Workshop on Nonlinear Wave Equations

March 15-19, 2004. The Fields Institute. Toronto

NLS with Small Viscosity. Lower Bounds for the Space Derivatives of the Solutions.

Andrei Biryuk.

In the late nineties Kuksin [1] introduces a method for obtaining lower bounds for spatial derivatives of solutions of Schrödinger type equation with small "viscosity". The bounds are in terms of negative powers of viscosity. Here we reconstruct his ideas and use them to obtain slightly better estimates.

$$-i\dot{u} = e^{i\psi(\dots)}\nu\Delta u + |u|^{2p}\cdot u \qquad (1)$$

$u = u^\nu(t,x)$ $\quad x\in\Omega$, $\dim x = n$, $\dim t = 1$, $p \neq 0$

$\dim_{\mathbb{C}} u = 1$, $\dim_{\mathbb{R}} u = 2$, $0<\nu\ll 1$

$\psi = \psi(t,x,u,\dots,\omega,\dots)$ is real.

so $|e^{i\psi}| \equiv 1$. probability event

Want (I) $\exists\, \varkappa > 0$ s.t.

for any big m (i.e. $\forall m > M_0$)

for small positive ν

(i.e. $\nu < \nu_0$ or, may be, $\nu < \nu_m$)

we have: Also want (II)

$$\frac{c_m}{\nu^{m\varkappa}} \overset{*}{\lesssim} \left\| \frac{D^m u^\nu}{Dx^m} \right\| \leq \frac{C_m}{\nu^{m\varkappa_2}}.$$

need to assume that $e^{i\psi} \equiv -i$

here $\|\cdot\|$ denotes a functional norm.

<You may think about your favourite one.>

Of course the example $u(t,x)\equiv 0$ breaks want I. So we need some non-degeneracy condition.

For want I need to assume an appropriate boundary conditions. e.g. periodic or zero Dirichlet / Neumann cond.

Q: why we want this?

A1. It is interesting in itself.

A2. application to Turbulence.

§ Answer 2: Motivation by Turbulence

Define $\bar\varkappa = \sup\{\varkappa : \varkappa \text{ satisfies } *\}$.

This is the exponent of the space scale and

$\nu^{\bar\varkappa}$ is the space scale.

there is a formula by Kuksin:

$$\boxed{\bar\varkappa = \liminf_{m\to\infty}\ \liminf_{\nu\to 0} \frac{\log \left\| \frac{D^m u}{Dx^m} \right\|}{m \log \nu^{-1}}}$$

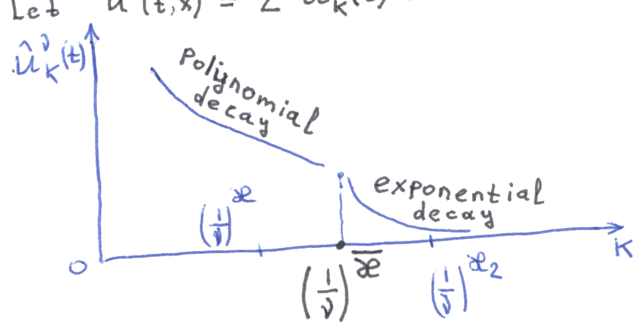
It is also remarkable that $\bar\varkappa$ does not depend on norm chosen. e.g. if you replace C^m norm with H^m norm the result will be the same. See Kuksin [1] to learn more about space scale.

Lower bounds for the derivatives = upper bounds for the space scale.

Kolmogorov-Obukhov type law.

Assume that the domain Ω is periodic. (torus)

Let $u^\nu(t,x) = \sum \hat{u}^\nu_k(t)\, e^{ikx}$



for eq. (1): $\frac{1}{3} \leq \bar\varkappa \leq \frac{1}{2}$.

See Kuksin [1] for more details.

End of § A2.

$$-i\dot{u} = e^{i\psi(\ldots)}\nu\Delta u + |u|^{2p}u \qquad (1)$$

If $e^{i\psi(\cdots)} \equiv a - bi$ with $b > 0$ and, say, we are considering eq.(1) on a bdd domain Ω with zero Dirichlet BC, then

$$|u(t,\cdot)|_{L^2} \leqslant \exp(-\lambda_1 b\nu t)|u(0,\cdot)|_{L^2}.$$

Here λ_1 is the first Dirichlet-eigenvalue for $(-\Delta)$ in Ω.

Hence the L^2 norm decays by a time $\gg \nu^{-1}$. Despite this fact, solutions of (1) develops a short scale (i.e. the derivatives become big) by the time $\sim \nu^{-1/3}$.

Theorem K3 (See [1, Thm 3]).
Consider a time-space cylinder $[0 \times \infty) \times \Omega$ where Ω is any open set.

Let $u = u^\nu(t,x)$ be any smooth complex-valued function on this cylinder that satisfies (1) in the interior.

No boundary conditions are assumed.

We assume a non-degeneracy cond.:

$$\boxed{\sup_{x\in\Omega} |u^\nu(0,x)| \leqslant C; \quad \operatorname{osc}_{\Omega} |u^\nu(0,\cdot)| \geq 1} \qquad (4.7)$$

where $\operatorname{osc}_{\Omega} |u^\nu(0,\cdot)| := \sup_{x,y\in\Omega} \big||u^\nu(0,x)| - |u^\nu(0,y)|\big|$.

Then $\forall \varepsilon > 0$ $\forall m \geq 2$ $\exists \nu_{\varepsilon,m}$ $\forall \nu < \nu_{\varepsilon,m}$

$$\boxed{\frac{1}{\nu^{(\frac13-\varepsilon)m}} \leqslant \left|\frac{D^m u^\nu(\ ,\ )}{Dx^m}\right|_{L^\infty([0;\nu^{-\frac13}]\times\Omega)}} \qquad (4.8)$$

Moreover $\exists$ $t_0 = t_0(\nu, u^\nu) \in [0; \nu^{-\frac13}]$ s.t. (4.8) happens at $t = t_0$ and

$$\operatorname{osc}_{\Omega} |u^\nu(t_0,\cdot)| \geq \frac12 \quad \text{for } \nu < \nu_{\varepsilon,m}.$$

<u>Remark 1.</u> $\nu_{\varepsilon,m}$ does not depend on $u(0,\cdot)$ [and $u(t,x)$] but does depend on "non-degeneracy parameters" C and 1 in (4.7)

<u>Remark 2</u> of course we can rescale all parametrs.
In particular we can replace the time interval $[0; \nu^{-\frac13}]$ with $[0; \text{Const}\,\nu^{-\frac13}]$ for any Const > 0.
$\nu_{\varepsilon,m}$ will be modified accordingly.

if Ω is a periodic domain (torus) then by $\|\cdot\|_m$ we denote the H^m Sobolev semi-norm i.e

$$\|u\|_m^2 = \int_\Omega ((-\Delta)^m u)\bar{u}\, dx \quad \text{for } m \in \mathbb{Z}_+$$

<u>Rmk</u> this formula also defines H^m-seminorm for $m \in \mathbb{R}$. Alternatively you can use Forier tranform to define H^m seminorm for real m.

Theorem K4 (See [1, Thm 4])
Let Ω be a periodic domain (torus)
Assume that $\|u^\nu(0,\cdot)\|_m \leqslant C_m$ $\forall m$
and $\operatorname{osc}_{\Omega} |u^\nu(0,\cdot)| \geq 1$.

Then $\forall m$ s.t. $\begin{pmatrix} m > \frac n2 + 2 \text{ and} \\ m \geq pn \end{pmatrix}$ $\forall \varepsilon > 0$
$\exists \nu_{m,\varepsilon}$ s.t. $\forall \nu < \nu_{m,\varepsilon}$ we have

$$\nu^{\frac13}\int_0^{\nu^{-\frac13}} \|u^\nu(t,\cdot)\|_m^2\, dt \geq \nu^{-\frac23[m-\frac n2-1-\varepsilon]\cdot\frac{m}{m+pn}}$$

Proof of Theorem K4 given Theorem K3.

$$\partial_t u = ie^{i\psi}\nu\Delta u + i|u|^{2p}u \tag{1'}$$

Multiplying (1′) by $\bar{u}$, integrating over period, and taking the real part we obtain $\frac{1}{2}\partial_t\|u\|_0^2 = -Re(ie^{i\psi}\nu)\|u\|_1^2 \leqslant 0$.
Similarly, multiplying (1′) by u in H^m we get

$$\tfrac{1}{2}\partial_t\|u\|_m^2 = -Re(ie^{i\psi}\nu)\|u\|_{m+1}^2 - Im\left\langle |u|^{2p}u, u\right\rangle_m \leqslant \left\| |u|^{2p}\, u\right\|_m \|u\|_m .$$

Hence $\frac{d}{dt}\|u\|_m \leqslant \left\||u|^{2p}\,u\right\|_m$ and $\left(\frac{d}{dt}\|u\|_m\right)_+ \leqslant \left\||u|^{2p}\,u\right\|_m$, where $(a)_+ = \max\{0,a\}$.
The **Key Lemma** is:
Let $f \in C^1([0,T];\mathbb{R})$ (or Lipschitz or even only absolutely continuous function) then

$$\int_0^T \left(\tfrac{df}{dt}(t)\right)_+ dt \geqslant \sup_{t\in[0,T]} f(t) - f(0).$$

From the Sobolev inequality $\|u\|_{H^{\frac{n}{2}+\varepsilon}} \geqslant c|u|_{L^\infty}$ we have

$$\|u(t_0,\cdot)\|_{m+\frac{n}{2}+\varepsilon} \geqslant c|u(t_0,\cdot)|_{C^m} \overset{\text{by Thm K3}}{\geqslant} c\nu^{-(\frac{1}{3}-\varepsilon')m}$$

for $m \geqslant 2$. Re-denoting $m + \frac{n}{2} + \varepsilon$ with m and "playing" with ε, ε' we obtain

$$\|u(t_0,\cdot)\|_m \geqslant c\nu^{-\frac{1}{3}(m-\frac{n}{2}-\varepsilon)} \qquad \text{for } m > 2 + \tfrac{n}{2}.$$

Applying the "Key Lemma" we obtain

$$\int_0^T \left(\tfrac{d}{dt}\|u\|_m\right)_+ dt \geqslant c\nu^{-\frac{1}{3}(m-\frac{n}{2}-\varepsilon)} \qquad \text{for small } \nu$$

since $\|u^\nu(0,\cdot)\|_m$ is bounded. Here $T = \nu^{-1/3}$; "for small ν" means $\nu \in (0, \nu_{m,\varepsilon})$.
Lemma 2. *Let $m > \frac{n}{2}$. Then* $\left\||u|^{2p}\,u\right\|_m \leqslant C\,\|u\|_0^{2p-\frac{pn}{m}}\,\|u\|_m^{1+\frac{pn}{m}}$.
Proof. Fix positive $\delta \leqslant \min\{\frac{n}{2}, m-\frac{n}{2}\}$. The proof follows from the chain of inequalities:

$$\tfrac{1}{C_m}\left\||u|^{2p}\,u\right\|_m \leqslant \|u\|_m\,|u|_{L^\infty}^{2p} \leqslant C'_\delta\,\|u\|_m\,\|u\|_{\frac{n}{2}+\delta}^p\,\|u\|_{\frac{n}{2}-\delta}^p \leqslant C'_\delta\,\|u\|_m\,\|u\|_0^{2p-\frac{pn}{m}}\,\|u\|_m^{\frac{pn}{m}} .$$

The second inequality is the Sobolev interpolation for L^∞. The last is just an interpolation of H^s norms (see appendix, Lemma A2). The first inequality follows from the inequality

$$\|fg\|_m \leqslant C_{m,n}(|f|_{L^\infty}\,\|g\|_m + \|f\|_m\,|g|_{L^\infty})$$

which will be proven in the appendix (Lemma A1). □

Since L^2 norm is non-increasing with time, we have from Lemma 2:

$$\int_0^T \|u\|_m^{1+\frac{pn}{m}}\,dt \geqslant c\nu^{-\frac{1}{3}(m-\frac{n}{2}-\varepsilon)}, \qquad \text{where } T = \nu^{-1/3}.$$

For $\alpha \leqslant 2$ we have $\frac{1}{T}\int_0^T f^2 dt \geqslant \left(\frac{1}{T}\int_0^T f^\alpha\right)^{\frac{2}{\alpha}}$. Using this for $\alpha = 1 + \frac{pn}{m}$ and $f(t) = \|u(t,\cdot)\|_m$ we get

$$\nu^{1/3}\int_0^{\nu^{-1/3}} \|u\|_m^2\,dt \geqslant C\nu^{-\frac{2}{3}(m-\frac{n}{2}-1-\varepsilon)\frac{m}{m+pn}}.$$

Since $\varepsilon > 0$ is arbitrary, we can kill the constant C by decreasing $\nu_{m,\varepsilon}$. Theorem K4 is proven. □

§ Some additional results.

Here we follow in part [2,chpt 4].

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be any smooth function. Consider the equation

$$\partial_t u = ie^{i\psi}\nu\Delta u + iuf(|u|). \tag{1''}$$

If $f(x) = x^{2p}$ we the get "standard" NLS equation (1). Before we start this presentation, we remark that all constants bellow also depend on the dimension n the nonlinearity f, but not on the real function $\psi = \psi(t, x, u, \ldots, \omega, \ldots)$ and small positive parameter ν.

Theorem K3$'$. *Let $\Omega \subset \mathbb{R}^n$ be any open set. Then for any real $A < \infty$ and $\sigma > 0$ there exist positive constants $\nu_0 > 0$ and $c_2, c_3, c_4, \ldots > 0$ such that for any smooth function $u : [0, +\infty) \times \Omega \to \mathbb{C}$ such that*

- $\sup\limits_{\Omega} |u(0,\cdot)| < A$,
- $\operatorname*{osc}\limits_{\Omega} f(|u(0,\cdot)|) > \sigma$,
- *u satisfies equation (1$''$) in the interior of the time-space cylinder $[0,\infty) \times \Omega$,*

for any $m \geqslant 2$ and for any positive $\nu < \nu_0$ we have:

$$\frac{c_m}{\nu^{m/3}} \leqslant \nu^{1/3} \int_0^{\nu^{-1/3}} \left|\tfrac{D^m u}{Dx^m}\right|_{L^\infty(\Omega)} dt.$$

Theorem K3$'$ follows from

Theorem K3$''$. *Under the assumptions of Theorem K3$'$ there exists $T = T(u) \leqslant \nu^{-1/3}$ such that*

$$\frac{c_m}{(T\nu)^{m/2}} \leqslant \tfrac{1}{T} \int_0^T \left|\tfrac{D^m u}{Dx^m}\right|_{L^\infty(\Omega)} dt \qquad \text{for } m \geqslant 2 \text{ and } \nu \in (0, \nu_0)$$

and

$$\sup_{[0,T]\times\Omega} |u(t,x)| < 2A \quad \text{and} \quad \forall t \in [0,T] \quad \operatorname*{osc}_{\Omega} |u(t,\cdot)| > \tfrac{\sigma}{2}.$$

Lemma. *Let $f : [0, A] \to \mathbb{R}$ be a continuous function. For any $\sigma > 0$ there exists $\sigma_2 > 0$ such that if there exist $a, b \in [0, A]$ such that $f(b) \geqslant f(a) + \sigma$ then there exists $c, d \in [a, b]$ such that*

$2\sigma_2$-neighborhood of the numbers 0, c and d do not intersect and

$$\sup_{x\in[d-\sigma_2, d+\sigma_2]} f(x) > \inf_{y\in[c-\sigma_2, c+\sigma_2]} f(y) + \tfrac{\sigma}{4}.$$

Theorem K3$''$ follows by a simple extrapolation from

Theorem K3$'''$. *Let the domain Ω, the positive real numbers A, σ and ν_0, and the function u satisfies the assumptions of Theorem K3$'$. Let σ_2 be as in previous lemma. Then there exists $k_1 > 0$ such that for any positive $\nu < \nu_0$ one of the following is true*

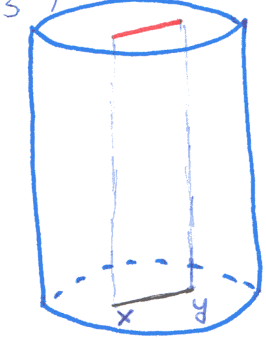
$$\int_0^T \nu\,|\Delta u|_{L^\infty(\Omega)}\,dt = \sigma_2 \quad \text{for some } T = T(u) \leqslant \nu^{-1/3}$$

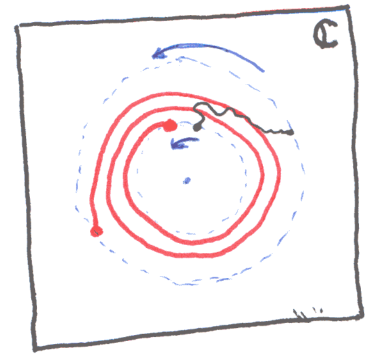
or

$$\int_0^{\nu^{-1/3}} \nu\,|\Delta u|_{L^\infty(\Omega)}\,dt < \sigma_2 \quad \text{and} \quad \nu^{\frac{1}{3}} \int_0^{\nu^{-1/3}} |\nabla u|_{L^\infty(\Omega)}\,dt \geqslant k_1\nu^{-1/3}.$$

We give a proof of Theorem K3$'''$ on the next page. For simplicity we will assume that $f(|u|) = |u|^{2p}$.

Idea of the proof. (of Thm K3''')





1) take our equation

$$\dot{u} = i|u|^{2p}u + \cancel{ie^{i\psi}\nu\Delta\psi}$$

2) delete the last term.

then the solution is given by

$$u(t,x) = u^0(x)\exp\left(it\,|u^0(x)|^{2p}\right)$$

So $|u^0(x)| \neq |u^0(y)|$ $\Rightarrow$ the angular velocity is different.

$\Rightarrow$ the "length of thread" $\geqslant \tilde{c}_1 t - \tilde{c}_2 \geq \tilde{\tilde{c}}_1 t$ for big t.

$\Rightarrow \displaystyle\int_x^y |\nabla u(t,\sigma)|\,d\sigma \geqslant$ length of thread at the time t $\geqslant \tilde{c}_1 t$ for big t

So if $t \sim \nu^{-1/3} \Rightarrow |\nabla u|_{L^\infty} \gtrsim \nu^{-1/3}$

3) because $|u|_{L^\infty} \sim 1$ we have $\left|\dfrac{D^m u}{Dx^m}\right| \gtrsim \left(\dfrac{1}{\nu}\right)^{m/3}$

"the real Proof" $\qquad \dot{u} = i|u|^{2p}u + ie^{i\psi}\nu\Delta u$

the key fact is $\big||u(0,x)| - |u(t,x)|\big| \leq \displaystyle\int_0^t |\nu\Delta u|_{L^\infty}\,dt$

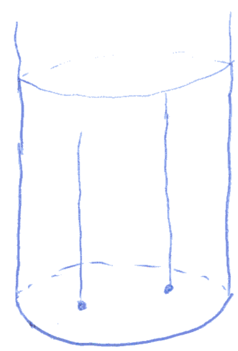
first want to prove that $\exists$ K s.t

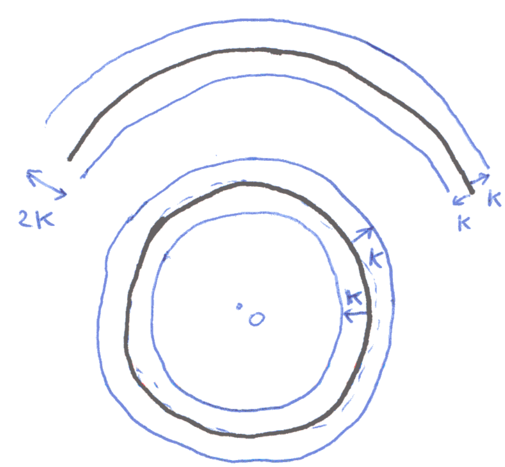
$$\boxed{\nu^{1/3}\int_0^{\nu^{-1/3}} |\Delta u|_{L^\infty}\,dt \geq K\nu^{-2/3}}$$

Suppose not then $\displaystyle\int_0^{\nu^{-1/3}} \nu|\Delta u|_{L^\infty}\,dt < K$

then the "Length of thread" becomes $\geq \tilde{c}t$.

consider $t \in [\frac{1}{2}\nu^{-1/3}, \nu^{-1/3}]$





done

References

[1] S Kuksin, *Spectral Properties of Solutions for Nonlinear PDEs in the Turbulent Regime*, GAFA 9 (1999) pp. 141-184. http://www.ma.hw.ac.uk/~kuksin/

[2] A Biryuk, *Estimates for Spatial Derivatives of Solutions for Parabolic Equations with Small Viscosity*, Heriot-Watt University Thesis, 2001.

Appendix.

For a smooth function f defined on an n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/(\ell\mathbb{Z})^n$ we define the H^m Sobolev seminorm as follows:

$$\|f\|_m^2 = \int ((-\Delta)^m f)\bar{f}\,dx = \sum_{|\alpha|=m} \binom{|\alpha|}{\alpha} |D^\alpha f|_{L^2} = \sum_{j_1,\dots,j_m=1}^{n} \left|\frac{\partial^m f}{\partial x_1 \cdots \partial x_m}\right|_{L^2}.$$

Here $\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha_1!\cdots\alpha_n!}$. Note that $\sum_{|\alpha|=m}\binom{|\alpha|}{\alpha} = n^m$.

Lemma A1. *Let f and g be smooth functions, defined on an n-dimensional torus. Let $m \geqslant 0$. Then*

$$\|fg\|_m \leqslant n^m 4^{m^2}(|f|_{L^\infty}\|g\|_m + \|f\|_m |g|_{L^\infty}).$$

Proof. It is sufficient to estimate every term of the form $\left|\frac{\partial^m(fg)}{\partial x_1\cdots\partial x_m}\right|_{L^2}$. We note that there are n^m many of them. Using the Leibniz rule of differentiating the products we find it is sufficient to estimate $\left|(D^\alpha f)(D^\beta g)\right|_{L^2}$ for any multi-indexes α and β such that $|\alpha+\beta| = m$. We note that we have that for any multi-index γ with $\gamma = m$ the Leibniz expansion of $D^\gamma(fg)$ has 2^m many terms of the form $(D^\alpha f)(D^\beta g)$ with $\alpha+\beta=\gamma$. Applying the Hölder inequality we obtain

$$\left|(D^\alpha f)(D^\beta g)\right|_{L^2} \leqslant |(D^\alpha f)|_{L^{\frac{2m}{|\alpha|}}} \left|(D^\beta g)\right|_{L^{\frac{2m}{|\beta|}}}.$$

Next we use Gagliardo-Nirenberg inequality

$$|(D^\alpha f)|_{L^{\frac{2m}{|\alpha|}}} \leqslant 4^{|\alpha|(m-|\alpha|)} |f|_{L^\infty}^{1-\frac{|\alpha|}{m}} \|f\|_m^{\frac{|\alpha|}{m}}$$

[see Lars Hörmander, "Lectures on nonlinear hyperbolic differential equations", Springer 1997; pp. 106-107] to obtain

$$|(D^\alpha f)|_{L^{\frac{2m}{|\alpha|}}} \left|(D^\beta g)\right|_{L^{\frac{2m}{|\beta|}}} \leqslant 4^{m^2-|\alpha|^2-|\beta|^2} (|f|_{L^\infty}\|g\|_m)^{\frac{|\beta|}{m}} (|g|_{L^\infty}\|f\|_m)^{\frac{|\alpha|}{m}}.$$

Using the inequality $A^sB^{1-s} \leqslant sA + (1-s)B \leqslant A+B$ we arrive at

$$\left|(D^\alpha f)(D^\beta g)\right|_{L^2} \leqslant 4^{m^2-|\alpha|^2-|\beta|^2} (|f|_{L^\infty}\|g\|_m + |g|_{L^\infty}\|f\|_m).$$

Finally, using $|\alpha|^2+|\beta|^2 \geqslant |\alpha|+|\beta| = m$, we have $2^m n^m 4^{m^2-|\alpha|^2-|\beta|^2} \leqslant 2^m n^m 4^{m^2-m} \leqslant n^m 4^{m^2}$. Lemma is proven. □

Lemma A2. *For fixed f the function $m \mapsto \|f\|_m$ is* log-*convex.*

Proof. Consider the Fourier representation $f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^n} \hat{f}_{\boldsymbol{k}} \exp(\frac{2\pi i \boldsymbol{k}\cdot\boldsymbol{x}}{\ell})$ then

$$\|f\|_m^2 = \ell^n \left(\tfrac{2\pi}{\ell}\right)^{2m} \sum_{\boldsymbol{k}\in\mathbb{Z}^n} |\boldsymbol{k}|^{2m}|\hat{f}_{\boldsymbol{k}}|^2.$$

It is sufficient to prove that the map $m \mapsto \sum |\boldsymbol{k}|^{2m}|\hat{f}_{\boldsymbol{k}}|^2$ is log-convex. This follows from the Hölder inequality

$$\sum |\boldsymbol{k}|^{2(sm_1+(1-s)m_2)}|\hat{f}_{\boldsymbol{k}}|^2 = \sum (|\boldsymbol{k}|^{2m_1}|\hat{f}_{\boldsymbol{k}}|^2)^s (|\boldsymbol{k}|^{2m_2}|\hat{f}_{\boldsymbol{k}}|^2)^{1-s} \leqslant$$
$$\left(\sum |\boldsymbol{k}|^{2m_1}|\hat{f}_{\boldsymbol{k}}|^2\right)^s \left(\sum |\boldsymbol{k}|^{2m_2}|\hat{f}_{\boldsymbol{k}}|^2\right)^{1-s}.$$

Here $s \in [0,1]$. □