Quasi-periodic Schrödinger operators and related

- Spectrum and eigenfunctions of Schrödinger operators

- Linear Schrödinger operators with time dependent potential

- KAM theory for nonlinear PDE

Quasi-periodic Schrödinger operators on $\mathbb{Z}$ (Formalism)

$$H = \lambda v(x + n\omega)\delta_{nn'} + \Delta \quad (n, n' \in \mathbb{Z})$$

$v=$ non-constant (real analytic) potential on $\mathbb{T}^d$

$\Delta=$ Laplacian on $\mathbb{Z}$

$$\Delta(n, n') = \begin{cases} 1 & \text{if } |n - n'| = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H\psi = E\psi \Leftrightarrow \begin{pmatrix} \psi_{N+1} \\ \psi_N \end{pmatrix} = M_N \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}$$

$$M_N = M_N(E; x) = \prod_N^1 \begin{pmatrix} \lambda v(x + n\omega) - E & -1 \\ 1 & 0 \end{pmatrix}$$

(transfer matrix)

$$L(E) = \lim_{N\to\infty} \frac{1}{N} \int \log \|M_N(E; x)\| dx$$

(Lyapounov exponent)

$$= \int \log |E - E'| d\mathcal{N}(E')$$

(Thouless formula)

The almost Mathieu case

$$H_\lambda = \lambda \cos(x + n\omega) + \Delta$$

(Bloch electron in a magnetic field – Peierls, Hofstadter)

For almost all (x, ω)

$\lambda > 2 : H_\lambda$ has pp spectrum with Anderson localization (Jitomirskaya)

$\lambda = 2 : H_\lambda$ has pure singular continuous spectrum (Gordon-Jitomirskaya-Last-Simon)

$\lambda < 2 : H_\lambda$ has pure absolutely continuous spectrum (Aubry-duality)

H_λ and $H_{\frac{4}{\lambda}}$ are dual

Phase transition at $\lambda = 2$.

General real analytic potential (pp spectrum)

$$H_\lambda = \lambda v(x + n\omega) + \Delta.$$

Theorem. *v real analytic, non-constant on $\mathbb{T}^d$ ($d \geq 1$). If $|\lambda| > \lambda_0(v)$, then for almost all (x, ω), H_λ satisfies A.L.*

(non-perturbative result)

Consequence of the following

Theorem. (B-Goldstein): *Assume $L(E) > 0$ for all ω and E. Then H_λ satisfies A.L. for almost all (x, ω).*

Theorem. *For $|\lambda| > \lambda_0(v)$,*

$$L(E) > \frac{1}{2} \log |\lambda| \textit{ for all } \omega, E$$

v = trigonometric polynomial (M. Herman)

= real analytic on $\mathbb{T}$ (Sorets–Spencer)

= real analytic on $\mathbb{T}^d$, $d > 1$ (B)

Almost Mathieu

$$L(E) = \max\left(0, \log\frac{|\lambda|}{2}\right) \text{ if } \omega \notin \mathbb{Q}, E \in \operatorname{Spec} H_{\lambda,\omega}$$

Theorem. *For any fixed* $\omega, L(E)$ *is a continuous function of* E

$d = 1$: Jitomirskaya, B

$d > 1$: B

Theorem. $L_\omega(E)$ *is jointly continuous at* (ω_0, E_0) *if*

$$n.\omega_0 \not\equiv 0 \qquad \forall n \in \mathbb{Z}^d$$

General real analytic potential (a.c. spectrum)

Theorem. (B-J) *$H = \lambda v(x+n\omega)\delta_{nn'} + \Delta$, v real analytic on $\mathbb{T}$, ω diophantine. Then for $|\lambda| < \lambda_1(v)$, H has pure a.c.-spectrum*

(non perturbabive)

Theorem. ($d > 1$): *For $|\lambda| < \lambda_1(v, \varepsilon)$ and $\omega \notin \Omega \subset \mathbb{T}^d$, $\operatorname{mes} \Omega < \varepsilon$, $H_{\lambda,\omega}$ has pure a.c.-spectrum*

(perturbative result)

Theorem. (B): *Let v be a trigonometric polynomial on $\mathbb{T}^2$ with non-degenerate maximum. There is $\Omega \subset \mathbb{T}^2$, $\operatorname{mes} \Omega > 0$ such that for $\omega \in \Omega$, $H_\omega = v(n\omega_1, n\omega_2) + \Delta$ has some point spectrum P, $\operatorname{mes} \bar{P} > 0$, with exponentially localized states.*

(possible coexistence of a.c. and point spectrum)

Quasi-periodic operators on $\mathbb{Z}^2$

$$H = \lambda v(x_1 + n_1\omega_1, x_2 + n_2\omega_2) + \Delta$$

$$\Delta(n, n') = \begin{cases} 1 & \text{if } |n_1 - n_1'| + |n_2 - n_2'| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Theorem. (B–Goldstein–Schlag)

Assume v real analytic on $\mathbb{T}^2$, none of the partial maps $v(x_1, \cdot), v(\cdot, x_2)$ constant. For all $\varepsilon > 0$, there is $\lambda_0 = \lambda_0(v, \varepsilon)$ such that for $|\lambda| > \lambda_0$ and $\omega \notin \Omega$, $\operatorname{mes} \Omega < \varepsilon$,

$$H = \lambda v(n_1\omega_1, n_2\omega_2) + \Delta$$

satisfies A.L

Perturbative result

Problem not solved on $\mathbb{Z}^3$

Initial arithmetic restrictions on ω of non-diophantine nature.

Initial frequency restriction

Proposition. *Fix N (large). There is subset* $\Omega_N \subset$ $[0,1]^2$

$$\text{mes}\,([0,1]^2 \backslash \Omega_N) < e^{-\sqrt{\log N}}$$

with the following property.

Let $S \subset [0,1]^2$ *be semi-algebraic of degree* B *such that*

$$\text{mes}\, S_{x_1} < \eta, \text{mes}\, S_{x_2} < \eta \text{ for all } x_1, x_2 \in [0,1]$$

$$\log B \ll \log N \ll \log \frac{1}{\eta}.$$

Then, for $\omega \in \Omega_N$

$$\{(n_1, n_2) \in \mathbb{Z}^2 |\, |n_i| \leq N \text{ and } (n_1\omega_1, n_2\omega_2) \in S (\text{mod}\, 1)\} < N^{1-}$$

(c > 0 a fixed constant).

Dynamical localization

H as above

$$i\frac{\partial\psi}{\partial t} = H\psi \qquad \psi(t) = e^{itH}\psi(0)$$

Dynamical localization

$$\sup_t \left(\sum_n (1 + |n|^2) \, |\langle e^{itH}\psi, \delta_n \rangle|^2 \right)^{1/2} < \infty$$

assuming

$$|\psi_n| < |n|^{-A} \text{ for } |n| \to \infty$$

(absence of diffusion $\Rightarrow$ no continous spectrum).

Theorem. *In the previous statements involving AL, dynamical localization holds as well.*

Skew-shift dynamics

$$T : \mathbb{T}^2 \to \mathbb{T}^2 : (x, y) \mapsto (x + y, y + \omega) \qquad \text{(skew shift)}$$

$$H_{\lambda,\omega,x,y} = \lambda\big(\cos(x + ny + \frac{n(n-1)}{2}\omega)\big)\delta_{nn'} + \Delta$$

Theorem. (B-Goldstein-Schlag) *For* $\lambda > \lambda_0(\varepsilon)$, $H_{\lambda,\omega,x,y}$ *satisfies A.L. for* $(\omega, x, y) \notin \Omega \subset \mathbb{T}^3$, mes $\Omega < \varepsilon$.

Problem. Does this hold for all $\lambda \neq 0$ and almost all (ω, x, y)?

Theorem. (B) For all $\lambda > 0$, there is a set $\Omega \subset \mathbb{T}$, mes $\Omega > 0$ such that for $\omega \in \Omega$, $H = \lambda(\cos \frac{n(n-1)}{2}\omega)\delta_{nn'} + \Delta$ has some point spectrum P, mes $\bar{P} > 0$.

($\Rightarrow$ different behaviour from almost Mathieu operator).

Quantum kicked rotor

Time dependent linear Schrödinger equation on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

$$i\frac{\partial\psi}{\partial t} + a\frac{\partial^2\psi}{\partial x^2} + ib\frac{\partial\psi}{\partial x} + V(t,x)\psi = 0$$

$$V(t,x) = \kappa(\cos 2\pi x)\left[\sum_{n\in\mathbb{Z}} \delta(t-n)\right]$$

$\Leftrightarrow$ periodic sequence of kicks

(quantum analogue of Chirikov standard map).

Monodromy operator

$$W = U_{a,b}W_\kappa$$

$$U_{a,b} = e^{i(a\frac{d^2}{dx^2} + ib\frac{d}{dx})} \leftrightarrow e^{i(4\pi^2 an^2 + 2\pi bn)}\delta_{nn'}$$

(related to skew shift)

W_κ= multiplication by $e^{i\kappa\cos 2\pi x}$.

Theorem. (B) *Fix* $\varepsilon > 0$. *For fixed* b, $\kappa < \kappa(\varepsilon)$ *and* $a \notin \Omega \subset \mathbb{T}$, $\operatorname{mes} \Omega < \varepsilon$, *flow of QKR is almost periodic in time (i.e.* **no** *diffusion).*

Precise formulation:

Given Sobolev exponent s, there is s_1 such that if $\psi(0) \in H^{s_1}(\mathbb{T})$, then $\psi(t)$ is almost periodic on $\mathbb{R}$ as $H^s(\mathbb{T})$-valued map.

Theorem. (B-J) *Previous statement holds in non-perturbative form, thus for* $\kappa < \kappa_0$ *(explicit constant) and almost all* a, b.

Problems.

- Establish result for all κ
- Estimate localization length

Quasi-periodic solutions in Melnikov problems and nonlinear PDE

Frohlich-Spencer-Wayne, Eliasson, Wayne, Kuksin, Craig–Wayne, B.

NLS-setting

$$iq_t = \mathcal{L}q + \varepsilon \frac{\partial H_1}{\partial \bar{q}}$$

$q = (q_n)_{n \in \mathbb{Z}^d}$ (Fourier modes)

Fix b modes $n_1, \dots, n_b \in \mathbb{Z}^d$

$\mathcal{L}$ given by multiplier $\{\mu_n\}_{n \in \mathbb{Z}^d}$

$$\begin{cases} \mu_{n_j} = \lambda_j & (1 \leq j \leq b) \\ \mu_n = |n|^2 \text{ for } n \in \mathbb{Z}^d \backslash \{n_1, \dots, n_b\} \end{cases}$$

$\lambda = (\lambda_1, \dots, \lambda_b)$ is parameter in interval Ω.

(perturbation of linear problem)

Unperturbed solution ($\varepsilon = 0$)

$$\begin{cases} q_{n_j}(t) = a_j e^{i\lambda_j t} & (1 \le j \le b) \\ q_n(t) = 0 \text{ if } n \notin \{n_1, \dots, n_b\}. \end{cases}$$

Theorem. (persistence) *ε small*

$$\Omega_\varepsilon \subset \Omega \qquad \text{mes}\,(\Omega \backslash \Omega_\varepsilon) \overset{\varepsilon \to 0}{\to} 0$$

$\exists$ smooth map $\lambda \mapsto \lambda'$ on Ω.

For $\lambda \in \Omega_\varepsilon$, there is quasi-periodic solution $q(t)$

$$q_n(t) = \sum_{k \in \mathbb{Z}^b} \hat{q}_n(k) e^{ik\lambda' t}$$

$$\widehat{q}_{n_j}(e_j) = a_j \qquad (1 \le j \le b)$$

$$|\widehat{q}_n(k)| < e^{-c(|n|+|k|)}$$

$$\sum_{(n,k) \notin \mathcal{R}} |\widehat{q}_n(k)| < \sqrt{\varepsilon} \text{ where } \mathcal{R} = \{(n_j, e_j) | j = 1, \dots, b\}$$

Proof proceeds by Lyapounov–Schmidt decomposition and Newton-iteration scheme

Linearized operator

$$(k.\lambda' - \mu_n)\delta_{(n,k),(n',k')} + \varepsilon S$$

Similar results for NLW

$$iq_t = Bq + \varepsilon B^{-1}\frac{\partial H_1}{\partial \bar{q}}$$

B = Fourier multiplier $(\mu_n)_{n\in\mathbb{Z}^d}$ (d arbitrary)

$$\begin{cases} \mu_{n_j} = \lambda_j > 0 & (1 \le j \le b) \\ \mu_n = |n| \text{ if } n \notin \{n_1, \dots, n_b\} \end{cases}$$

$B \leftrightarrow \sqrt{-\Delta}$.

Remark. Admissible frequencies in perturbed equation depend on nonlinearity, in particular ε.

Methods

Subharmonic function theorey

(⇒ measure estimate)

Theorey of semi-algebraic sets

(⇒ complexity bounds, elimination of variables)

Large deviation type estimates

Proposition.

$$H = v(x + n\omega)\delta_{nn'} + \Delta$$

v real analytic on $\mathbb{T}$

$$\|k\omega\| > c\frac{1}{|k|(\log(1+|k|))^3} \text{ for } k \in \mathbb{Z}\backslash\{0\}$$

For $\kappa > N^{-\frac{1}{10}}$

$$\text{mes}\left[x \in \mathbb{T} \left| \left|\frac{1}{N}\log\|M_N(x)\| - L_N(E)\right| > \kappa\right.\right] < Ce^{-c\kappa^2 N}$$

Similar results in multi-frequency case.

Matrix-valued Cartan theorem

Theorem. $A(\sigma)$ (self-adjoint) $N \times N$ matrix function, satisfying

(i) $A(\sigma)$ real analytic with analytic extension to

$$|\operatorname{Re} z| < \delta, |\operatorname{Im} z| < \gamma < \delta$$

such that

$$\|A(z)\| < B_1$$

(ii) For each $\sigma \in [-\delta, \delta]$, there is $\Lambda \subset [1, N]$, such that

$$|\Lambda| < M < N$$

$$\|(R_{[1,N]\setminus\Lambda} A(\sigma) R_{[1,N]\setminus\Lambda})^{-1}\| < B_2$$

(iii) $\text{mes}\, [\sigma \in [-\delta, \delta] |\, \|A(\sigma)^{-1}\| > B_3] <$ $10^{-3}\gamma(1 + B_1)^{-1}(1 + B_2)^{-1}$.

Then, for

$$K > (1 + B_1 + B_2)^{10M}$$

$$\text{mes}[\sigma \in [-\delta, \delta] |\, \|A(\sigma)^{-1}\| > K] < \exp - \frac{c \log K}{M \log(M + B_1 + B_2 + B_3)}$$

Uniformization theorem for semi-algebraic sets (Yomdin-Gromov)

Theorem. $S \subset [0,1]^d$ *semi-algebraic of degree* B *(*d *fixed,* B *large)*

Fix r. *There is constant* $C = C(d, r)$ *s.t.*

S *can be triangulated in* $\leq B^C$ *simplices* Δ *(=* k*-simplex)* $\subset S$ *and for each* Δ, *there is homeomorphism*

$$h_\Delta : \Delta^k \to \Delta \qquad (\Delta^k \subset \mathbb{R}^k \textit{ is regular simplex})$$

satisfying

$$\|D_{r'} h_\Delta\| \leq 1 \textit{ for } r' \leq r.$$