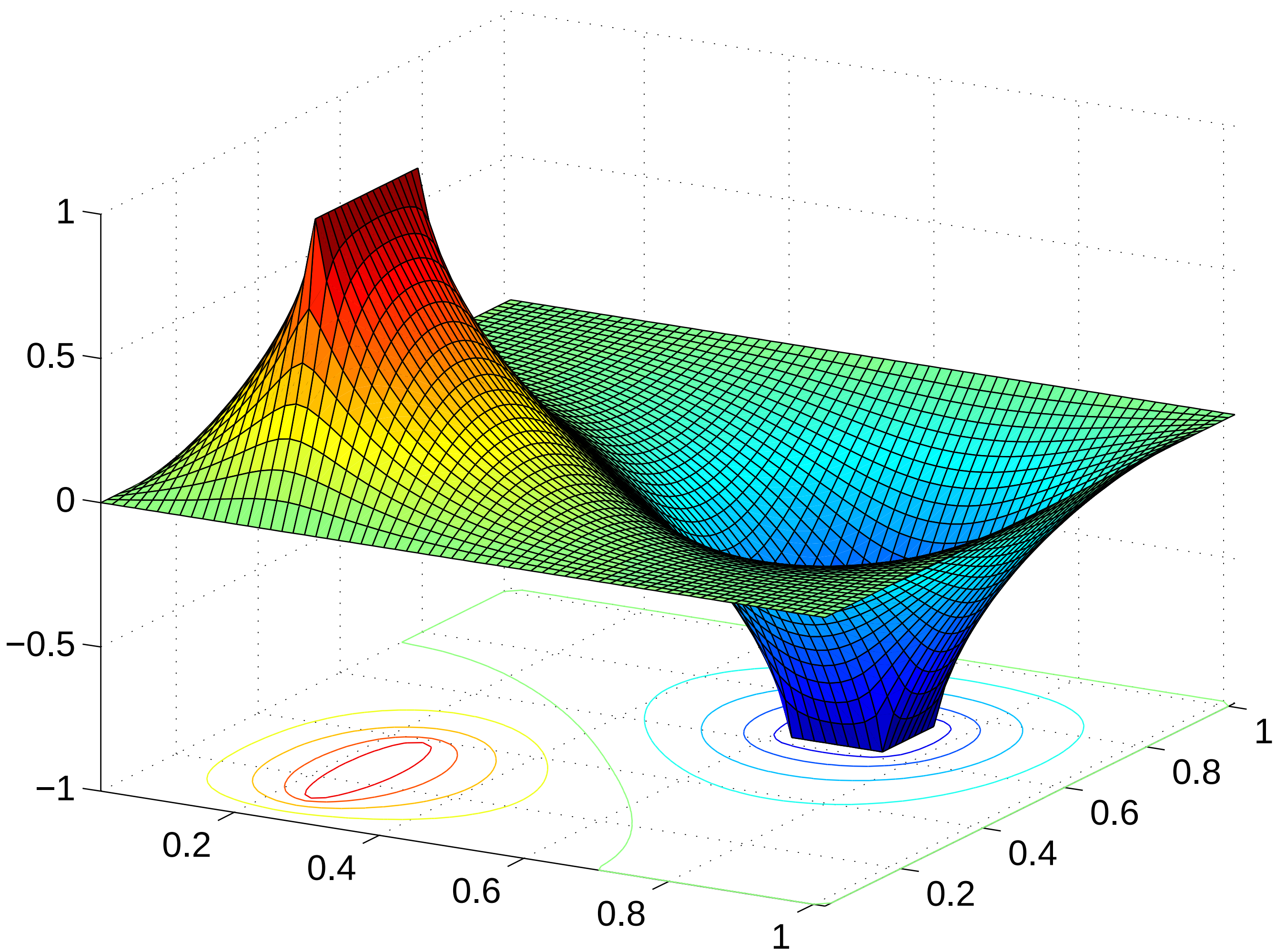
is degenerate elliptic, and the scheme

$$\mathcal{F} = \min(\mathcal{F}_1, \mathcal{F}_2)$$

is consistent and discretely elliptic.

Computations

Double obstacle problem



Solution of the double obstacle problem.

Toy HJB equation

Non-convex Hamilton-Jacobi-Bellman equation.

Stochastic Differential Games.

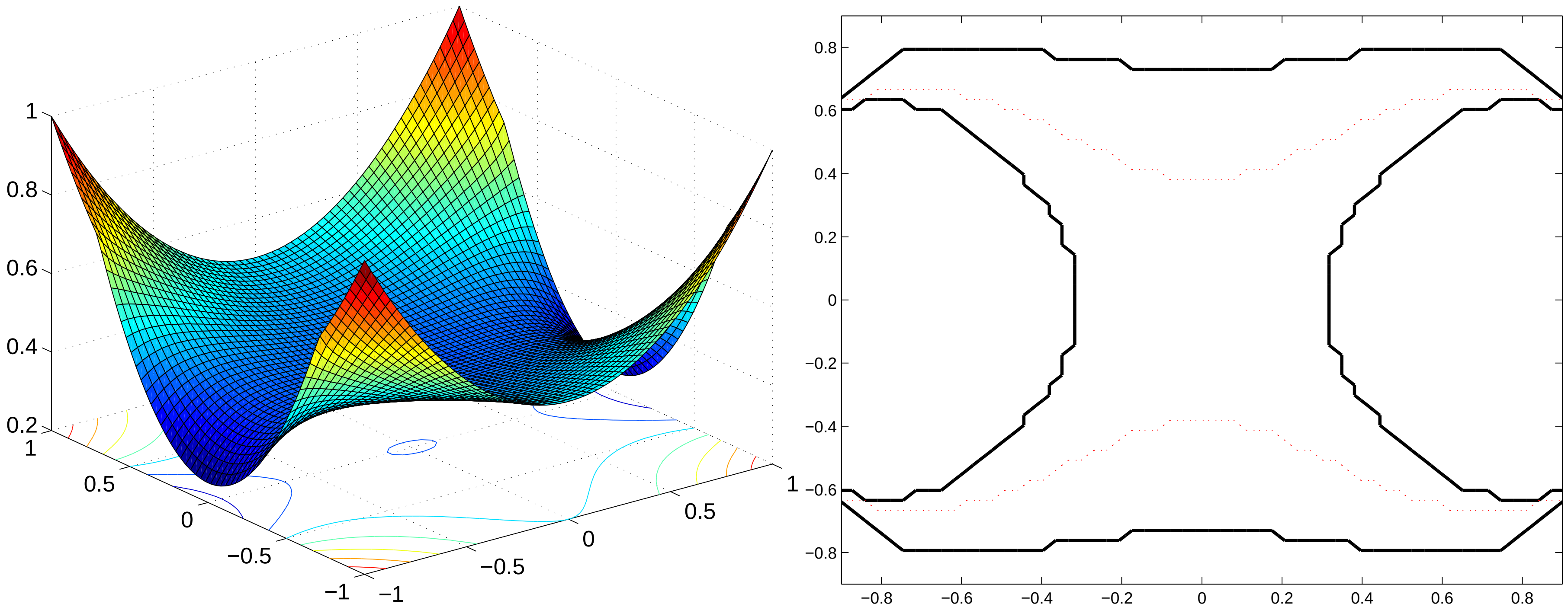
$$\begin{cases} \max(\min(L^1u, L^2u)L^3u) = 1 & \text{in } \Omega = \{-1 \leq x, y \leq 1\} \\ u = f & \text{on } \partial\Omega \end{cases}$$

where

$$\begin{aligned} L^1u &= \ \ u_{xx} + \ \ u_{yy}, \\ L^2u &= .5u_{xx} + 2u_{yy}, \\ L^3u &= .5u_{xx} + \ \ u_{yy} \end{aligned}$$

and

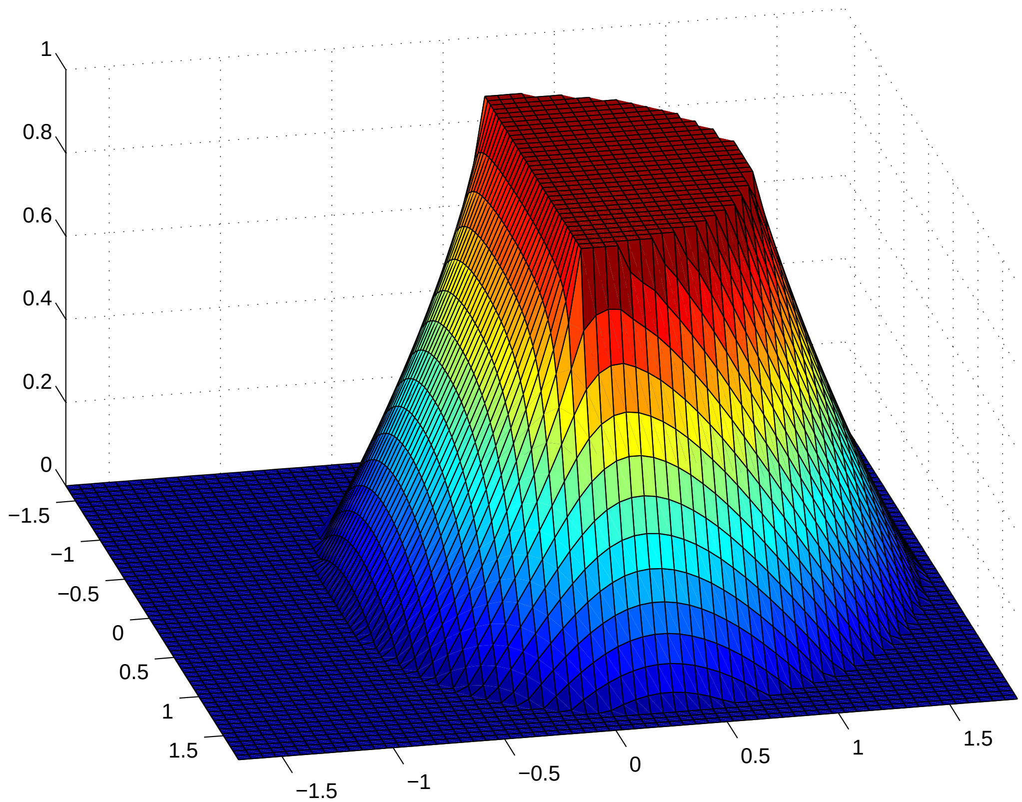
$$f(x,y) = .5\max(\min(x^2 + y^2, .5x^2 + 2y^2), .5x^2 + y^2)$$

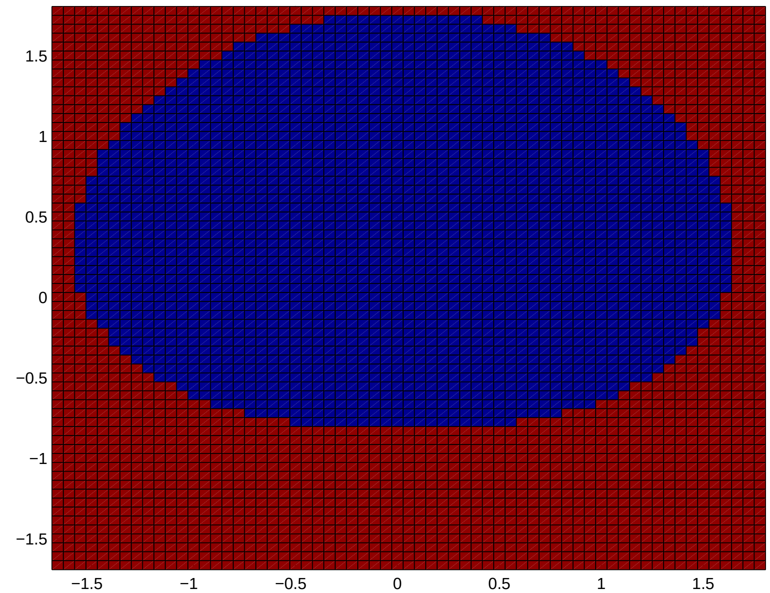


Solution and free boundary for the nonconvex, fully nonlinear second order equation max(min($L^1u, L^2u)L^3u) - 1$.

Combustion FBP

These methods can be generalized to other kinds of free boundary problems.





Early Exercise Price in Incomplete Markets

The equation for the buyer's indifference price is (A.O.-T.Zariphopolou)

$$\min(-h_t - L + H, h - g(y)) = 0$$

where $L = L(h_{yy}, h_y, y, t)$ is a linear elliptic equation

$$L(h_{yy}, h_y, y, t) = \frac{1}{2}a^2(y,t)h_{yy} + \left(b(y,t) - \rho\frac{\mu}{\sigma}a(y,t)\right)h_y$$

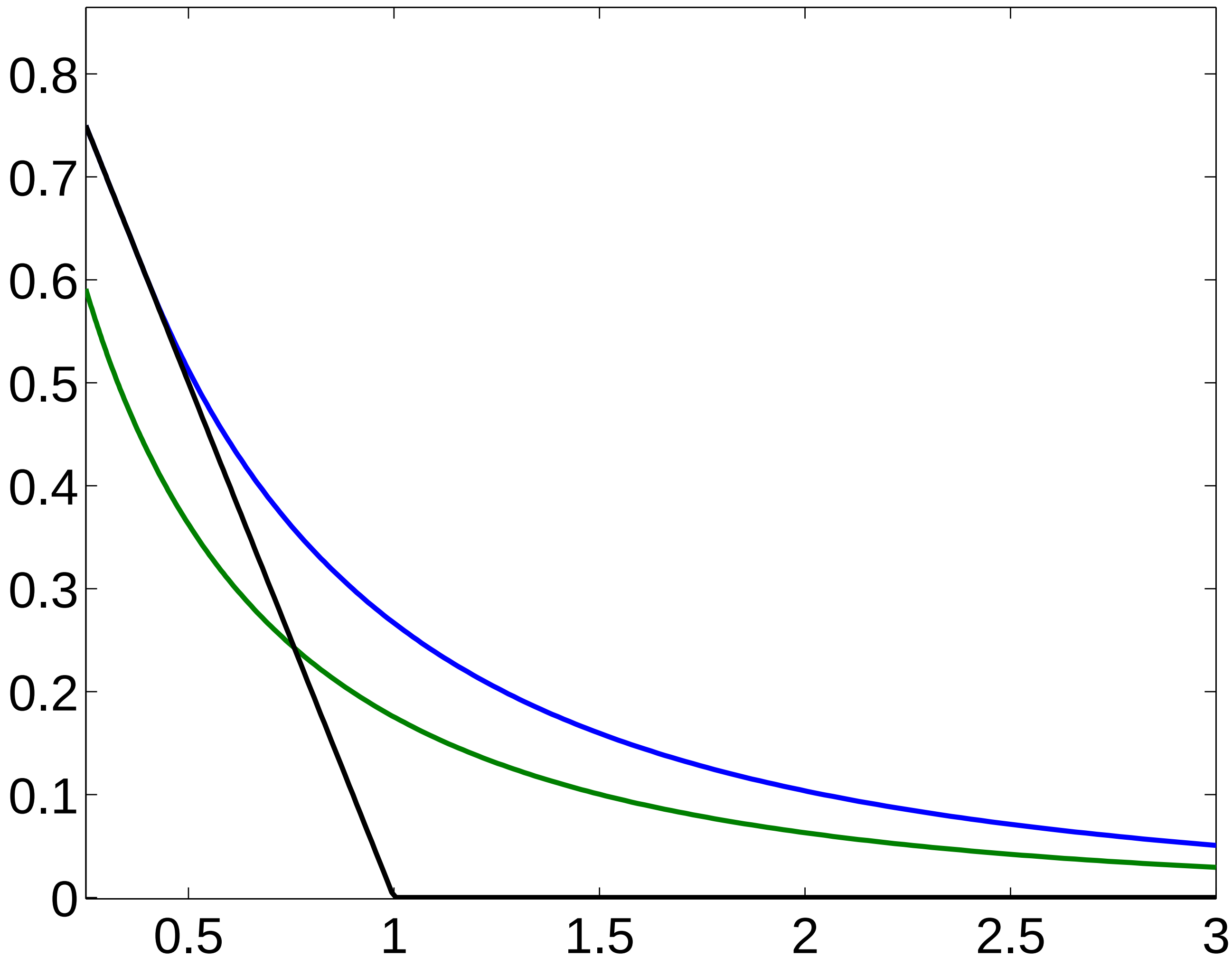
and $H = H(h_y, y, t)$ is a nonlinear first order operator

$$H(h_y, y, t) = \frac{1}{2}a^2(y,t)\gamma(1-\rho^2)h_y^2$$

and $\min(L + H, h - g)$ is an obstacle problem.

- μ, σ drift, volatility of the tradeable asset,
- ρ correlation of the untradeable with tradeable asset,
- b, a drift, volatility of the untradeable asset,
- $\gamma > 0$ the risk aversion.

Solution: American Option



Comparison of European and American options after time 1, with initial data and Sharpe = 1, $a_0 = 1, b_0 = 0.3, \rho = .1, \gamma = 1$

Mean Curvature and Infinity Laplacian

Motion of Level sets by mean curvature:

Osher-Sethian. hundreds of papers. Search google for “Level Set method” get thousands of hits.

Applications: interface motion in physics, medical imaging, movie special effects, . . .

Infinity Laplacian

‘Best’ solution of classical Whitney-McShane problem of extending a Lipschitz function.

Also used in image processing for inpainting.

Regularity: one of the last open problems in elliptic PDE,

Mean Curvature: More Background

Motion of Level Sets by Mean Curvature.

Numerics: (selected)
Bence-Merriman-Osher scheme. Alternate heat operator with a thresholding operator.

Phase field approach: singular limit of reaction diffusion eqn. (indirect).

Walkington, finite element method: (direct PDE, but no uniqueness).

M.Crandall-P.L. Lions: direct method. (impractical: requires large stencil, size $O(1/dx)$).

Theory: Evans-Spruck, Chen-Giga-Goto: existence and uniqueness of viscosity solutions.

Infinity Laplacian: introduction

(Aronsson, Crandall, Evans, Gariepy)

$$\Delta_\infty u = \frac{1}{|Du|^2} \sum_{i,j=1}^{m} u_{x_i x_j} u_{x_i} u_{x_j} = 0 \qquad \text{(IL)}$$

rewrite as

$$\Delta_\infty u = \frac{d^2 u}{dv^2}, \qquad \text{where } v = \frac{Du}{|Du|}.$$

Appears in the definition of mean curvature:

$$\Delta_1 u = \Delta u - \Delta_\infty u, \qquad \text{where } \Delta_1 \text{ is M.C.}$$

Interpretations

1. Formally limit as $p \to \infty$ of *p-Laplacian*, which is

$$\min \int |Du|^p$$

2. Minimal Lipschitz extension of boundary data: absolute minimizer.
(App: Inpainting, edge enhancement)

Difficulties in building schemes for M.C. and I.L.

Problem: Degeneracy. Even for linear elliptic eqns. may be impossible to build monotone, second order schemes (Motzkin-Wasow '53).

Solution: Drop the requirement of second order accuracy.

Problem: Quasilinearity

$$\Delta_1 u = \frac{d^2 u}{dv^2} \qquad \Delta_\infty u = \frac{d^2 u}{dn^2}, \quad n = \frac{Du}{|Du|}, v = n^\perp \text{ in } \mathbb{R}^2$$

Naive approach: compute gradient, compute 2nd derivative in the direction perp to gradient (or of gradient in case of IL). Not monotone.

Solution: Find a discrete analog of the underlying principles (variational, geometric) of the PDE to build a monotone scheme.

Variational interpretation

Given boundary data,
Dirichlet Intergral for Laplacian

$$\Delta u = 0 \text{ found by } \min \int |Du|^2 \, dx.$$

Formally

$$\Delta_1 u = 0 \text{ found by } \min \int |Du|^1 \, dx,$$

and

$$\Delta_\infty u = 0 \text{ found by } \min \int |Du|^\infty \, dx$$

Convex optimization problems

finite dimensional analogy of the variational problems

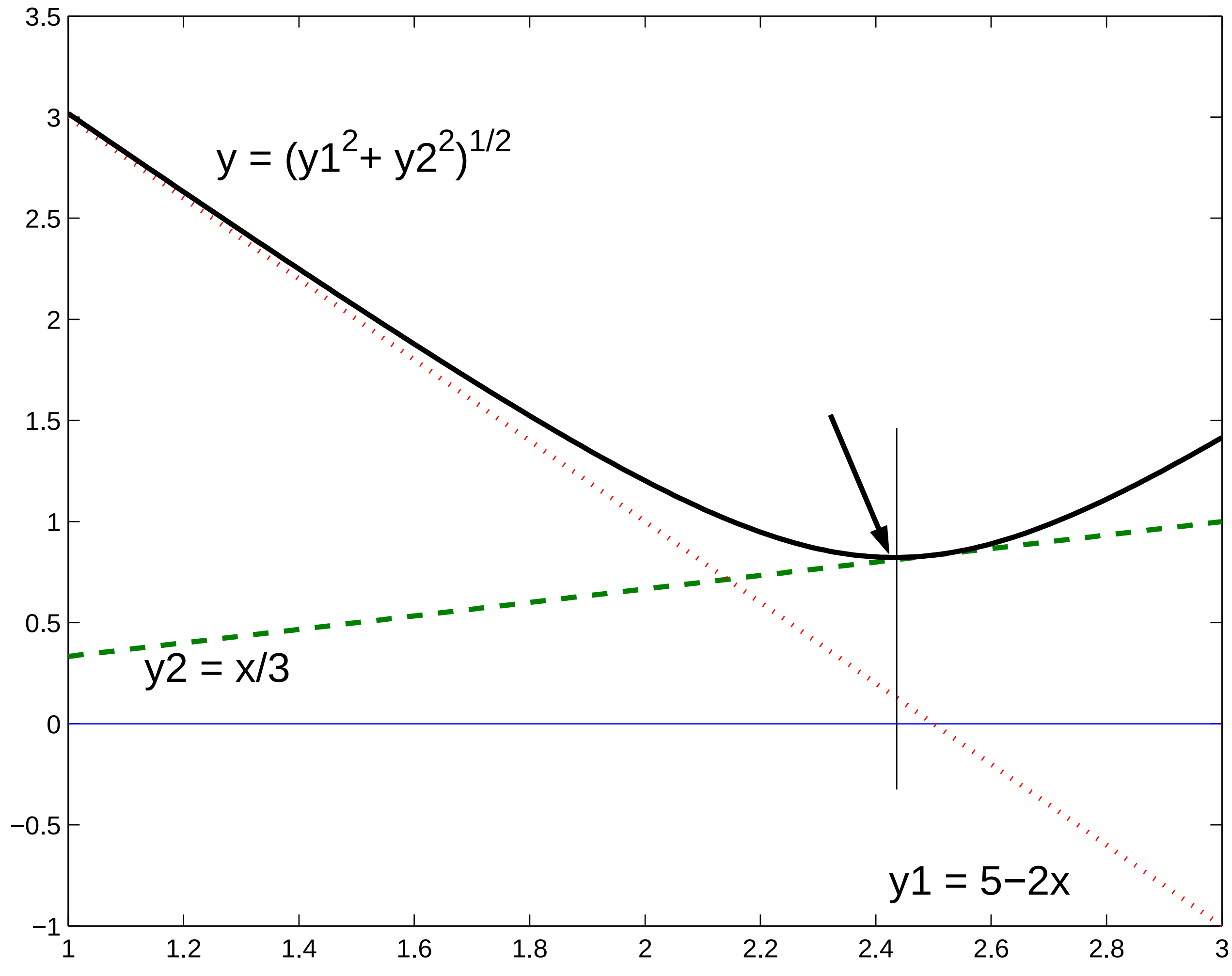
Convex optimization plays an important role in nonlinear difference schemes.

Compare with the role that solution of linear systems plays for the finite element method.

Finite Elements in L^2

Smooth ($\sim$ quadratic) optimization (classical conditions for minimum)

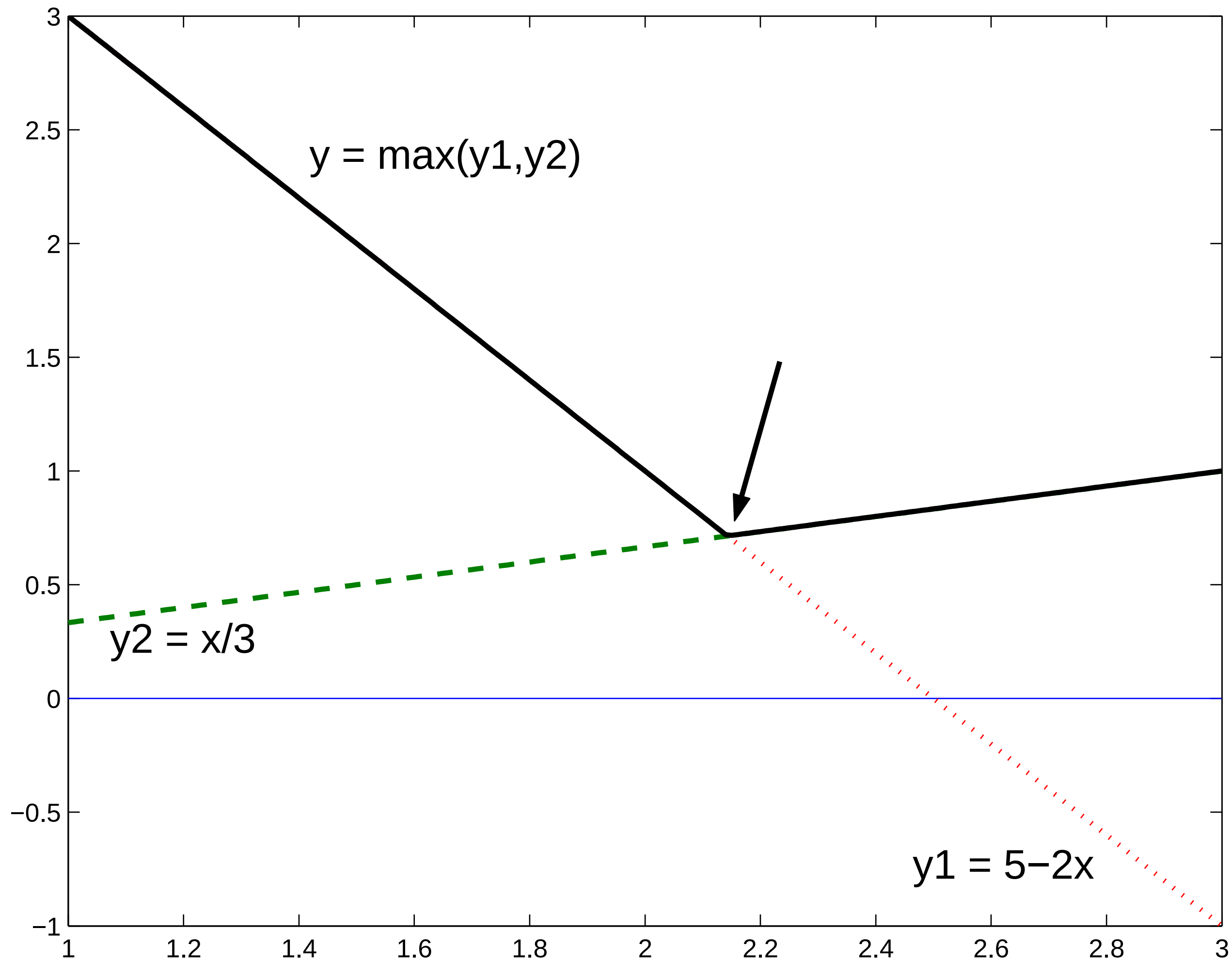
$$\min_{x \in \mathbb{R}^n} \|Ax - F\|_2$$



Finite Elements in L^∞

With non-smooth ($\sim$ p.w. linear) optimization (more difficult)

$$\min_{x \in \mathbb{R}^n} \|Ax - F\|_\infty$$



Finite differences for points on a circle

Given $x_1, \dots x_{2n}$ points equally spaced on a circle of radius dx in $\mathbb{R}^2$, $u_i = u(x_i), u_0 = u(0)$, for u a smooth function.

Write $d\theta = \frac{1}{2n}$. From Taylor series,

$$\Delta u = \frac{1}{dx^2}\left(u_0 - \text{average}(u_i)\right) + O(dx^2),$$

$$|Du|\Delta_1 u = \frac{1}{dx^2}\left(u_0 - \text{median}(u_i)\right) + O(dx^2 + d\theta),$$

$$\Delta_\infty u = \frac{1}{dx^2}\left(u_0 - (\max u_i + \min u_i)/2\right) + O(dx^2 + d\theta).$$

These discretizations do indeed give monotone schemes, but the scheme is not fully discretized.

Generalize to non-equidistant neighbors

Motivation: Discretize the minimization $\int |Du|^p$, locally.

Continuous case: variational problem.
Discretize: get a convex optimization problem.

At every grid point, solve 1d the convex optimization problem

$$\min_u \sum_{i=1}^{n} \left| \frac{u - u_i}{d_i} \right|^p A_i, \qquad 1 \le p < \infty$$

$$\min_u \max_{i=1}^{n} \left| \frac{u - u_i}{d_i} \right|, \qquad p = \infty$$

where the u_i are the values at the neighbors x_i, A_i area of triangle i.

Solution, and consistent scheme

For $1 < p < \infty$: minimize using calculus:

$$\min_u \left(\sum_{i=1}^n w_i |u - u^i|^p \right) \qquad \text{gives} \qquad 0 = \sum_{i=1}^n (w_i |u - u^i|^p)'$$

In particular, for $p = 2$,

$$u^* = \frac{1}{n} \sum_{i=1}^n u_i \qquad\qquad \text{(average)}.$$

Non-smooth convex 1d optimization problem for $p = 1, p = \infty$.
For $p = \infty$,

$$\text{Find } i, j \text{ which } \max_{i,j} \frac{|u_i - u_j|}{|d_i + d_j|}, \qquad \text{then } u^* = \frac{d_j u_i + d_i u_j}{d_i + d_j}.$$

linear interp. of values which maximize the "relaxed discrete gradient".

For $p = 1$, $u^* =$ (weighted) median of the data.
(median: sort values, take average of middle two)

Monotonicity and Consistency

For $1 \leq p \leq \infty$, if u^* is solution of problem for a given p,

$$\frac{u - u^*}{dx^2} \qquad \text{gives a monotone scheme.}$$

Furthermore,

$$\begin{aligned}
\Delta u = u_{nn} + u_{tt} &= \frac{u^* - u_0}{dx^2} + O(dx^2) \\
\Delta_\infty = u_{nn} \qquad &= \frac{u^* - u_0}{dx^2} + O(dx^2 + d\theta), \\
|Du|\Delta_1 = \qquad u_{tt} &= \frac{u^* - u_0}{dx^2} + O(dx^2 + d\theta),
\end{aligned}$$

Observation:

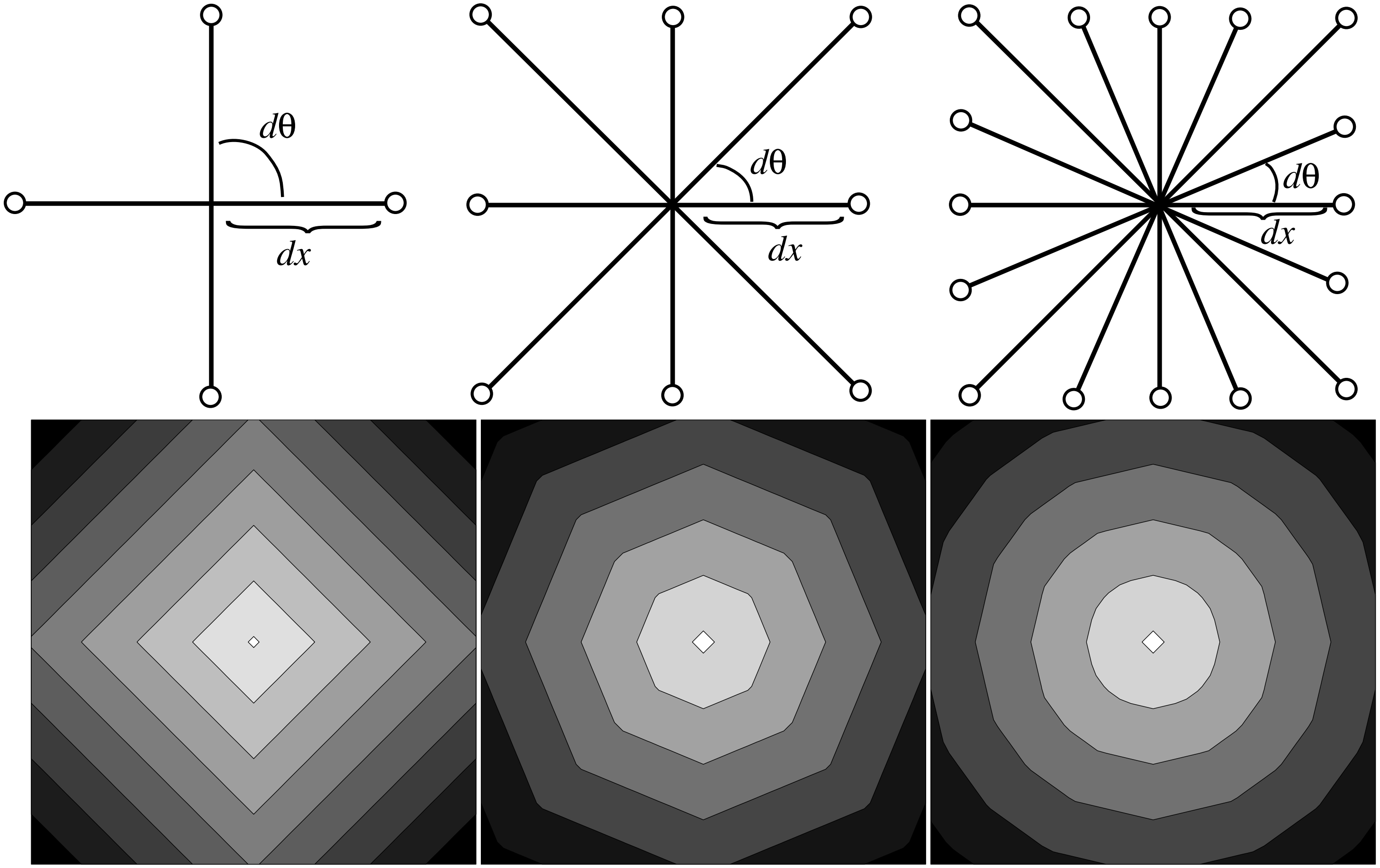
$$\Delta u = |Du|\Delta_1 + \Delta_\infty$$

$$\text{average } \approx (\text{median } + \text{ range}/2)\,/2$$

Numerics for Infinity Laplacian

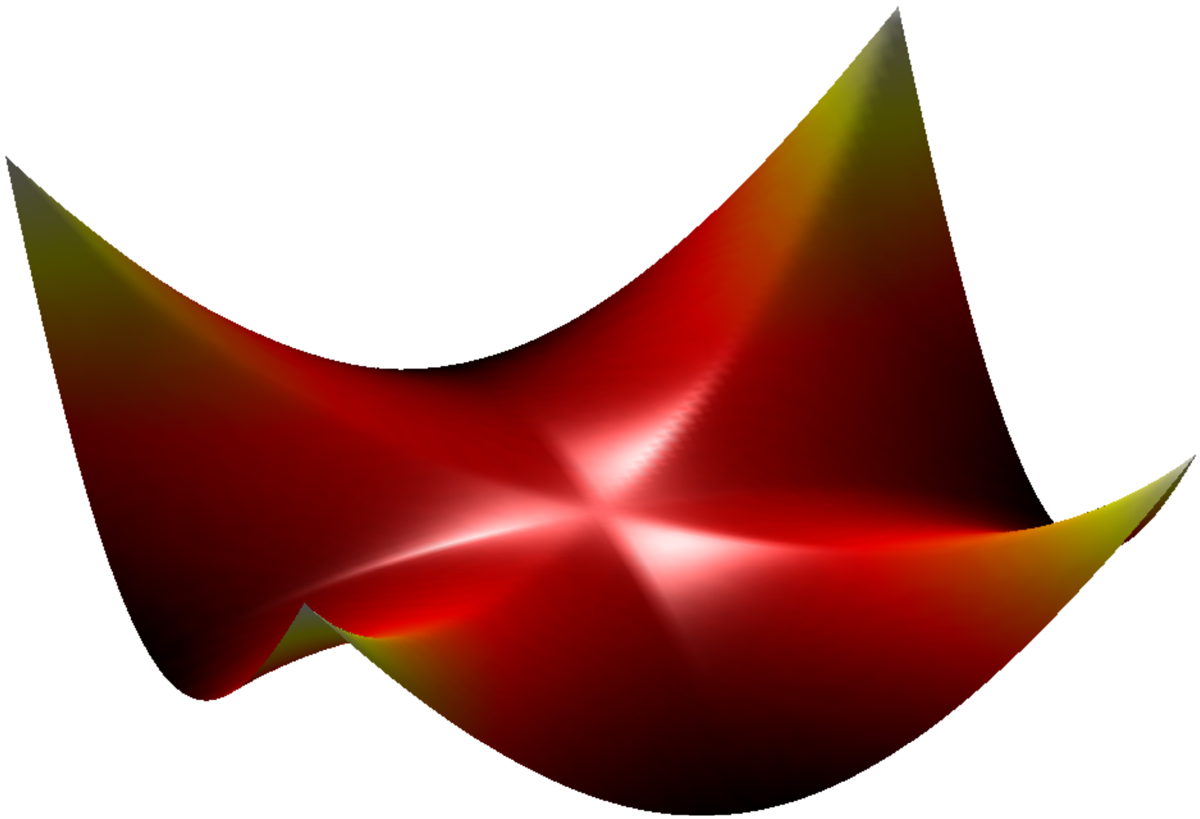
Details

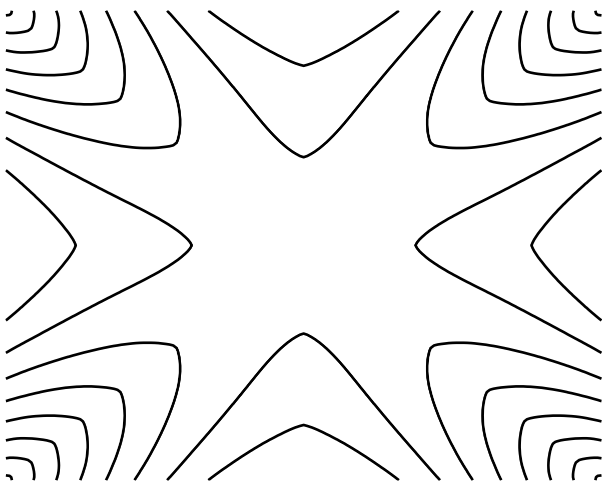
Theoretical convergence requires that we sent $dx \to 0$ and $d\theta \to 0$.



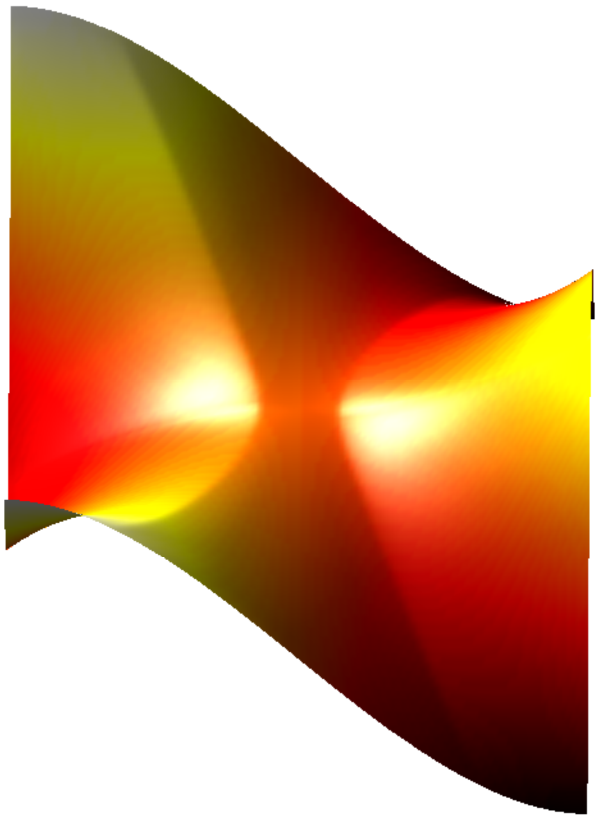
Grids for the 5, 9, and 17 point schemes, and level sets of the cones for the corresponding schemes.

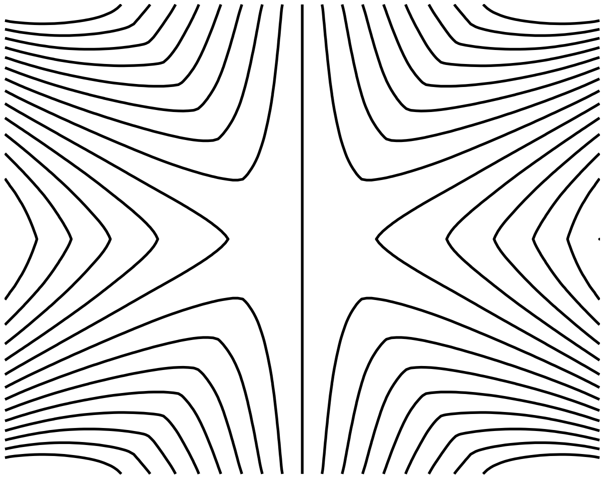
Boundary data cone: $\sqrt{x^2 + y^2}$



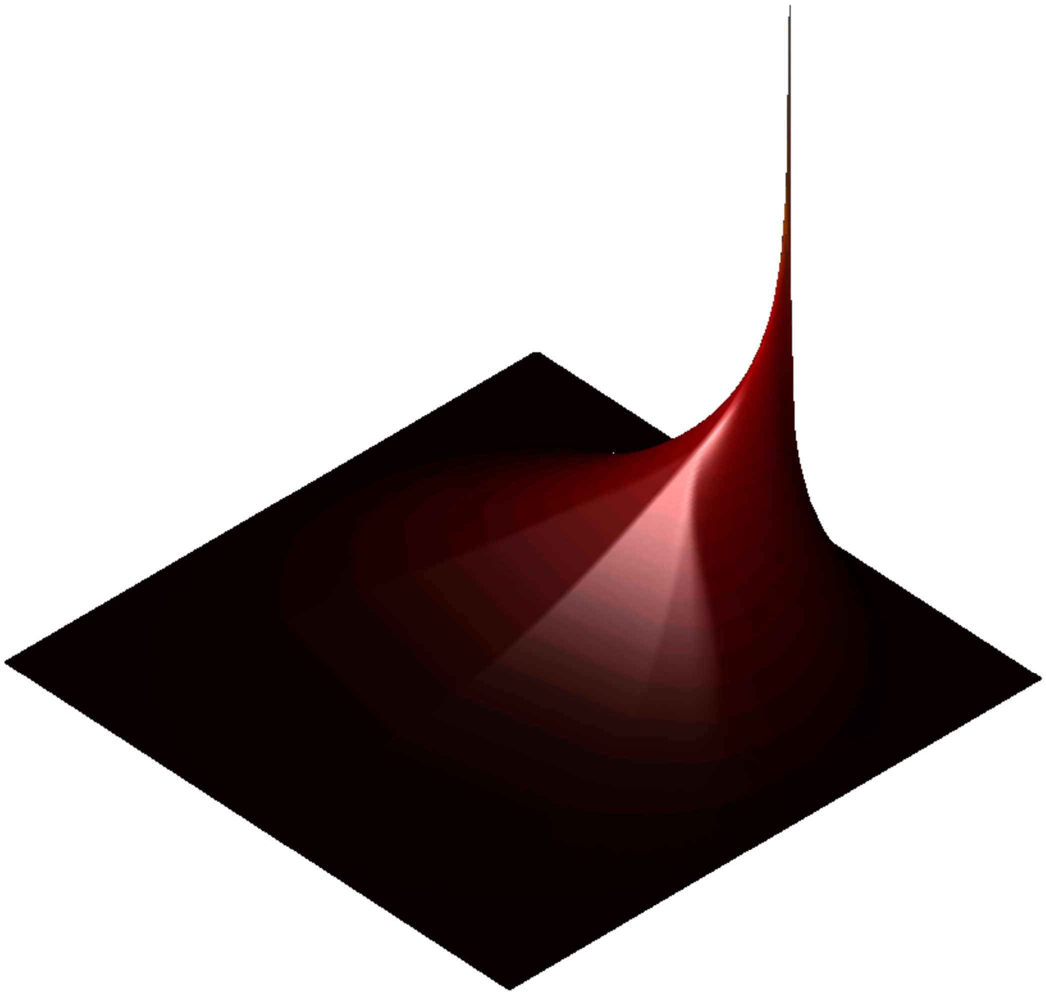


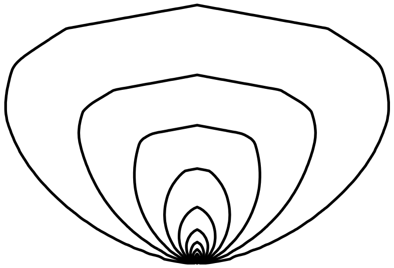
Triple symmetry





Point disturbance





Numerics for Motion by Mean Curvature

(explain the level set method)

Details

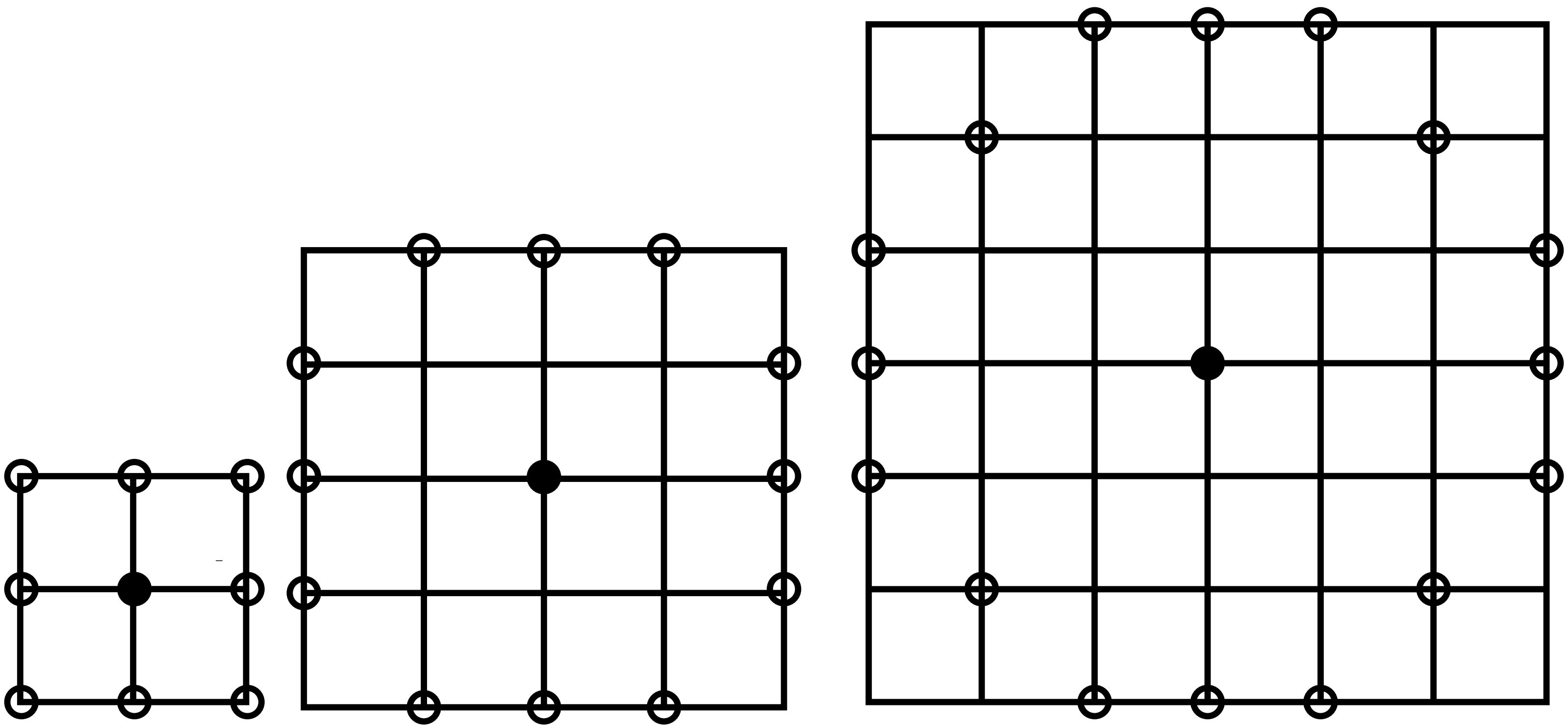
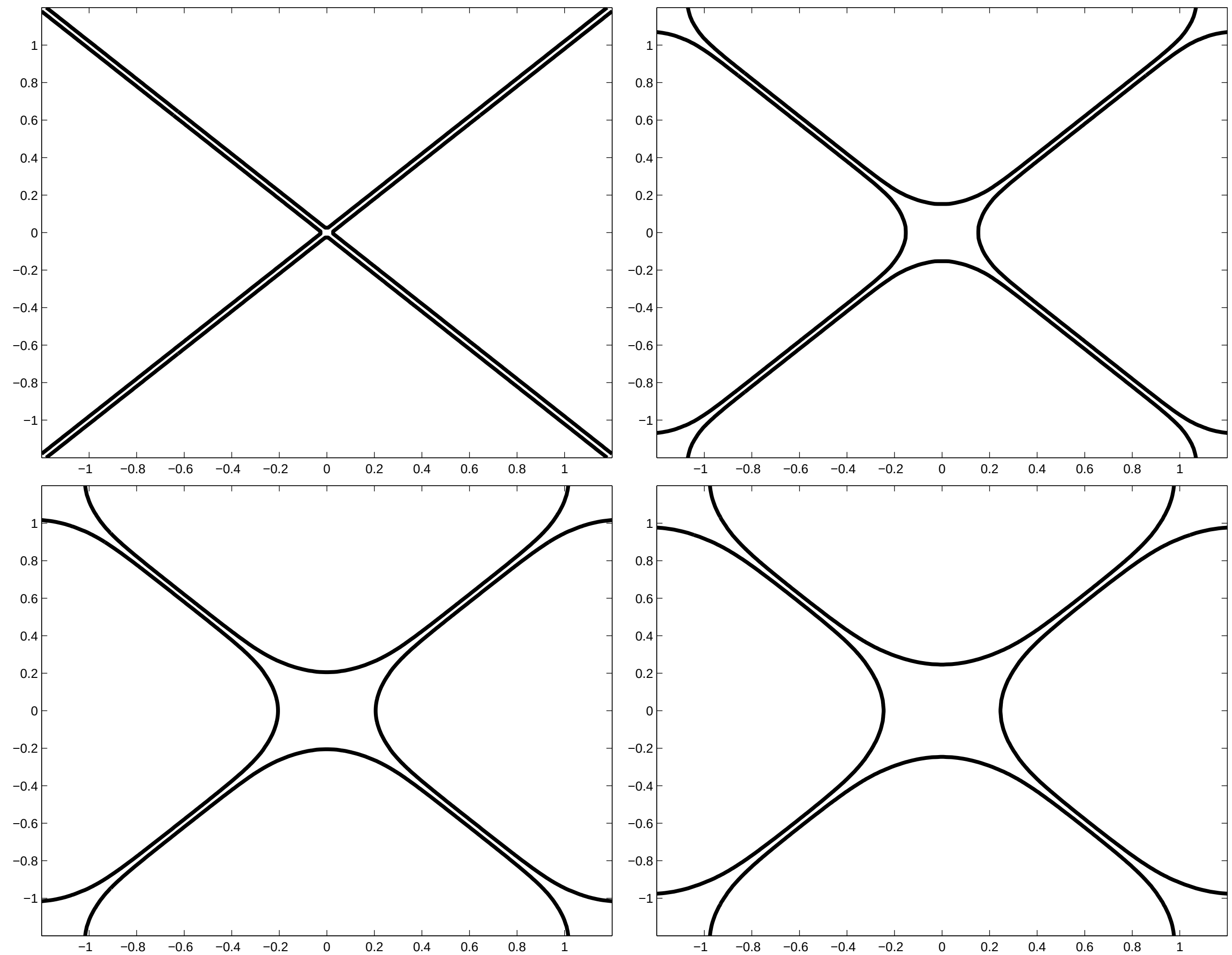
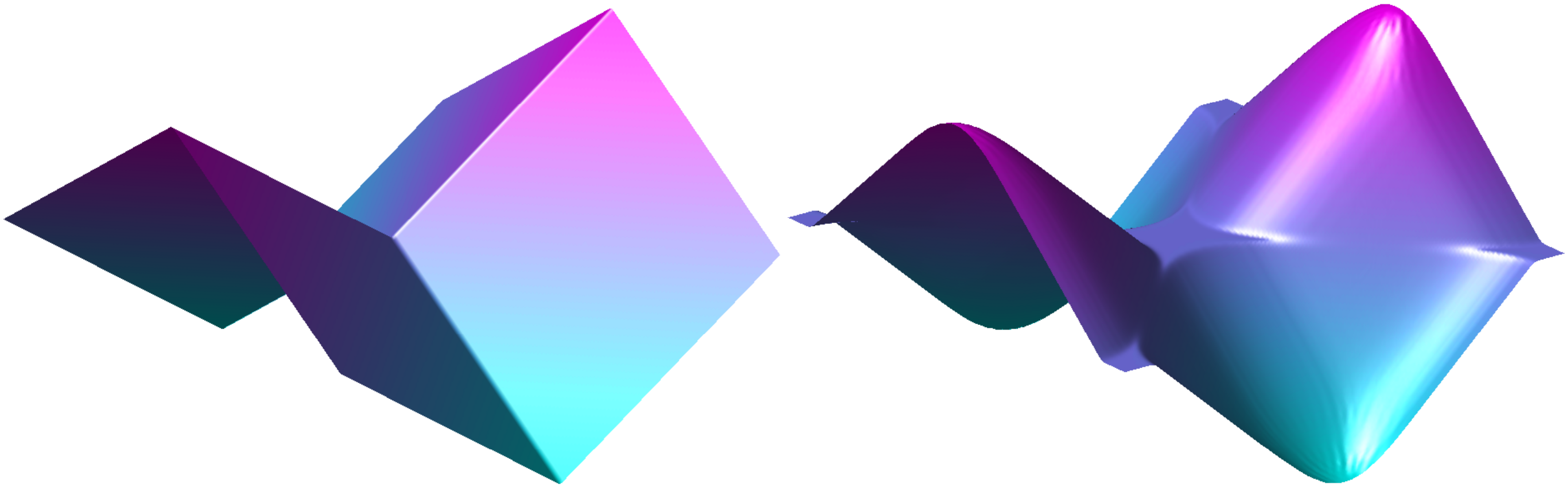


Illustration of the schemes used for $n_\theta = 1, 2, 3$



Contour plots of the $-.02$, and $.02$ contours at times $0, .015, .03, .045$.



Surface plot: initial data, and solution at time .03.

End