NONLINEAR RANDOM SCHRODINGER EQUATIONS

Nonlinear lattice Schrodinger equation

$$d = 1$$

$$i\dot{q}_j + V_j q_j + \varepsilon(q_{j-1} + q_{j+1}) + \delta q_j |q_j|^2 = 0$$

Frohlich-Spencer-Wayne (J.Stat.Ph., 1986)

Problems

(i) Diffusion problem

Growth of $D(t) = \left(\sum j^2 |q_j(t)|^2\right)^{1/2}$ **for** $t \to \infty$**?**

(ii) Nonlinear localization?

(iii) Construction of almost periodic and quasi-periodic solutions

Recent (W. Wang + B): general theory of quasi-periodic solutions for ε, δ **small (in arbitrary dimension), (to appear in CMP).**

More general finite-range models

$$H(q,\bar q) = \sum V_j |q_j|^2 + \varepsilon \sum_j (\bar q_j q_{j+1} + q_j \bar q_{j+1})$$

$$\mathrm{Re} \sum_j \lambda_j \prod_{k \in S_j} q_k^{n_k} \bar q_k^{n'_k}$$

where

$$S_j \subset [j-C, j+C],$$

$$\sum n_k = \sum n'_k, 4 \le \sum (n_k + n'_k) < C$$

(systems of coupled harmonic oscillators)

Theorem: (F-S-W) $(\varepsilon = 0)$

Existence of invariant tori of 'FULL' dimension with action variables of decay

$$I_j < e^{-|j|^\alpha} \qquad (\alpha > d = 1)$$

J. Poschel (CMP, 1990)

Milder decay assumptions

$$I_j < e^{-(\log |j|)^\alpha} \qquad (\alpha > 1)$$

CASE $\varepsilon \neq 0$

Underlying linear operator $H = V + \varepsilon\Delta$

Problems

- **Diophantine properties of eigenvalues**
- **Need to consider more general non-linearities**

Quasi-periodic solutions

$\lambda = (\lambda_1, \dots, \lambda_b)$ **tangential frequencies**

$\{\mu_\alpha\}$ **normal frequencies**

1^e **Melnikov conditions** $k.\lambda + \mu_\alpha \neq 0$ **(+ lower bounds)**

2^e **Melnikov conditions** $k\lambda + \mu_\alpha - \mu_\beta \neq 0 (\alpha \neq \beta)$

Improved KAM technology requires only 1^e **Melnikov condition (invariant tori may have positive Lyapounov exponents)**

At each step of iteration, need to control Green's function of (non-diagonal) linearized operator on $\mathbb{Z}^{b+d}$ with mixed random and quasi-periodic features

$$
\underbrace{(k.\lambda + V_j)\delta_{(k,j),(k',j')}}_{\text{diagonal}} + \varepsilon\Delta + \underbrace{\delta S}_{\substack{\text{Toeplitz in } k \\ \text{rapid decay in } j}}
$$

Use of methods from B, Goldstein, Schlag (Acta Math. 2002)

Subharmonic function technique

Semi-algebraic sets

APPLICATION

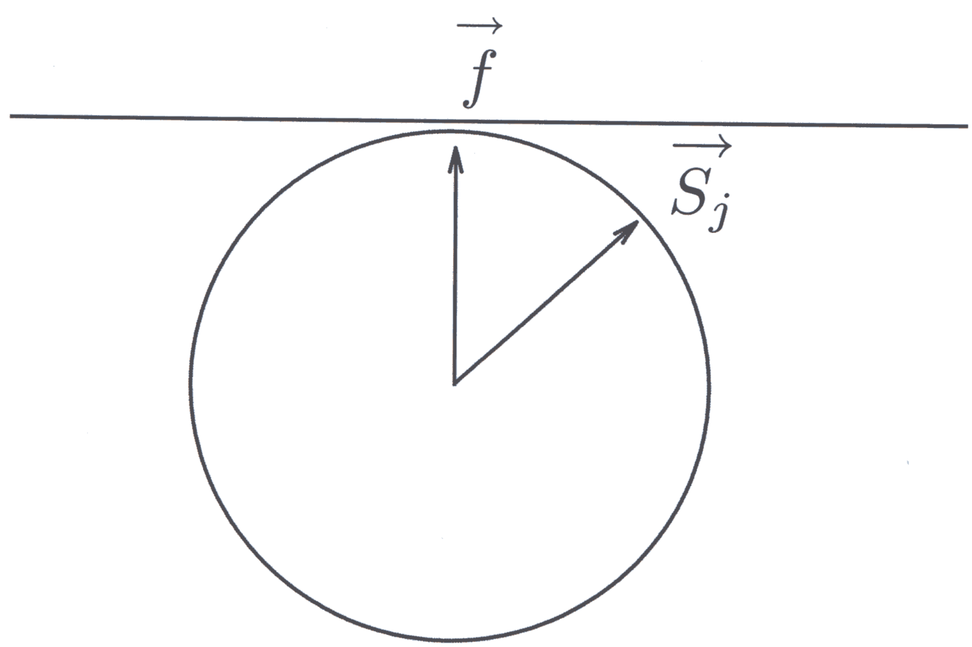
Construction of quasi-periodic solutions of Landau-Lipschitz equations for nonlinear classical spin waves

$$\dot{S}_j = S_j \wedge \left((\Delta S)_j + h_j \vec{f}\right)$$

$j \in \mathbb{Z}^d$, (h_j) **large random**

$$S_j = \vec{S}_j \in \mathbb{R}^3 \qquad |S_j| = 1$$

$\vec{S}_j$ **appears as quasi-periodic perturbation of** $\vec{f}$



DIFFUSION PROBLEM ($d = 1$)

$$
\begin{aligned}
H(q,\bar{q}) = & \sum_j V_j |q_j|^2 + \\
& \varepsilon \sum_j (\bar{q}_j q_{j+1} + q_j \bar{q}_{j+1}) + \\
& \operatorname{Re} \sum_j \lambda_j \prod_{k \in S_j} q_k^{n_k} \bar{q}_k^{n'_k}
\end{aligned}
$$

where

$$S_j \subset [j - C, j + C]$$

$$\sum n_k = \sum n'_k ; 4 \leq \sum (n_k + n'_k) < C.$$

Linear Case: $\lambda_j = 0$

Anderson and dynamical localization

$$\sup_t D(t) < \infty.$$

Theorem. Assume $|\lambda_j| < \varepsilon |j|^{-\tau}$ **and** $\varepsilon < \varepsilon(\tau, \kappa)$ **then**

$$D(t) < t^{\kappa} \textbf{ for } t \to \infty.$$

Use of Nekhoroshev type methods.

Problem of construction of invariant tori of full dimension under explicit/realistic decay conditions on action variables

NLW $y_{tt} - y_{xx} + Vy + 0(y^3) = 0$ **(B; GAFA, 1996)**

NLS $iu_t + u_{xx} + Vu + \varepsilon W(Wu|Wu|^2) = 0$

(Poschel; EDS, 2002)

1D (Dirichlet boundary conditions)

Typical potential V

Dirichlet Spectrum

$$\mu_n(V) = \pi^2 n^2 + c_0(V) + c_1(V)\frac{1}{n} + \cdots + C_R(V)\frac{1}{n^R} + 0(n^{-R-1})$$

W **= smoothing operator**

Method: Consecutive perturbations of finite dimensional tori

$\Rightarrow$ **Very fast (non-explicit) decay of action-variables**

REALISTIC DECAY CONDITION

Teorem: *1D NLS with random multipliers*

$$iu_t + u_{xx} + Mu + \varepsilon \frac{\partial}{\partial \bar{u}} P(|u|^2) = 0$$

$$Mu = \sum_{j \in \mathbb{Z}} V_j \ \ \hat{u}(j) e^{2\pi i j x}$$

$V_j \in [-1, 1]$ *independent parameters*

$\Rightarrow$ invariant tori of full dimension with action variables

$$I_n \sim e^{-|n|^\alpha} \qquad\qquad (0 < \alpha < 1)$$

monomials in nonlinearity of the form

$$q_{n_1} \bar{q}_{n_2} q_{n_3} \bar{q}_{n_4} \cdots q_{n_{2s-1}} \bar{q}_{n_{2s}}$$

not short range but satisfying

$$n_1 - n_2 + n_3 - n_4 \cdots + n_{2s-1} - n_{2s} = 0$$

and, if resonant

$$n_2^2 - n_2^2 + n_3^2 \cdots - n_{2s}^2 = 0$$

$n_1^* \geq n_2^* \geq \cdots$ **decreasing rearrangement of modes in monomial**

$\Rightarrow \; n_1^* \leq n_2^* + n_3^* + \cdots$

also

$$\sum \sqrt{|n_j|} \geq 2\sqrt{n_1^*} + \frac{1}{4}(\sqrt{n_3^*} + \sqrt{n_4^*} + \cdots)$$

for resonant monomials with $n_1 \neq n_2$

$$|n_1| + |n_2| < 2\big((n_3^*)^2 + (n_4^*)^2 + \cdots\big)$$

normal forms expressed in $q_n, \bar{q}_n$ **series**

$$\sum_{\ell,k,k'} B_{\ell,k,k'} \prod_n |q_n(0)|^{2\ell_n} \, q_n^{k_n} \, \bar{q}_n^{k'_n}$$

$$|B_{\ell,k,k'}| < e^{\rho(\sum \sqrt{|n_j|} - 2\sqrt{n_1^*})}$$

NEKHOROSHEV STABILITY IN LARGE DIMENSION

LONG-TIME STABILITY (= NEAR QUASI-PERIODIC BEHAVIOUR) IN ε–PERTURBATIONS OF LINEAR OR INTEGRABLE HAMILTONIAN SYSTEMS

EXAMPLES

- **INFINITE SYSTEMS OF COUPLED HARMONIC OSCILLATORS**

  (BENETTIN–FROHLICH–GIORGILI)

$$T_\varepsilon \sim \exp\left\{ c \frac{(\log \frac{1}{\varepsilon})^2}{\log\log \frac{1}{\varepsilon}} \right\}$$

- $iu_t + u_{xx} + V(x)u + \varepsilon \frac{\partial H}{\partial \bar{u}} = 0$
- $y_{tt} - y_{xx} + \rho y + \varepsilon f'(y) = 0$
- $iu_t + u_{xx} \pm u|u|^2 + \varepsilon \frac{\partial H}{\partial \bar{u}} = 0$

'NON-TRIVIAL' STABILITY TIMES $T_\varepsilon \gg \frac{1}{\varepsilon}$

2 MECHANISMS

- **PERTURBATIONS OF LINEAR NONRESONANT SYSTEMS** (QUASI-NEKHOROSHEV)

EXAMPLE: $y_{tt} - y_{xx} + \rho y + \varepsilon f'(y) = 0$ **(DIRICHLET BC)**

IF $y(0), y'(0) \in H^s[0,1]$, **THEN** $y(t)$ **IS 'CLOSE' TO QUASI-PERIODIC FOR** $t < (\frac{1}{\varepsilon})^A$

$$A = A(s) \underset{s \to \infty}{\longrightarrow} \infty$$

•• **PERTURBATIONS OF STRICTLY CONVEX INTEGRABLE HAMILTONIAN**

ADJUSTMENT OF P. LOCHAK'S ARGUMENT BASED ON RATIONAL APPROXIMATIONS

REQUIRES STRONG FINITE DIMENSIONAL APPROXIMATIONS /

COMBINATION OF THE TWO MECHANISMS

Example: (1D finite range)

$$H(q,\bar{q}) = \sum_j (V_j I_j + I_j^2) + \varepsilon \sum_{n,n' \in \mathbb{Z}^\infty} c(n,n') \mathrm{Re}\left[\prod q_j^{n_j} \bar{q}_j^{n'_j}\right]$$

$$\sum n_j = \sum n'_j$$

$$2 \leq \sum (n_j + n'_j) < C$$

$$\mathrm{diam}\,(\mathrm{supp}\, n \cup \mathrm{supp}\, n') < C \qquad \mathrm{supp}\, n = \{j \in \mathbb{Z} | n_j \neq 0\}$$

Theorem: *(with large probability in V)*

Let $\sigma > 0$ and $q(0) = q|_{t=0}$ satisfy

$$\|q\|_{H^\sigma} = \left(\sum |j|^{2\sigma} |q_j|^2\right)^{1/2} < C$$

Then

$$\sup_j |I_j(0) - I_j(t)| < \sqrt{\varepsilon} \text{ and } \|q(t)\|_{H^\sigma} < C + 1$$

for $t < T_\varepsilon, \log T_\varepsilon \gg \log \frac{1}{\varepsilon}$.

(Compare with Benettin-Frohlich-Giorgili, CMP (1989))

ROLE OF DIMENSION OF THE TORI

- **MARCO–SAUZIN**

$$|I(t) - I(0)| < \varepsilon^b \qquad |t| < \exp c\frac{1}{\varepsilon^a}$$

EXAMPLES IN GEVREY CLASS SHOWING THAT $a \to 0$ **FOR** $d \to \infty (a \sim \frac{1}{d})$.

- **CONSTRUCTION OF EXAMPLES WITH** $d \sim \log \frac{1}{\varepsilon}$ **AND**

$$T_\varepsilon \sim \frac{1}{\varepsilon} \quad \textbf{(GEVREY CLASS)}$$

$$T_\varepsilon \sim \frac{1}{\varepsilon^{1+\tau}} \quad \textbf{(ANALYTIC)}$$

$$H = \tfrac{1}{2}(I_0^2 + I_1^2 + \cdots + I_d^2) + \varepsilon \cos\theta_0 \sum_{k=1}^{K}$$

$$\sigma_k \Big[\varphi^2\Big(\tfrac{1}{d}\sum_{j=1}^{d}\cos(\theta_j - k\lambda_j)\Big) - 2\varphi\Big(\tfrac{1}{d}\sum_{j=1}^{d}\cos(\theta_j - k\lambda_j)\Big)v(t)\Big]$$

$v(t)$ **TIME PERIODIC** $\qquad |I_0(T_\varepsilon) - I_0(0)| > c$

$\lambda_j \in [0,1], \sigma_k \in [-1,1], K \sim \frac{1}{\varepsilon}$

$$d \sim \log \frac{1}{\varepsilon} \Rightarrow T_\varepsilon \sim \frac{1}{\varepsilon}$$

optimal in real analytic category.

PROBLEM.

Improve T_ε if perturbation is short range or polynomial.

CONJECTURE. (topological) genericity (in smooth and analytic category) of diffusion phenomenon in infinite-dimensional Hamiltonian systems.