Hamiltonian Theory of the General Rational Isomonodromic Deformation Problem

(Workshop on integrable & near-integrable Hamiltonian PDE's, Fields Institute, Toronto, May 17-21, 2004)

I. Rational isomonodromic deformations

Rational covariant derivative operator on over $\mathbf{CP}^1$:

$$\mathcal{D}_x = \frac{\partial}{\partial x} - A(x), \quad A(x) = \sum_{r=0}^{n} A_r(x)$$

$$A_r(x) = \sum_{j=1}^{d_r} \frac{A_{rj}}{(x-\alpha_r)^{j+1}}, \quad r = 1, \dots n$$

$$A_0(x) = \sum_{j=1}^{d_r} A_{0j}(x^{j-1}), \quad A_{0j}, A_{rj} \in \mathfrak{sl}(m, C)$$

(Assume also leading term singularity spectrum is simple.)

$$\lambda_{ra} \neq \lambda_{rb} \quad \text{if} \quad a \neq b, \quad \{\lambda_{ra}\} = \text{spec}\{A_{r,d_r}\}$$

Fundamental system

$$\mathcal{D}_x \Psi(x) = 0, \quad \Psi(x) \in \mathfrak{Sl}(M, C)$$

Formal asymptotics, generalized monodromy data:

1. Monodromy representation:

$$M : \pi_1(\mathbf{CP}^1/\{\infty, \alpha_1, \dots \alpha_n\}) \to \mathfrak{Sl}(n, C)$$

$$M : \gamma_r \mapsto M_r := M(\gamma_r)$$

$$\gamma : \Psi \mapsto \Psi M(\gamma)$$

2. Stokes matrices (irregular singular points)

$$\Psi_{r,j+1}(x) = \Psi_{rj}(x) S_{rj}, \quad x \sim \alpha_r, \quad j = 1, \dots 2d_r$$

3. Connection matrices

$$C\,\cancel{T}_i : \Psi_\infty \mapsto \Psi_r$$

4. Local formal asymptotics:

$$\Psi_r \underset{x \sim \alpha_r}{\sim} Y_r(x) e^{T_r(x)}, \quad r = 0, \dots n$$

$$Y_r(x) = C_r \left(\mathbf{I} + \mathcal{O}(x - \alpha_r) \right), \quad r = 1, \dots n$$

$$T_r(x) = \sum_{a=1}^{m} \left(\sum_{j=1}^{d_r} \frac{t^a_{rj}}{(x - \alpha_r)^j} + t^a_{r0} \ln(x - \alpha_r) \right) E_a$$

$$Y_0(x) = C_0 \left(\mathbf{I} + \mathcal{O}(x^{-1}) \right)$$

$$T_0(x) = \sum_{a=1}^{m} \left(\sum_{j=1}^{d_0} t^a_{0j} x^j + t^a_{00} \ln(x) \right) E_a$$

$$(E_a)_{bc} := \delta_{ab} \delta_{ac}$$

Deformations preserving generalized monodromy

Question: Allowing $\{\alpha_r, t^a_{rj}\}$ to vary, what are the most general differentiable deformations in A(x) that preserve $\{M_r, S_{rj}, T_r\}$?

Answer: (Schlesinger (1905) - Jimbo, Miwa, Ueno (1981)) The necessary and sufficient condition is the compatibility of the overdetermined system:

$$\mathcal{D}_x \Psi = 0, \qquad \frac{\partial}{\partial \alpha_r}\Psi = \Omega_r \Psi, \qquad \frac{\partial}{\partial \alpha_{t^a_{rj}}}\Psi = \Omega^a_{rj}\Psi,$$

where

$$\Omega^a_{rj} = \left(Y_r \frac{\partial T_r}{\partial t^a_{rj}} Y_r^{-1} \right)_{\mathcal{P}_{\alpha_r}}, \quad \Omega_r = \left(Y_r \frac{\partial T_r}{\partial a_r} Y_r^{-1} \right)_{\mathcal{P}_{\alpha_r}}$$

Equivalently,

$$\widetilde{d}\Psi = \widetilde{\Omega}\Psi$$

$$\widetilde{\Omega} := A(x)dx + \Omega, \quad \widetilde{d} := dx \frac{\partial}{\partial x}$$

$$d := \sum_{j=0}^{n} \left(\sum_{r=1}^{d_r} \sum_{a=1}^{m} dt^a_{rj} \frac{\partial}{\partial t^a_{rj}} + d\alpha_r \frac{\partial}{\partial \alpha_r} \right)$$

$$\Omega := \sum_{j=0}^{n} \left(\sum_{r=1}^{d_r} \sum_{a=1}^{m} \Omega^a_{rj} + d\Omega_r \right)$$

Compatibility ("zero-curvature") condition:

$$\widetilde{d}\widetilde{\Omega} + \frac{1}{2}\left[\widetilde{\Omega}, \widetilde{\Omega}\right] = 0$$

Isomonodromic tau function

Define a 1-form on the space of deformation parameters $\{\alpha_r, t^a_{rj}\}$

$$\omega := \sum_{j=0}^{n} \left(\sum_{r=1}^{d_r} \sum_{a=1}^{m} dt^a_{rj} h^a_{rj} + d\alpha_r h_r \right)$$

where

$$h^a_{rj} := \operatorname*{res}_{x=\alpha_r} \operatorname{tr} \left(Y_r^{-1} \frac{\partial Y_r}{\partial x} \frac{\partial T_r}{\partial t^a_{rj}} \right)$$

$$h_r := \operatorname*{res}_{x=\alpha_r} \operatorname{tr} \left(Y_r^{-1} \frac{\partial Y_r}{\partial x} \frac{\partial T_r}{\partial \alpha_r} \right)$$

If the zero curvature equations are satisfied, this is a closed 1−form ([JMU, 1981]):

$$d\omega = 0$$

Therefore, there exists, at least locally, a function τ^{IM} determined up to a parameter independent normalization factor by:

$$d \ln(\tau^{IM}) = \omega.$$

(Actually, τ^{IM} is globally defined, away from collision points of the poles and resonances, as a section of a line bundle, and is holomorphic where defined [Miwa, 1981].)

2. Examples

1. "Schlesinger equations" (Fuchsian systems): all $d_r = 0,\ d_0 = 0$

$$A(x) = \sum_{j=1}^{n} \frac{A_r}{x-\alpha_r}, \qquad \Omega_r = -\frac{A_r}{x-\alpha_r},$$
$$\frac{\partial A_r}{\partial \alpha_r} = \frac{[A_r, A_s]}{\alpha_r - \alpha_s}, \qquad r \neq s,$$
$$A_\infty := -\sum_{r=1}^{n} A_r = \text{cst.}$$

Moreover $h_r = \frac{\partial \ln(\tau^{IM})}{\partial \alpha_r} = \frac{1}{2}\,\mathrm{res}_{x=\alpha_r} \operatorname{tr}(A^2(x))|_{\mathbf{int}}$

In particular, for $m = 2$, $n = 3$, set $\alpha_1 = 0,\ \alpha_2 = 1,\ \alpha_3 = t$ and eigenvalues$(A_r) = \pm\theta_r$, $\quad A_\infty = \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}$,
this reduces to Painlevé P_{VI}:

$$\begin{aligned}\frac{d^2u}{dt^2} &= \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)\left(\frac{du}{dt}\right)^2 \\ &\quad - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)\frac{du}{dt} \\ &\quad + \frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{u^2} + \gamma\frac{t-1}{(u-1)^2} + \delta\frac{t(t-1)}{(u-t)^2}\right),\end{aligned}$$

where

$$\alpha = (\theta_\infty + \frac{1}{2})^2, \quad \beta = -\frac{1}{2}\theta_1^2, \quad \gamma = \frac{1}{2}\theta_2^2, \quad \delta = -\frac{1}{2}\theta_3^2 + \frac{1}{2}.$$

Hamiltonian structure of Schlesinger equations

Lie Poisson structure on $[\mathfrak{sl}^*(m, \mathbf{C})]^n$

$$(A_1, \ldots, A_n) \in [\mathfrak{sl}^*(m, \mathbf{C})]^n$$

$$\{f,\ g\} = \sum_{r=1}^{n} \mathrm{tr}\left(A_i, \left[\frac{\partial f}{\partial A_i}, \frac{\partial g}{\partial A_i}\right]\right)$$

Commutative Hamiltonians (non-autonomous):

"time variables" = location of poles $\{\alpha_r\}$

$$H_r := \frac{1}{2} \operatorname*{res}_{x=\alpha_r} \mathrm{tr}(A^2(x)),$$

$$\{H_r,\ H_s\} = 0, \quad r, s = 1, \ldots, n$$

Hamiltonian vector field:

$$X_{H_r}(A) = \{A(x), H_r\} = [\Omega_r, A], \quad \Omega_r = \frac{A_r}{x - a_r}$$

Total "derivative":

$$\widetilde{X}_{H_r} = X_{H_r} + X_r^0,$$

$$X_r^0 := \frac{\partial^0}{\partial \alpha_r}, \qquad \text{"explicit derivative''}$$

Since

$$X_r^0(A) = -\frac{A_r}{(x - \alpha_r)^2} = \frac{\partial \Omega_r}{\partial x},$$

Hamilton's equations $\equiv$ Schlesinger $\equiv$ zero curvature:

$$\frac{\partial A}{\partial x} = \{A, H_r\} + X_r^0(A) = [\Omega_r, A] + \frac{\partial \Omega_r}{\partial x}$$

2. Second order pole at ∞: all $d_r = 0, d_0 = 1$
[JMU (1981), JH (1994)]

$$A(x) = T_0 + \sum_{j=1}^{n} \frac{A_r}{x - \alpha_r},$$

$$T_0 := \mathrm{diag}(t_1, \ldots, t_m)$$

$$A_\infty := -\sum_{r=1}^{n} A_r$$

$$\Omega = \sum_{r=1}^{n} \Omega_r d\alpha_r + \sum_{a=1}^{m} \Omega_0^a dt_a$$

$$\Omega_r = -\frac{A_r}{x - \alpha_r},$$

$$\Omega_0^a = xE_a + \sum_{b \neq a}^{n} \frac{E_a A_\infty E_b + E_b A_\infty E_a}{t_a - t_b}$$

τ function and Hamiltonians:

$$d\ln(\tau^{IM}) = \sum_{r=1}^{n} h_r d\alpha_r + \sum_{a=1}^{n} k_a dt_a$$

$$h_r = H_r|_{\mathbf{Int}}, \quad k_a = K_a|_{\mathbf{Int}}$$

where

$$H_r := \frac{1}{2} \operatorname*{res}_{x=\alpha_r} \mathrm{tr} A^2$$

$$K_a := \frac{1}{2} \operatorname*{res}_{x=\infty} \operatorname*{res}_{y=t_a} \mathrm{tr} \left(A(x)(T_0 - y\mathbf{I})^{-1} - \mathbf{I}\right)^2$$

3. General case

Poisson structure: Rational R-matrix structure

$$\{A(\xi) \overset{\otimes}{,} A(y)\} := [r(x-y),\ A(x) \otimes \mathbf{I} + \mathbf{I} \otimes A(y)].$$

where $A(x),\ A(y) \in \mathrm{End}(\mathbf{C^m})$

$$r(x-y) := \frac{P_{12}}{x-a} \in End(\mathbf{C^m} \otimes \mathbf{C^m}), \quad \mathbf{P_{12}(u \otimes v) := v \otimes u}$$

(Equivalent to $\oplus_{r=0}^{n} \left[\mathfrak{sl}^{d_r}(m, C)\right]^*$.)

$$\{A(x)\} \subset (L\mathfrak{sl}(m,C)^*)_R = \mathfrak{sl}_+(m,C) \ominus \mathfrak{sl}_+(m,C)$$

Classical split, rational R-matrix: $R := \frac{1}{2}P_+ - P_-$

$$[X,\ Y] = [RX,\ Y] + [X,\ RY] = [X_+,\ Y_+] - [X_-,\ Y_-]$$
$$X_\pm := P_\pm(X), \quad Y_\pm := P_\pm(Y), \quad (P_\pm = \text{projectors})$$

Casimir invariants: $\{t^a_{rj}, a_r\}$ (centre of Poisson algebra)
Spectral curve: $\det(y\mathbf{I} - A(x)) = 0$ (M-sheeted branched cover of CP^1).
Hamiltonian vector field for: $h \in \mathcal{I}\,(L\mathfrak{sl}(m,C))$ (spectral invariant)

$$X_h(A) = [R\,(\delta(H))\,,\ A]$$

Theorem 1.

1) $t^a_{rj} = \mathrm{res}_{y_a(x=\alpha_r)}(x-\alpha_r)^{j-1} y dx, \quad j = 1, \dots d_r$

2) $h^a_{rj} := \frac{\partial \ln \tau^{IM}}{\partial t^a_{rj}} = \mathrm{res}_{y_a(x=\alpha_r)}(x-\alpha_r)^{-j} y dx, \quad j = 1, \dots d_r$

$h_r := \frac{\partial \ln \tau^{IM}}{\partial \alpha_r} = \mathrm{res}_{y_a(x=\alpha_r)}(x-\alpha_r)^{-j} y dx = \frac{1}{2}\mathrm{res}_{x=\alpha_r} \mathrm{tr} A^2(x)$

3) $\Omega^a_{rj} = (\delta h^a_{rj})_{\mathcal{P}_{\alpha_r}}, \qquad \Omega_r = (\delta h_r)_{\mathcal{P}_{\alpha_r}}$

4) $X^0_{t^a_{rj}}(A) = \frac{\partial \Omega^a_{rj}}{\partial x}, \qquad X^0_{\alpha_r}(A) = \frac{\partial \Omega_r}{\partial x}$

Theorem 2.

Trace formula: $\quad h^a_{rj} = H^a_{rj}|_{\mathbf{Int}}$ where

$$H^a_{rj} = \frac{1}{2} \mathop{\mathrm{res}}_{x=\alpha_r} \mathop{\mathrm{res}}_{y=t^a_{r,d_r}} (x-\alpha_r)^{-(d_r+j+1)} \times$$

$$\times \sum_{p=1}^{d_r+j} \frac{1}{p} \mathrm{tr}\left(\left[(x-\alpha_r)^{d_r+1}A(x) - T_r\right](T_r - y\mathbf{I})^{-1}\right]^p$$

References

[BHHP] M. Bertola, J. Harnad, J. Hurtubise and G. Pusztai, "Hamiltonian structure of the general rational isomonodromic deformation problem", preprint CRM (2004. in preparation)

[BEH] M. Bertola, B. Eynard, J. Harnad,"Matrix models, orthogonal polynomials with semiclassical measures and isomonodromic tau functions", preprint CRM Saclay-T04/019 (2004)

[JH] J. Harnad, "Dual Isomonodromic Deformations and moment maps into Loop Algebras", *Commun. Math. Phys.* **166**, 337-365 (1994).

[JMU] M. Jimbo, T. Miwa and K. Ueno, "Monodromy Preserving Deformation of Linear Ordinary Differential Equations with Rational Coefficients I.", *Physica* **2D**, 306-352 (1981).

[Miwa] T. Miwa "Painlevé property of monodromy preserving deformation equations and the analyticity of τ funcrtions", *Publ. Res. Inst. Math. Sci.* **17**, 703-721 (1981).

[W] N. Woodhouse, "The symplectic and twistor geometry of the general isomonodromic deformation problem", *J. Phys. Geom.* **39**, 97-128 (2001).