ON RANDOM SCHRÖDINGER OPERATORS $(D > 1)$

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Lattice version.

$$\Delta + \sum_{n \in \mathbb{Z}^d} V_n \delta_{nn'}$$

$$\{V_n | n \in \mathbb{Z}^d\} \; i.i.d.'s$$

$$\Delta(n, n') = \begin{cases} 1 \text{ if } |n - n'| = 1 \\ 0 \text{ otherwise} \end{cases} \qquad \text{(lattice Laplacian)}$$

Continuum version.

$$-\Delta + \sum_{n \in \mathbb{Z}^d} V_n \varphi(x - n) \qquad (x \in \mathbb{R}^d)$$

$$\varphi \text{ 'bumpfunction'}$$

Rough spectral picture

$d = 1$ Almost surely Anderson localization:

Pure point spectrum + exponentially decaying eigenstates

$d > 1$ Lattice version: AL for large disorder or at edge of spectrum

Continuum version: AL at the bottom of the spectrum

For $d \geq 3$: presence of a.c. spectrum expected??

We will also consider the case of decaying potentials $W_n = |n|^{-\alpha} V_n, \alpha > 0$ and $\{V_n\}$ as above.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Part 1: The Anderson Bernoulli model

1. This is the situation where the V_n are discrete valued, say

$$V_n \in \{0, \lambda\}.$$

These models are significantly harder to analyze, due to the fact that certain basic techniques in this field do depend on a continuous distribution (for instance, eigenvalue variation arguments).

For $d = 1$: theory is reasonably well understood.

For $d \geq 2$: only very recent result.

2. Anderson-Bernoulli in 1D.

Proof of AL.

Carmona-Klein-Martinelli (CMP, 1987).

transfer matrix + Furstenberg-Lepage method

Shubin-Vakilian-Wolff (GAFA, 1988)

super symmetric approach

Damanik-Sims-Stolz (Duke Math, 2002) (continuum model).

Problem. Behaviour of the density of states of

$$H_\lambda = \Delta + \lambda \sum_{n \in \mathbb{Z}} \varepsilon_n \delta_{nn'}$$

for small λ.

Denote $N(E)$ the IDS (Integrated density of states) related to the Lyapounov exponent by Thouless formula

$$L(E) = \int \log |E - E'| dN(E')$$
$$L(E) = \lim_{N \to \infty} L_N(E)$$
$$L_N(E) = \frac{1}{N} \int \log \|M_N(E; \varepsilon)\| d\varepsilon$$

where $M_N(E;\varepsilon)$ denotes the 'transfer-matrix', in our case

$$M_N(E;\varepsilon) = \prod_{n=N}^{1} \begin{pmatrix} \lambda\varepsilon_n - E & -1 \\ 1 & 0. \end{pmatrix}$$

It is known that the IDS for 1D Bernoulli-model is Hödler continuous

$$|N(E) - N(E')| \leq C|E - E'|^\alpha$$

for some exponent $\alpha = \alpha(\lambda) > 0$ (see [S-V-W] for instance).

$\alpha(\lambda) \leq \frac{2\log 2}{\text{Arc cosh }(1+\frac{|\lambda|}{2})}$ (Simon-Taylor-Halperin).

Is $N(E)$ Lipschitz for λ small enough?

$a(\lambda) > \frac{1}{5} - 0(\lambda)$ for λ small and E inside $]-2,2[$ (B, LNM?)

3. Bernoulli for $D \geq 2$.

Theorem. *(B+C. Kenig, 2004)*

$$H_\varepsilon = -\Delta + V$$

$$V = V_\varepsilon(x) = \sum_{n\in\mathbb{Z}^d} \varepsilon_n \varphi(x-n) \qquad \varepsilon \in \{0,1\}^{\mathbb{Z}^d}$$

$$0 \leq \varphi \leq 1, \varphi \textit{ smooth}, \operatorname{supp}\varphi \subset B\left(0, \frac{1}{10}\right)$$

At energies near the bottom of the spectrum $(E > 0, E \approx 0)$, H_ε displays Anderson localization a.s. in ε.

Some ingredients of the proof.

(i) In order to perform eigenvalue variation, we need to consider the natural extension of $V_\varepsilon(x)$ to $\varepsilon \in [0,1]^{\mathbb{Z}^d}$. First order eigenvalue variation gives then that

$$\partial_n E = \int_{\mathbb{R}^d} \xi(x)^2 \varphi(x-n)dx \tag{3.1}$$

with $\xi = \xi_{E,\varepsilon}$ the corresponding normalized eigenfunction, i.e. $H_\varepsilon \xi = E\xi$, $\|\xi\|_2 = 1$.

The 'influence' I_n on site n of E, defined as

$$\begin{aligned} I_n(\varepsilon) &= E(\varepsilon_n = 1; (\varepsilon_{n'})_{n' \neq n}) - E(\varepsilon_n = 0; (\varepsilon_{n'})_{n' \neq n}) \\ &= \int_0^1 (\partial_n E) d\varepsilon_n \\ &= \int_0^1 \int \xi_\varepsilon(x)^2 \varphi(x-n) dx d\varepsilon_n > 0 \end{aligned} \tag{3.2}$$

There are 2 issues $\begin{cases} \text{upperbounds} \\ \text{lower-bounds} \end{cases}$ for $|n| \to \infty$

Upperbounds are gotten from Green's function estimates, the usual way

$$|G^{(N)}(x_0, x)| < e^{-c|x|} \tag{3.3}$$

where say $x_0 \in B(0,1)$, assuming $\max_{|y|\leq 1} |\xi(y)| > c$, and $|x|$ 'large'. (we denote $G^{(N)}$ the Green's function corresponding to $H^{(N)}$, obtained by restriction of H to $[-N-\frac{1}{2}, N+\frac{1}{2}]^d$, imposing Dirichlet conditions).

In view of the preceding, we will need estimates on $G^{(N)}(x_0, x; \varepsilon)$ not only for $\varepsilon \in \{0,1\}^{(\mathbb{Z}\cap[-N,N])^d}$ but also allowing the range $\varepsilon_n \in [0,1]$ for a sufficiently large collection of sites $n \Rightarrow$ 'FREE SITES'.

The need for these free sites complicates considerably matters at the probabilistic level, when formulating Wegner estimates.

Since $I_n > 0, E(\varepsilon)$ is a monotone function on $\{0,1\}^{\mathbb{Z}^d}$.

Under suitable restriction of ε (outside set of 'small' measure), (3.3) holds (at a fixed energy E and hence for $|E' - E| < e^{-c_1 N}$). From (3.1), (3.3), $|\partial_n E'| < e^{-\frac{c}{10}N} < e^{-c_1 N}$ for $|n| > \frac{N}{10}$ and in particular

$$I_n < e^{-c|n|}$$

(ii) The main difficulty is to obtain a lower bound on I_n.

In view of (3.2), this amounts to insuring certain lower bounds on

$$\max_{|x-n|\leq 1} |\xi(x)|, \quad H\xi = E\xi.$$

These lower bounds are independent of the randomness ε and obtained from *Carleman inequalities* (only available in the continuum case).

Clearly

$$|\Delta\xi| \leq C|\xi|. \tag{3.4}$$

Proposition. *(B-K). If $\xi(0) = 1$, $|\xi| \leq C$ and (3.4) then $\forall x, |x| > 10$*

$$\max_{|x-x'|\leq 1} |\xi(x')| > c' \exp(-c'(\log|x|)|x|^{4/3}). \tag{3.5}$$

We will discuss inequality (3.5) later.

It clearly yields the lower bound

$$I_n > e^{-c|n|^{4/3}\log|n|}. \tag{3.6}$$

(iii) A basic difficulty with the Bernoulli model is that, unlike in the case of a continuous site distribution, small probability of events can not be established by variation of V on a single site but requires many sites.

Our basic tool is the following immediate consequence of *Sperner's Lemma* on sets of incomparable elements of $\{0,1\}^M$.

Lemma. *Let $E = E(\varepsilon_1, \ldots, \varepsilon_M)$ be a function on $\{0,1\}^M$ such that for all $j = 1, \ldots, M$*

$$I_j > \kappa > 0.$$

Then, for all $E_0 \in \mathbb{R}$

$$\text{mes}_{\{0,1\}^M}\left[|E - E_0| < \frac{\kappa}{4}\right] < M^{-1/2}. \tag{3.7}$$

4. The Wegner Inequality.

The main statement in our analysis is the following

Proposition A. *Denote $\Lambda_\ell \subset \mathbb{R}^d$ an ℓ-cube.*

There is a subset $\Omega \subset \{0,1\}^{\Lambda\cap\mathbb{Z}^d}$ s.t.

$$|\Omega| > 1 - \ell^{-\rho} \tag{4.1}$$

(ρ any number $< \frac{3}{8}d$) such that for $\varepsilon \in \Omega$, the resolvent R_Λ (E fixed) satisfies

$$\|R_\Lambda\| < e^{\ell^{1-}} \tag{4.2}$$

$$\|\chi_x R_\Lambda \chi_{x'}\| < e^{-c\ell} \text{ for } |x - x'| > \frac{\ell}{10}. \tag{4.3}$$

Comments.

(i) The estimate (4.1) gives a weaker estimate on the exceptional set than what is usually appearing in a multi-scale analysis. The recent work of Klein and Germinet however does process in particular a statement as in Prop. A to establish the statement on A.L.

(ii) The usual Wegner estimate refers only to the resolvent bound (4.2) and is gotten from first order eigenvalue variation *without* relying on multiscale arguments. In the [B-K] paper, we were unable to prove a Wegner estimate directly and had to rely on a multi-scale process involving both conditions (4.2), (4.3) already at this stage.

(iii) Due to the need of 'free sites' as discussed earlier, Proposition A is actually replaced by a more technical statement verified by induction on the scale ℓ. A more precise description of the set Ω is needed and we assume Ω to be a disjoint union of 'cylinders' of the form

$$C = \{0,1\}^S \times \prod_{n \in \Lambda \setminus S} \{\varepsilon_n\}$$

where $S \subset \Lambda \cap \mathbb{Z}^d$ and $(\varepsilon_n)_{n \in \Lambda \setminus S} \in \{0,1\}^{\Lambda \setminus S}$ depend on C. The set S will then provide 'free sites' and estimates (4.2), (4.3) remains true extending the range $\varepsilon_n \in [0,1]$ for $n \in S$.

(iv) (As usual), one verifies Proposition A first at an initial large scale ℓ_0. The argument is perturbative and uses the assumption $E \approx 0$ (it is only used here).

(v) A rough check of the estimates leading to Proposition A and the consistency of the statement displays a remarkable numerology!

First, at scale ℓ, we need to choose κ in (3.7) with

$$\kappa > e^{-\ell^{1-}}. \tag{4.4}$$

By (3.6), this means that our free sites n need to satisfy

$$|n|^\sigma < \ell^{1-}$$

hence

$$|n| < \ell^{\frac{1}{\sigma}-} \tag{4.5}$$

where σ is the exponent in (3.6),

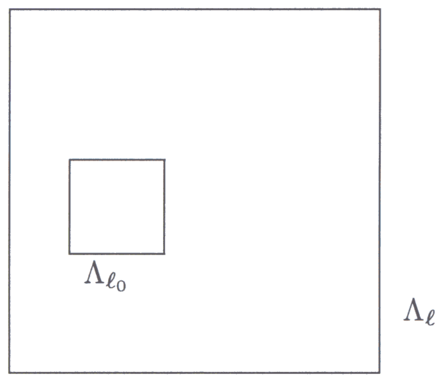
$$\sigma = 4/3.$$

In the range (4.5), the number of 'free sites' is obviously at most $\ell^{\frac{d}{\sigma}}$ (in dimension d), so that the best probabilistic estimate one may hope for in (3.7) is

$$\text{mes}\,[|E - E_0|] < e^{-\ell^{1-}}| < \ell^{-\frac{d}{2\sigma}+} < \ell^{-\frac{3}{8}d+}. \tag{4.6}$$

This justifies the exponent ρ in Proposition A.

We verify the consistency in the multi-scale analysis



Assume Proposition A (and strengthening) available at scale $\ell_0 = \ell^\alpha, \alpha < 1$. Cover Λ_ℓ appropriately with $\sim (\frac{\ell}{\ell_0})^d$ ℓ_0-cubes. The probability for R_{Λ_0} *not* to satisfy the estimates (4.2), (4.3) is, by (4.6), at most

$$\ell_0^{-\frac{d}{2\sigma}} < \ell^{-\frac{\alpha}{2\sigma}d}. \tag{4.7}$$

Our aim is to ensure that in our covering of Λ_ℓ, there are at most a *bounded* (by some fixed constant) 'bad' ℓ_0-boxes. This may clearly be insured provided for some $\delta > 0$

$$\ell_0^{-\frac{d}{2\sigma}} \left(\frac{\ell}{\ell_0}\right)^d < \ell^{-\delta} \tag{4.8}$$

which forces the condition

$$\ell_0 = \ell^\alpha \text{ with } \alpha\left(1 + \frac{1}{2\sigma}\right) > 1 \tag{4.9}$$

(a lower bound on ℓ_0).

Let $\Lambda_{\ell_0} \subset \Lambda_\ell$ be a bad box. Consider then a neighborhood

$$\Lambda_{\ell_0} \subset \Lambda_{\ell_1} \subset \Lambda_\ell \text{ with } \ell_1 \sim \ell$$

and establish a bound on R_{Λ_1} of the form

$$\|R_{\Lambda_1}\| < e^{\ell_1^{1-}}$$

using the eigenvalue variation argument described earlier. In view of condition (4.5) we need to satisfy

$$\ell_0 < \ell_1^{\frac{1}{\sigma}-} \tag{4.10}$$

hence

$$\alpha < \frac{1}{\sigma}. \tag{4.11}$$

By (4.9), (4.11), we conclude that σ must satisfy

$$\frac{1}{\sigma}\left(1+\frac{1}{2\sigma}\right) > 1$$

i.e.

$$1,333 = \frac{4}{3} = \sigma < \frac{1+\sqrt{3}}{2} \approx 1.36. \tag{4.12}$$

5. Carleman inequalities.

Inequality (3.5) results from the following Carleman type inequality.

Proposition. *There are constants C_1, C_2, C_3 depending only on d (the dimension $d \geq 2$ arbitrary) and an increasing function $w = w(r)$ for $0 < r < 10$ such that*

$$\frac{1}{C_1} < \frac{w(r)}{r} < C_1$$

and for all functions $f \in C_0^\infty(B_{\frac{1}{10}} \backslash \{0\}), \alpha > C_2$, we have

$$\alpha^3 \int w^{-1-2\alpha} f^2 \leq C_3 \int w^{2-2\alpha} (\Delta f)^2. \tag{4.13}$$

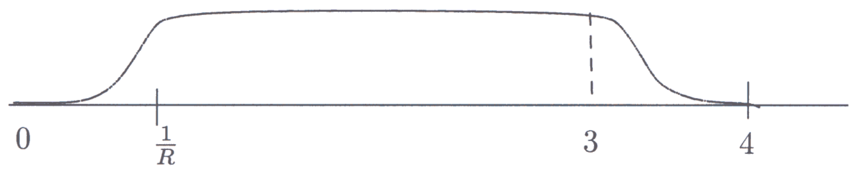
Essentially contained in [Escauriaza-Vessela, Contemp. Math. (2003)] and going back to [Hörmander, Comm. PDE (1983)].

Application of (4.13) to deduce (3.5) is standard. First rescale

$$u(x) = \xi(Rx)$$

and localize to complement of $\frac{1}{R}$-neighborhood of 0

$$f(x) = u(x)\psi(x)$$



$$\begin{aligned}\Delta f &= \Delta u.\psi + 0(|\nabla u| \cdot |\nabla \psi| + |u|\,|\Delta \psi|)\\ &= 0(R^2|f|) + 0(R^2|u| + |\Delta u|)\chi_{\frac{1}{2R}<|x|<\frac{2}{R}}) + 0(R)\end{aligned}$$

Contribution of first term in (4.13)

$$CR^4 \int w^{2-2\alpha} f^2$$

may be absorbed in left side of (4.13) taking

$$\alpha \sim R^{4/3}$$

$\Rightarrow \max_{|x|\leq \frac{1}{R}} [|u(x)| + |\nabla u|(x)] \gtrsim \left(\frac{1}{R}\right)^{\alpha} > e^{-CR^{4/3}\log R}$.

This gives (3.5).

Remarks.

(a) The exponent $\frac{4}{3}$ is the limitation of the Carleman method.

Proposition. *(V. Meshkov. Math. Sbornik (1991)).*

(i) Assume $|\Delta u| \leq C|u|$ *on* $\mathbb{R}^d$ *and* $|u(x)|$ *decays faster than*

$$\exp(-a|x|^{4/3}) \text{ for } |x| \to \infty, \forall a > 0.$$

Then $u \equiv 0$.

(ii) There is an example of <u>*complex*</u>*-valued* u *s.t.* $|\Delta u| \leq C|u|$ *on* $\mathbb{R}^2$, $u \neq 0$, *and*

$$|u(x)| \leq C\exp(-c|x|^{4/3}), \ \forall x \in \mathbb{R}^2.$$

Notice that the Carleman method does not distinguish between real and complex case. However, the following problem remains unanswered.

Problem. *Assume u* <u>*real*</u>*-valued and* $|\Delta u| \le C|u|$ *on* $\mathbb{R}^d$. *What may be said about*

$$\underline{\lim}_{|x|\to\infty} \frac{\log\log\frac{1}{|u(x)|}}{\log|x|} \quad \textit{and} \quad \overline{\lim}_{|x|\to\infty} \min_{|x-x'|\le 1} \frac{\log\log\frac{1}{|u(x')|}}{\log|x|}?$$

(b) Is there a discrete analogue of the Theorem for the lattice model

$$H_\varepsilon = \Delta + \lambda \sum_{n\in\mathbb{Z}^d} \varepsilon_n \delta_{nn'} \qquad (d \ge 2)?$$

No Carleman-type inequalities for the discrete Laplacian seem known and there is little understanding of 'unique continuation' on the lattice, when $d \ge 2$ (the case $d = 1$ is immediate from the transfer-matrix formulation).

Part II. Decaying Random Potential

1. Consider lattice version

$$H = \Delta + \lambda \sum_{n\in\mathbb{Z}^d} (1+|n|)^{-\alpha} \omega_n \delta_{n,n'}$$

$\{\omega_n\}$ i.i.d's of mean 0.

$0 < \alpha$

<u>$d = 1$</u> Assume moreover ω_n bounded with absolutely continuous distribution

$\alpha < \frac{1}{2}$: (a.s.) dense pure point spectrum in $[-2, 2]$.

$\alpha > \frac{1}{2}$: (a.s.) a.c. spectrum in $[-2, 2]$.

$\alpha = \frac{1}{2}$: (a.s.) no a.c.-spectrum

region of purely singular continuous spectrum of fractional Hausdorff dimension for small λ

(Kiselev, Last, Simon; CMP (1998)).

<u>$d > 1$</u> **Conjecture**

$\forall\alpha > 0$: $[-2d, 2d] \subset$ a.c. spectrum

Open

2. <u>$d = 2$</u>

Proposition 1. *(B, DCDS (2002)). For $\alpha > \frac{1}{2}$, $[-4,4] \subset$ a.c. spectrum (a.s.)*

Proposition 2. *(idem)*

Fix any $\tau > 0$ and denote $I = \{E \in [-4,4] | \tau < |E| < 4 - \tau\}$

Then, a.s., the wave operator

$$W(H,\Delta)E_0(I) = \lim_{t\to\infty} e^{-itH} e^{it\Delta} E_0(I)$$

and generalized wave operator

$$W(\Delta, H)E(I) = \lim_{t\to\infty} e^{-it\Delta} e^{itH} E(I)$$

exist and are complete.

Remark. Restriction $|E| > \tau$ possibly unnecessary.

Remark. For $\alpha < \frac{1}{2}$, establishing wave operators will likely require renormalization, where Δ is replaced by $\tilde{\Delta} = \Delta +$ deterministic potential.

3. Proof based on estimates of the Born series expansion of resolvent

$$R(z) = \sum_{s\geq 0} (-1)^s [R_0(z)V]^s R_0(z)$$

$H = \Delta + V, R_0(z) = (\Delta - z)^{-1}$ and $R(z) = (H - z)^{-1}$.

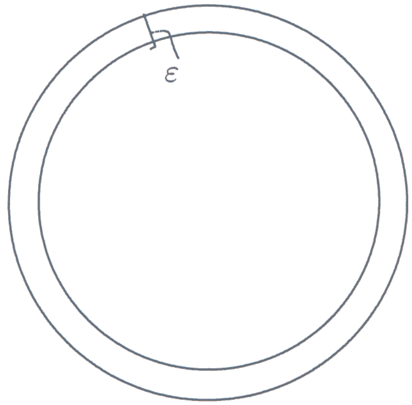
Key estimate in Analysis due to T. Wolff (see Schlag-Shubin-Wolff, J. Analyse (2002)).

Consider random Fourier multiplier

$$(M_\omega \varphi)(\theta) = \sum_{\substack{n\in\mathbb{Z}^2 \\ |n|<N}} \omega_n \hat{\varphi}(n) e^{2\pi i n.\theta} \tag{3.1}$$

and restriction operators R_ε

$$R_\varepsilon \varphi = \varphi \chi_{[1-\varepsilon<|x|<1]}$$



Then

$$\mathbb{E}_\omega \|R_\varepsilon M_\varepsilon R_\delta\| \lesssim (\sqrt{\varepsilon} + \sqrt{\delta}) \log N. \tag{3.2}$$

Remarks.

(i) (3.2) holds in arbitrary dimension

(ii) Used in [S-S-V] to show that in $d = 2$ the localization length for $H_\lambda = \Delta + \lambda \sum_{n\in\mathbb{Z}^d} \omega_n \delta_{nn'}$ is at least λ^{-2+}.

4. In view of decay properties of the free-resolvent, relaxation of the decay condition on the potential is expected to be easier in higher dimension.

In dimension d, rough behaviour of free resolvent

$$\frac{\cos \gamma |n|}{|n|^{\frac{d-1}{2}}} \text{ away from the edges} \tag{4.1}$$

$$\frac{1}{|n|^{d-2}} \text{ at the edge.} \tag{4.2}$$

Exploiting (4.2), the following was shown in [B], LNM 1807.

Proposition. *Let $d \geq 5$. Consider*

$$H = H_\omega = \Delta + \lambda \sum_{n\in\mathbb{Z}^d} (1 + |n|)^{-\alpha} \omega_n \delta_{nn'}$$

where $\alpha > \frac{1}{3}$ (and likely may be taken arbitrary small by elaboration of the argument). Then a.s., H has a proper extended state.

Remark. Condition $d \geq 5$ ensures tht $\int_{\mathbb{R}^d} \frac{1}{(1+|x|)^{2(d-2)}} < \infty$. To control the resolvent by a Born series, we do rely on renormalization. It consists in removing the divergent graphs by introducing appropriate corrections of Δ that appears as a (smooth) deterministic potential of faster decay ($|V_n| = 0(|n|^{-2\alpha})$). Possibly, for smaller α, these corrections may be non-diagonal. The general idea is close to an unpublished work of T. Spencer 'Lipschitz tails and localization'.