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# QUALITATIVE ASPECTS OF HAMILTONIAN PDE'S AND LATTICE MODELS

# IMPLEMENTATION OF METHODS FROM DYNAMICAL SYSTEMS THEORY

## Long time behaviour of higher Sobolev norms

Hamiltonian PDE's on a bounded domain (no dispersion and scattering).

Periodic boundary conditions ($x \in \mathbf{T}^{\mathbf{d}}, \mathbf{T} = \mathbf{R}/\mathbf{Z}$).

Example of the quintic defocusing NLS in 1D

$$\begin{cases} iu_t + u_{xx} - u|u|^4 = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Hamiltonian

$$H(\phi) = \int_{\mathbf{T}} \left( \frac{1}{2} |\nabla \phi|^2 + \frac{1}{6} |\phi|^6 \right)$$

gives apriori bound on

$$\|u(t)\|_{H^1}.$$

Cauchy problem is globally wellposed for $\phi \in H^s(\mathbf{T}), \mathrm{s} \geq 1$ and solution $u$ satisfies $u(t) \in H^s$ for all time.

**Problem 1.** *Assume* $\phi \in H^s$, $s > 1$.

$$\overline{\lim_{t\to\infty}} \|u(t)\|_{H^s} < \infty?$$

**Problem 2.** *If Problem 1 has negative answer, how fast may* $\|u(t)\|_{H^s}$ *grow when* $t \to \infty$?

General Gronwall estimates

$$I(t) = \|u(t)\|_{H^s}^2$$

From the equation

$$\begin{aligned} \dot{I} &= \operatorname{Im}\langle u|u|^4, \partial_x^{(2s)} u\rangle \\ |\dot{I}| &\le C\|u\|_\infty^4 \; I \end{aligned}$$

In 1D

$$\|\phi\|_\infty \le \|\phi\|_2^{1/2} \|\phi\|_{H^1}^{1/2}$$

$$|\dot{I}| \le C(\phi) I \Rightarrow I(t) \le I(0)e^{Ct}$$

$$\|u(t)\|_{H^s} \leq \|\phi\|_{H^s}\ e^{Ct}$$

Best result to date

$$\|u(t)\|_{H^s} < t^{\frac{s-1}{2}} \quad \text{for} \quad t \to \infty$$

Example of the defocusing cubic NLS in $d = 2$

$$iu_t + \Delta u - u|u|^2 = 0$$

$$H(\phi) = \int_{\mathbf{T}^2} \left[\frac{1}{2}|\nabla\phi|^2 + \frac{1}{4}|\phi|^4\right]$$

BREZIS-GALLOUET

$$2D \qquad \|\phi\|_\infty \leq C\|\phi\|_{H^1}\left[\log\frac{\|\phi\|_{H^2}}{\|\phi\|_{H^1}}\right]^{1/2}$$

$$|\dot{I}| \leq C(\log I)\ I$$

$$\Rightarrow \|u(t)\|_{H^s} < \exp\exp Ct \quad \text{for} \quad t \to \infty$$

Best result here

$$\|u(t)\|_{H^s} < t^{s-1}$$

**Conjecture.**

*In both NLS-examples*

$$\|u(t)\|_{H^s} \ll t^{\varepsilon} \text{ for all } \varepsilon > 0$$

*assuming $u(0)$ smooth.*

Case of *linear* Schrödinger equation with smooth time dependent potential

$$iu_t + \Delta u + V(x,t)u = 0$$

$x \in \mathbf{T}^{\mathbf{d}}$ ($d$ arbitrary)

$V$ smooth in $x, t$ with uniform bounds in time

**Theorem.** $\|u(t)\|_{H^s} \ll t^{\varepsilon} \|u(0)\|_{H^s}$ for $t \to \infty$

**Remark 1.** Examples of slow growth in $H^s (s > 0)$ even for smooth time periodic potential.

**Remark 2.**

In some cases, no improvements beyond Gronwall estimates are known.

Example of the 2D Euler equation

$$\begin{cases} u_t + (u.\nabla)u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases}$$

Classical solutions exist for all time.

But only double exponential bounds known for $\|u(t)\|_{H^s}$.

**Lattice Models** (coupled harmonic oscillators)

$$q = (q_n)_{n \in \mathbf{Z}}$$

Lattice Schrödinger equations

$$i\dot{q}_n + V_n q_n + \varepsilon(\Delta q)_n + \delta |q_n|^2 q_n = 0$$

$(V_n)_{n \in \mathbf{Z}}$: quasi-periodic or random

$\Delta$ = lattice Laplacian

$$\Delta(n, n') = \begin{cases} 1 \text{ if } \max |n_j - n'_j| = 1 \\ 0 \text{ otherwise} \end{cases}$$

More general: finite range models

$$i\dot{q}_n + V_n q_n + \varepsilon(\Delta q)_n + \frac{\partial}{\partial \bar{q}_n} Re \sum \lambda_n [\prod_m q_m^{k_m} \bar{q}_m^{k'_m}] = 0$$

$$|m - n| < C, 4 \leq \sum (k_m + k'_m) < C, \sum k_m = \sum k'_m$$

## The linear case: Schrödinger operators

$$H = \varepsilon\Delta + V$$

**Anderson localization:** Pure point spectrum with exponentially localized states.

**Dynamical localization:** Absence of diffusion in $e^{itH}$

$$\sup_t \sum_{n\in\mathbf{Z}^\mathrm{d}} (1+|n|^2)|(e^{itH}q)_n|^2 < \infty$$

$q = (q_n)_{n\in\mathbf{Z}^\mathrm{d}}$

$V = (V_n)_{n\in\mathbf{Z}^\mathrm{d}}$ i.i.d $\quad d=1 \quad \varepsilon$ arbitrary

$\qquad\qquad d>1 \quad \varepsilon$ small

$V_n = V(n.\omega)$ quasi-periodic, $V$ non-constant real analytic

$d = 1, 2 \qquad \varepsilon$ small

**Problems.** (Frohlich-Spencer-Wayne)

(1) Is there a nonlinear counterpart of localization?

(2) Estimate the rate of diffusion

$$D(t) = \left(\sum n^2 |q_n(t)|^2\right)^{1/2}$$

for $t \to \infty$.

Few rigorous results

$$i\dot{q}_n + V_n q_n + \varepsilon(q_{n-1} + q_{n+1}) + \frac{\partial}{\partial \bar{q}_n} \sum_{n,k,k'} \lambda_n Re\left[\prod_m q_m^{k_m} \bar{q}_m^{k'_m}\right] = 0$$

$d = 1$, $V$ random.

**Theorem:** For all $\gamma > 0, \kappa > 0$ if $\lambda_n < |n|^{-\gamma}$ and $\varepsilon < \varepsilon(\gamma, \kappa)$, then

$$D(t) < t^\kappa \text{ for } t \to \infty$$

Based on Nekhoroshev type methods.

## (II) Construction of quasi-periodic solutions

Perturbations of linear and integrable equations

$$iu_t + \Delta u + V(x)u + \varepsilon \frac{\partial H}{\partial \bar{u}} = 0$$

$$y_{tt} - \Delta y + \rho y + V(x)y + \varepsilon \partial_y F(x, y) = 0$$

$$iu_t + u_{xx} \pm u|u|^2 + \varepsilon \frac{\partial H}{\partial \bar{u}} = 0$$

$$y_t + \partial_x^3 y + y\partial_x y + \varepsilon \partial_x[\partial_y F(x, y)] = 0$$

Persistency problem of invariant tori

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Small solutions: Reduction to perturbations of linear equations with parameters by normal forms techniques.

Large solutions: Use of 'integrable coordinates' on Riemann surfaces.

Perturbations of linear equations with parameters. Example of NLS

$$(*)\quad iu_t+Mu+\varepsilon\frac{\partial H}{\partial \bar{u}}=0 \quad H(u,\bar{u}) = \text{polynomial}$$

$x \in \mathbf{T}^d$, arbitrary

$M=$ Fourier-multiplier

$$M\phi = \sum \widehat{\phi}(n) M_n e^{in.x}$$

$\{n_1, \dots, n_b\} \subset \mathbf{Z}^d$ distinguished modes

$$M_{n_j} = \lambda_j \qquad (j = 1, \cdots, b)$$
$$M_n = |n|^2 \text{ for } n \notin \{n_1, \dots, n_b\}$$

$\lambda = (\lambda_1, \dots, \lambda_b) = b$-dimensional parameter on $[0, 1]^b$.

$\varepsilon = 0 \Rightarrow$ quasi-periodic solution

$$u_0(x,t) = \sum_{j=1}^{b} a_j \; e^{i(n_j.x+\lambda_j t)}.$$

**Theorem.** *There is a Cantor subset* $\mathcal{C}_\varepsilon \subset [0,1]^b$, *mes* $\mathcal{C}_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} 1$ *and a smooth map* $\lambda \to \lambda'$ *such that for* $\lambda \in \mathcal{C}_\varepsilon$, $(*)$ *has real analytic quasi-periodic solution* $u_\varepsilon$ *near* $u_0$ *with time frequencies* $\lambda'_1, \dots, \lambda'_b$.

## REMARKS

(i) Similar results for NLW

$$M_n = |n| \quad \text{for} \quad n \notin \{n_1, \dots, n_b\}$$

(ii) Problem of multiplicities

$$\{n \in \mathbf{Z^d} |\ |\mathbf{n}|^2 = \mathbf{R}^2\}$$

For $d \geq 2$, unbounded for $R \to \infty$

(iii) No parametrization of $u_\varepsilon$ with fixed frequency $\lambda'$ as real analytic function of $\varepsilon$

(unlike in usual KAM theory)

Finite dimensional model problem:
persistency of lower dimensional invariant tori

(MELNIKOV PROBLEM)

$$\begin{cases} i\dot{q}_j = \lambda_j q_j + \varepsilon \frac{\partial H}{\partial \bar{q}_j} & (1 \le j \le b) \\ i\dot{q}_j = \mu_j q_j + \varepsilon \frac{\partial H}{\partial \bar{q}_j} & (b+1 \le j \le N) \end{cases}$$

$\lambda = (\lambda_1, \ldots, \lambda_b)$=tangential frequency vector = parameter

$\mu_{b+1}, \ldots, \mu_N$ = normal frequencies (possible multiplicities)

Persistency of $b$-dimensional torus

$$\begin{cases} |q_j| = a_j & (1 \le j \le b) \\ q_j = 0 & (b < j \le N) \end{cases}$$

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Invariant tori may have positive Lyapounov exponents.

## Nekhoroshev stability

Long-time stability (= near quasi-periodic behavior) in $\varepsilon$-perturbations of linear or integrable Hamiltonian systems.

2 mechanisms

• Perturbations of linear non-resonant systems (quasi-Nekhoroshev)
•• Perturbations of (quasi)-convex integrable Hamiltonians

$$\bullet \qquad H_\varepsilon(I,\theta) = \sum_{j=1}^{d} \lambda_j I_j + \varepsilon H'(I,\theta)$$

$\lambda = (\lambda_1, \ldots, \lambda_d)$ diophantine

$$\bullet\bullet \qquad H_\varepsilon(I,\theta) = (I_0) + \sum_{j=1}^{d} I_j^2 + \varepsilon H'(I,\theta)$$

$H'$ real analytic (or Gevrey) Stability times $t < T_\varepsilon$

$$\log T_\varepsilon > \left(\frac{1}{\varepsilon}\right)^\sigma$$

$$\sigma = \sigma(d) \sim \frac{1}{d}$$

**Implementation In Large and Infinite Dimension**
(Benettin, Frohlich, Giorgili, Bambusi, Herman, Marco, Sauzin, Kaloshin, B, ...)

($\bullet$) In preceding generality, no nontrivial stability times ($T_\varepsilon \gg \frac{1}{\varepsilon}$) unless $d \lesssim \log \frac{1}{\varepsilon}$
(B-Kaloshin)

($\bullet\bullet$) Infinite systems of coupled harmonic oscillators
(Benettin-Frohlich-Giorgili)

$$T_\varepsilon \sim \exp\left\{ c \frac{(\log \frac{1}{\varepsilon})^2}{\log\log \frac{1}{\varepsilon}} \right\}$$

(quasi-Nekhoroshev type theorem)

($\bullet\bullet\bullet$)PDE's

$$\begin{aligned} & iu_t + u_{xx} + V(x)u + \varepsilon \frac{\partial H}{\partial \bar{u}} = 0 \\ & y_{tt} - y_{xx} + \rho y + \varepsilon f'(y) = 0 \\ & iu_t + u_{xx} \pm u|u|^2 + \varepsilon \frac{\partial H}{\partial \bar{u}} = 0 \end{aligned}$$

Nontrivial stability times: $\log T_\varepsilon \gg \log \frac{1}{\varepsilon}$
Depending on phase-space topology and nonlinearity