Steepest Descent and the g-function Mechanism in Rigorous Semiclassical Focusing NLS Asymptotics

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joint work with

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Semi-classical focusing Cubic Schroedinger Equation (NLS)

$$\boxed{i\epsilon q_t + \frac{1}{2}\epsilon^2 q_{xx} + |q|^2 q = 0, \quad \varepsilon \to 0}.$$

One-parameter family of initial conditions (contains both solitons and radiation),

$$\boxed{q(x,0) = A(x)e^{iS(x)/\epsilon}}$$

$$\begin{cases} A(x) = -\text{sech}\, x, \\ S'(x) = -\mu \tanh x, S(0) = 0, \text{parameter} : \mu \geq 0. \end{cases}$$

<u>Goal</u>: Asymptotic calculation of $q(x,t,\varepsilon)$, $\varepsilon \to 0$.

<u>Results</u>:

- Pure radiation: Global leading behavior of $q(x,t,\varepsilon)$ as $\varepsilon \to 0$; explicit large time asymptotic behavior.
- Solitons+Radiation: Leading behavior till second break
- Both cases: Rigorous error estimate

Cai, McLaughlin, McLaughlin

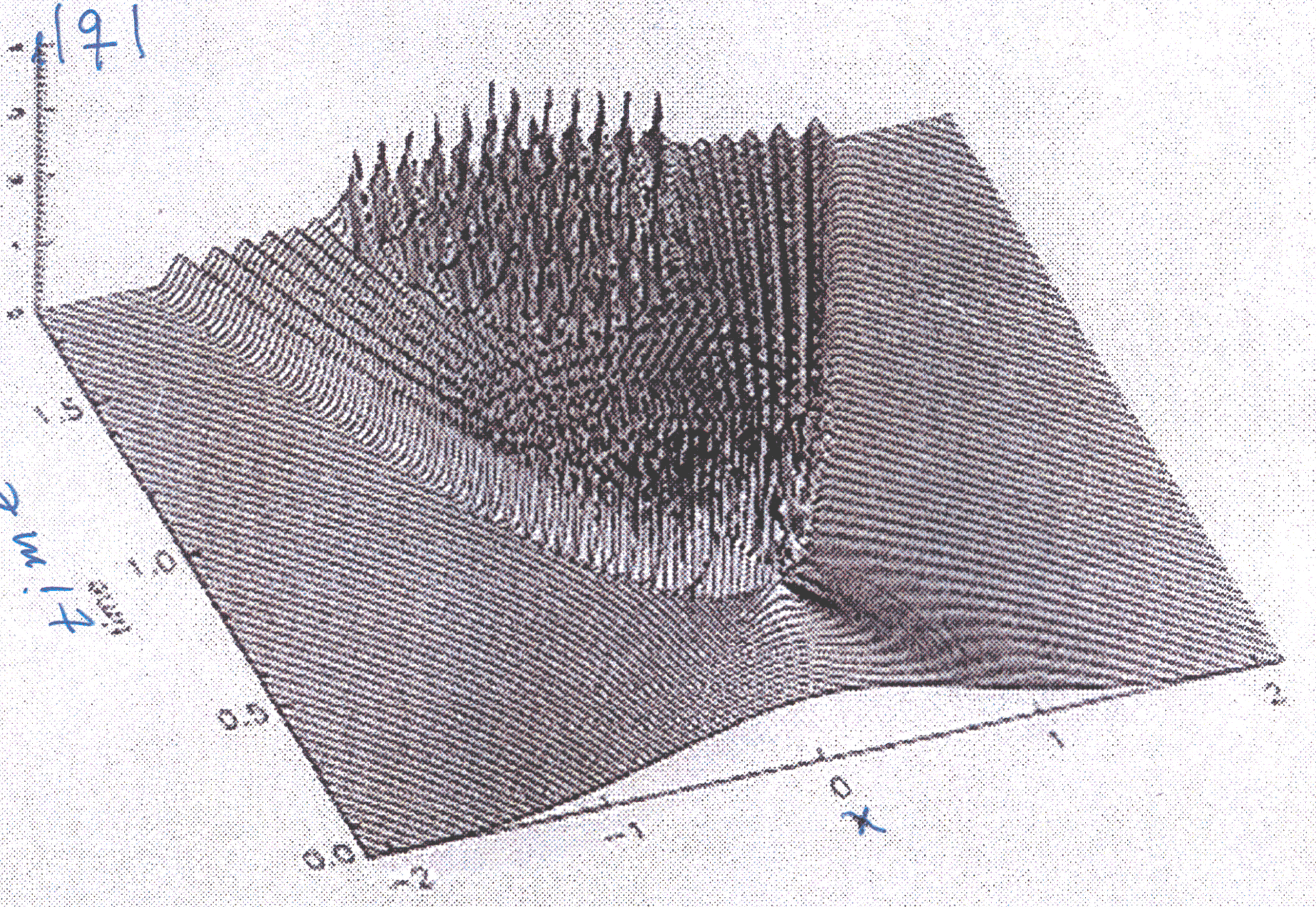


Figure 13: Semi-classical behavior: Focusing nonlinearity

LINEAR PROBLEM $(\varepsilon \to 0)$

$$i\varepsilon q_t + \frac{\varepsilon^2}{2} q_{xx} = 0 \quad , \quad q(x,0) = A(x)\, e^{iS(x)/\varepsilon}$$

Solution by Fourier Transform

$$q(x,t) = \frac{1}{2\pi\varepsilon} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(y)\, e^{\frac{i}{\varepsilon}\underbrace{\left[S(y) + z(x-y) - \frac{z^2}{2}t\right]}_{\varphi(y,z)}} dy\, dz$$

Stationary Phase/steepest descent

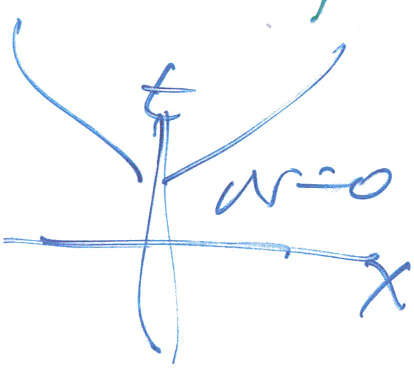
(x, t : parameters)

Eikonal:

$$\nabla\varphi = 0 \qquad \frac{\partial}{\partial y}\varphi = S'(y) - z = 0$$

$$\frac{\partial}{\partial z}\varphi = x - y - zt = 0$$

$z_t + z z_x = 0$ (Burgers)



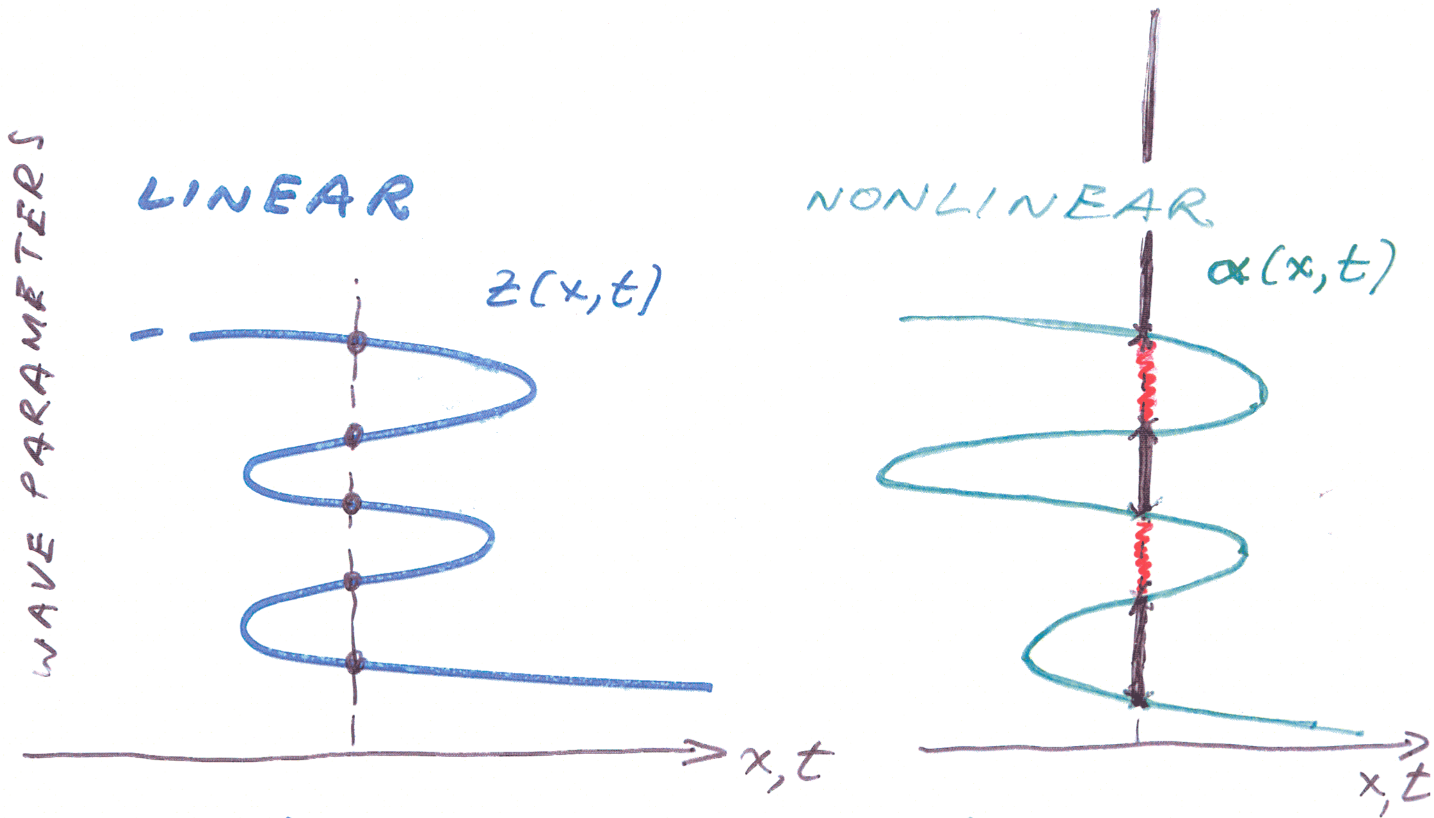
Solutions: $\left(y_j(x,t), z_j(x,t)\right)_{j=0}^{N(x,t)}$

$$q(x,t) \sim \sum_{j=0}^{N} c_j \underbrace{A(y_j)\, e^{\frac{i}{\varepsilon}\left[S(y_j) - z_j(x-y_j) - \frac{1}{2} z_j^2 t\right]}}$$

modulated exponential solution

(plane wave if y_j, z_j real)

CAUSTICS: $N(x,t)$ = discontinuous



five linear wave phases

two nonlinear wave-phases

MAIN CONTRIBUTION TO FOURIER INTEGRAL

MAIN CONTRIBUTIO TO RIEMANN HILBERT PROBLEM

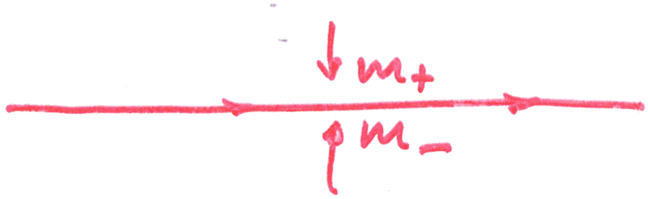
POINTS

INTERVALS (ARCS

LINEAR	INTERABLE NONLINEAR
Fourier Transform (FT)	Scattering Transform (ST)
Evolution of FT	Evolution of ST
Inverse FT (integral)	Inverse ST (Riemann-Hilbert Problem (RHP)
Steepest Descent (SD) for integrals	S.D for RHP Deift, Zhou
	g-function mechanism Deift, V, Zhou
fully nonlinear waveforms → g-function mechanism	↓
Linear Superposition of phases	Θ-function superposition
	rigorous treatment of endpoints: Deift, Kriecherbauer McLaughlin, V, Zhou.

RIEMANN-HILBERT PROBLEM (RHP)

m



Given V find m

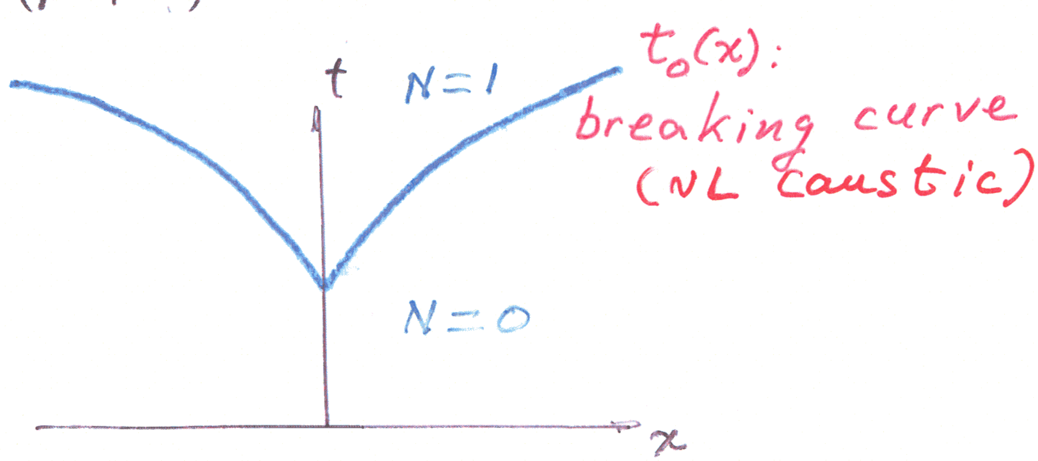
m $\begin{cases} m_+ = m_- V \\ m(z) \to I \text{ as } z \to \infty \end{cases}$

Main Theorem. *There exists a breaking curve* $t = t_0(x)$, $x \in \mathbb{R}$, *with the following properties:*

- *The genus of the solution* $q_0(x, t, \varepsilon)$ *is zero below the curve, i.e. in the region* $0 \leq t < t_0(x)$;
- *In the solitonless (pure radiation) case* $\mu \geq 2$ *the solution in the entire region above the breaking curve has genus exactly two. In the presence of solitons* ($\mu < 2$), *there exists some function* $t_1(x)$, $x \in \mathbb{R}$, $t_0(x) < t_1(x) \leq \infty$, *such that the genus equals two in the region* $t_0(x) < t < t_1(x)$,
- *The breaking curve is an even function, smooth and monotonically increasing for* $x > 0$ *with the asymptotic behavior*

$$t_0(x) \sim \frac{x}{2\mu}, \quad x \to +\infty,$$

$$t_0(x) = \frac{1}{2(\mu + 2)} + \frac{2\pi\sqrt{\mu + 2}}{5} x + o(x), \quad x \to 0^+;$$



- *In the genus zero region* $(0 \leq t < t_0(x))$, *the solution is controled by a point* α_0 *in the upper complex half plane that depends on* x *and* t.

$$q_0(x,t,\varepsilon) = \Im\alpha_0(x,t)e^{-2\frac{i}{\varepsilon}\int_0^x \Re\alpha_0(s,t)ds}$$

- *In the genus two region* $(t_0(x) < t < t_1(x))$ *the solution is controled by three points in the upper half plane* $\alpha_0, \alpha_2, \alpha_4$ *that depend on* x *and* t.

$$q_0(x,t,\varepsilon) = \Theta e^{\frac{2i}{\varepsilon}\Omega_1}\Im(\alpha_2 - \alpha_0 - \alpha_4),$$

$$\Theta = -\frac{\theta(-\frac{\hat{W}}{2\pi\varepsilon} - u_\infty + d)\theta(u_\infty + d)}{\theta(-\frac{\hat{W}}{2\pi\varepsilon} + u_\infty + d)\theta(-u_\infty + d)}.$$

Here the quantities $\Omega_1, \hat{W}, u_\infty, d$ *are explicit (quadratures) functions of* $\alpha_0, \alpha_2, \alpha_4$, *in which the hyperelliptic function below plays a crucial part.*

$$R(z) = \left(\prod_{j=0}^{2}(z - \alpha_{2j})(z - \bar{\alpha}_{2j})\right)^{1/2}$$

- *Accuracy loc. unif. in* x, t *not on breaking curve*

$$|\, q(x,t,\varepsilon) - q_0(x,t,\varepsilon)\, | = O(\varepsilon),$$

- **modulational instability:** *The evolution system for the* $2N+1$ *functions* $\alpha_0, \alpha_2, \cdots, a_{4N}$ *of* x *and* t *is elliptic.*

Previous work

Small Dispersion Weak Limits from Dyson Determinant

- Lax, Levermore: KdV, pure soliton
- Venakides: KdV, pure radiation and periodic
- Deift, K T-R McLaughlin: Toda
- Jin, Levermore, D. W. McLaughlin: Defocusing NLS

Strong Limit (Leading Behavior)

- Venakides: KdV small dispersion (fully nonlinear waveform, Ansatz)

Steepest Descent for RHP (SD)

- Deift, Zhou: mKdV, Painleve I, II, ...

g-function mechanism for Steepest Descent (fully nonlin. waves)

- Deift, Venakides, Zhou: small dispersion KdV

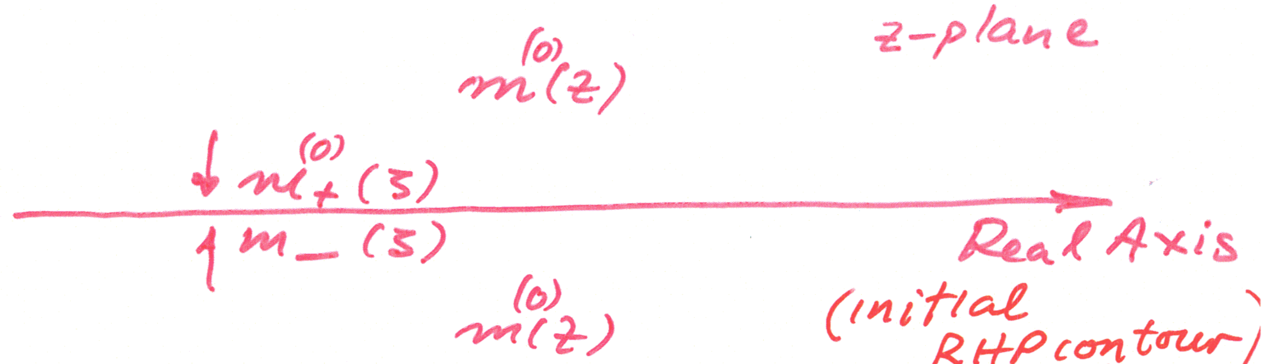
Focusing NLS, early work

- Zakharov, Shabhat: Integration of focusing NLS
- D. W. McLaughlin: semiclassical analysis
- ZS Satsuma, Yajima: spectral analysis for special potential $\mu = 0$

Semiclassical Focusing NLS

- Bronski: distribution of e-values
- Kamvissis, Miller: NLS numerics reveal structure
- Ceniseros, Tian: NLS numerics reveal structure
- Tovbis, Venakides: calculation of scattering data, $\mu > 0$
- Kamvissis, K. T-R. McLaughlin, Miller: first breakthrough, pure soliton case ($\mu = 0$), use of steepest descent and g-function mechanism, complex contributing contours; variational formulation, small time leading behavior, connection between genus zero and two assuming breaking curve.
- Miller: continuum limit of pole structure

1) Find the enemy: RHP $m^{(0)}, \Sigma^{(0)}, V^{(0)}$

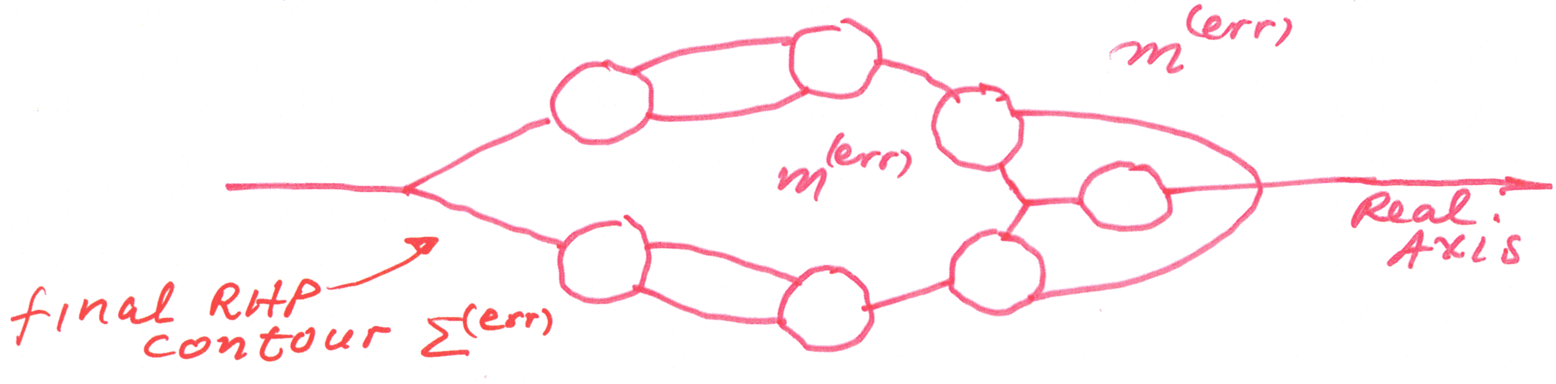


$$m_+^{(0)} = m_-^{(0)} V^{(0)}(\zeta)$$

m, V : 2×2 matrices

2) Weaken the enemy:

- Relocate the enemy
  Give battle on ground of your choice
- Peel off contributions in the process



3) Crush the enemy:

Final RHP jump matrix: $V^{err} = I + O(\varepsilon)$

Integrability of NLS (ZS, Zhou): LAX PAIR

$$\partial_x \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = -\frac{i}{\varepsilon} \begin{pmatrix} z & q(x,t) \\ \bar{q}(x,t) & -z \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad \textbf{ZS system}$$

$$\partial_t \phi = B(z, q, q_x)\phi, \quad \phi = (\phi_1, \phi_2)^T \quad \textbf{Evolution System}$$

NLS EMERGES AS THE COMPATIBILITY CONDITION:

$$\partial_x \partial_t \phi = \partial_t \partial_x \phi \iff i\varepsilon q_t + \varepsilon^2 q_{xx} + |q|^2 q = 0$$

(Cancellations reduce vector condition to a scalar one)

SOLUTION VIA DIRECT/INVERSE SCATTERING

- **Work on the associated ZS system (LINEAR)**
  Scattering matrix $S = S(z,t) : \phi_{out} = S\phi_{in}$
  determined by the reflection coefficient $r_0 = r_0(z,t)$.
  Bound States $\longmapsto$ Solitons.
- **Direct Scattering Transformation (ZS):**
  $q(x,0) \longmapsto r_0(z,0)$ ($|x| \to \infty$ asympt. of ϕ at $t=0$)
- **Evolution of Scattering Data (Evol. System):**
  $r_0(z,t) = r_0(z,0)e^{4iz^2t/\varepsilon}$:) $\lim_{|x|\to 0} q(x,t) = 0$
- **Inverse Scattering Transformation (ZS):**
  $r_0(z,t) \longmapsto q(x,t)$ (Riemann-Hilbert Problem for fund. matrix $\mathbf{\Phi}$ of ZS as a function of spectral variable z)

Inverse scattering of ZS via RHP

Goal: To construct $q(z)$ from the **reflection coefficient** $r_0(z)$ (for simplicity assume no bound states).

$\Phi(z,x)$: ZS fundamental matrix ($\Im z \neq 0$) determined by,

$$\Phi(z,x) \sim \begin{pmatrix} e^{-\frac{izx}{\varepsilon}} & 0 \\ 0 & e^{\frac{izx}{\varepsilon}} \end{pmatrix}, \text{ as } x \to +\infty$$

AND $\begin{cases} \text{column 1} \to 0 \text{ when } \Im z > 0, \\ \text{column 2} \to 0 \text{ when } \Im z < 0. \end{cases}$ as $x \to -\infty$,

NON-ENTIRE NATURE OF Φ: RIEMANN-HILBERT PROBLE

ODE theory: for $z \in \mathbb{R}$, we have $\boxed{\Phi_+ = \Phi_- \mathrm{C}}$, where $\mathrm{C} = \mathrm{C}(z)$ is a 2×2 matrix that is independent of x.
Normalization: $\Phi(z,x) \longrightarrow m(z,x)$ (remove oscillations),

$$m(z,x) \sim \mathrm{I} + \frac{\tilde{m}(x)}{z} + \mathrm{O}(z^{-2}), \quad \boxed{m_+ = m_- \mathrm{V}}$$

$$\boxed{\mathrm{V}|_{z\in\mathbb{R}} = \begin{pmatrix} 1+|r|^2 & \bar{r} \\ r & 1 \end{pmatrix}; \quad r = r_0(z) e^{2izx/\varepsilon}.}$$

Recovery of q from equation $\boxed{q(x) = -2\tilde{m}_{12}(x)}$

Rigorous asymptotic solution of the RHP with x and t fixed

Tool: Jump matrix factorization, contour deformation

Rules

- $+$ to the left, $-$ to the right of the oriented contour.
- A RIGHT factor having a LEFT analytic continuation splits off to the LEFT on its own contour.
- A LEFT factor having a RIGHT analytic continuation splits off to the RIGHT on its own contour.

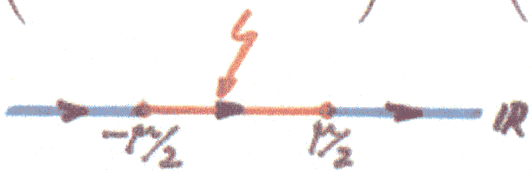
Goals

1. Achieve a contour that is the union of arcs over each of which the jump matrix is either constant (independent of z) or decays as $\varepsilon \to 0$.
2. Solve the model RH problem that neglects the decaying contours
3. Estimate the error

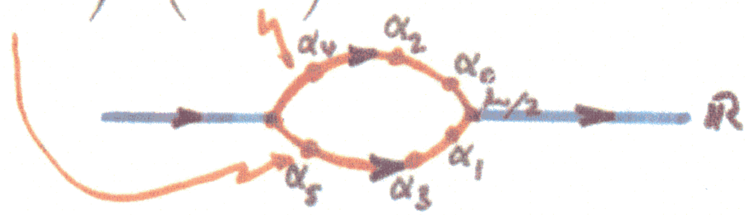
Overview of the Procedure

- green contour: V is constant in z piecewise.
- blue contour: V decays as $\varepsilon \to 0$

$$\begin{pmatrix} 1+|r|^2 & \bar{r} \\ r & 1 \end{pmatrix} = \begin{pmatrix} 1 & \bar{r} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}; \; r = \quad e^{2if/\varepsilon}$$



$\Sigma^{(0)}$



$\Sigma^{(1)} = \Sigma^{(2)} = \Sigma^{(3)}$

(0) → (1) : Initial Factorization/Deformation

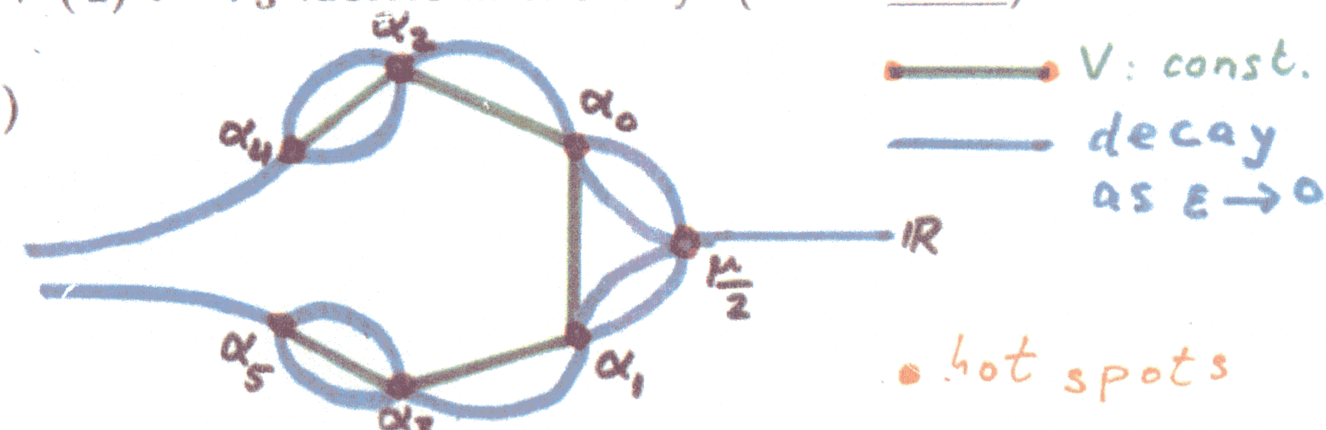
(1) → (2) : Stirling Approx. of $r_0(z)$

(2) → (3) : g-function transf: $m^{(3)} = m^{(2)} e^{2ig(z)\sigma_3/\varepsilon}$

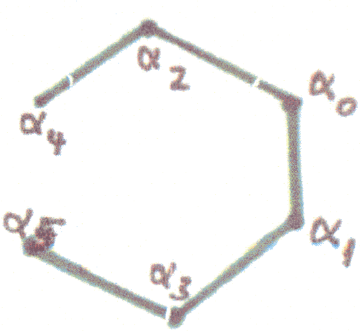
(3) → (4) : Factor./Deform.

(3) → (4) : V_3 factors in two ways (recall Goals)

$\Sigma^{(4)}$



$\Sigma^{(mod)}$



Explicit solution through theta function

The Initial Data enter in function f

$r_0(z;\varepsilon)$ is the reflection coefficient of the initial data.

$r(z;x,t,\varepsilon) = r_0(z,\varepsilon)e^{2i(xz+2tz^2)/\varepsilon}$

$$r(z;x,t,\varepsilon) \sim \begin{cases} e^{-\frac{2i}{\varepsilon}f(z,\varepsilon)} & \text{when } z < \mu/2, \\ e^{-\frac{2i}{\varepsilon}(f(z,\varepsilon)+2\pi i(\frac{\mu}{2}-z))} & \text{when } z > \mu/2 \end{cases}$$

$$f(z;x,t,\varepsilon) = (\frac{\mu}{2} - z)\left[\frac{i\pi}{2} + \ln(\frac{\mu}{2} - z)\right]$$
$$+ \frac{z+T}{2}\ln(z+T) + \frac{z-T}{2}\ln(z-T)$$
$$- T\tanh^{-1}\frac{T}{\frac{\mu}{2}} - xz - 2tz^2 + \frac{\mu}{2}\ln 2 + \frac{\pi}{2}\varepsilon, \quad \text{when } \Im z \geq$$

where positive values have real logarithm, and $f = f(z,\varepsilon;x,t)$ has analytic extension into the upper complex half-plane. $T = \sqrt{(\frac{\mu}{2})^2 - 1}$.

BOTTOM LINE: Given f find h.

Error Analysis

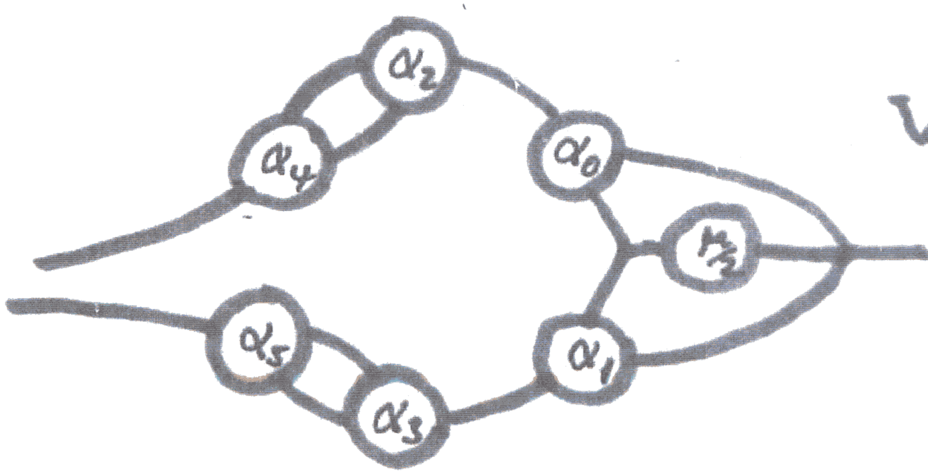
Construction of $m^{(app)}(z)$ (to peel off $m^{(4)}$).

1. $m^{(app)}(z)$ equals the solution of the model problem $m^{(mod)}(z)$ outside circles centered at the points α_j, $\bar{\alpha}_j$, and $\mu/2$ with radii $\delta > 0$, small but independent on ε;
2. $m^{(app)}(z)$ is a paramatrix of $V^{(4)}$ inside each circle i.e. it satisfies the jump conditions of the RHP $P^{(4)}$ inside these circles exactly;
3. the jump $m_+^{(app)} m_-^{(app)-1}$ of $m^{(app)}(z)$ across the circles is of order $I + O(\varepsilon)$ uniformly on the circles.

An easy calculation gives for $q(x, t, \varepsilon)$

$$q = -2 \lim_{z\to\infty} z(M(z) - I)_{12} - 2 \underbrace{\lim_{z\to\infty} z(m^{(err)}(z) - I)_{12}}_{O(\varepsilon)}$$

where $M(z) = m^{(mod)} e^{-\frac{2i}{\varepsilon} g(z)\sigma_3}$.



$V = I + O(\varepsilon)$

The g function mechanism

Introduce the transformation $m^{(2)} \to m^{(3)}$, ($h(z)$=TBD),

$$m^{(3)} = m^{(2)} \begin{pmatrix} e^{\frac{2i}{\varepsilon} g(z)} & 0 \\ 0 & e^{-\frac{2i}{\varepsilon} g(z)} \end{pmatrix} ; \quad g(z) = \frac{h(z) + f(z}{2}$$

where the analytic in $\bar{\mathbb{C}} \setminus \Sigma^{(2)}$ complex valued function $g(z)$ is to be determined. The symmetry of the problem requires the Schwartz reflection invariance of g, i.e. $g(\bar{z}) = \overline{g(z)}$.

$$V^{(3)}|_{z \in \Sigma^+} = \begin{pmatrix} e^{\frac{i}{\varepsilon}(h_+ - h_-)} & 0 \\ -e^{\frac{i}{\varepsilon}(h_+ + h_-)} & e^{-\frac{i}{\varepsilon}(h_+ - h_-)} \end{pmatrix}$$

Alternative factorizations of the jump matrix

Two types of factorization are given by the formulae,

$$\begin{pmatrix} a & 0 \\ -b & a^{-1} \end{pmatrix} \begin{cases} = \begin{pmatrix} 1 & -ab^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & b^{-1} \\ -b & 0 \end{pmatrix} \begin{pmatrix} 1 & -a^{-1}b^{-1} \\ 0 & 1 \end{pmatrix} \\ = \begin{cases} \begin{pmatrix} 1 & 0 \\ -a^{-1}b & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \\ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -ab & 1 \end{pmatrix} \end{cases} \end{cases}$$

The g function mechanism

Constancy and Decay Conditions

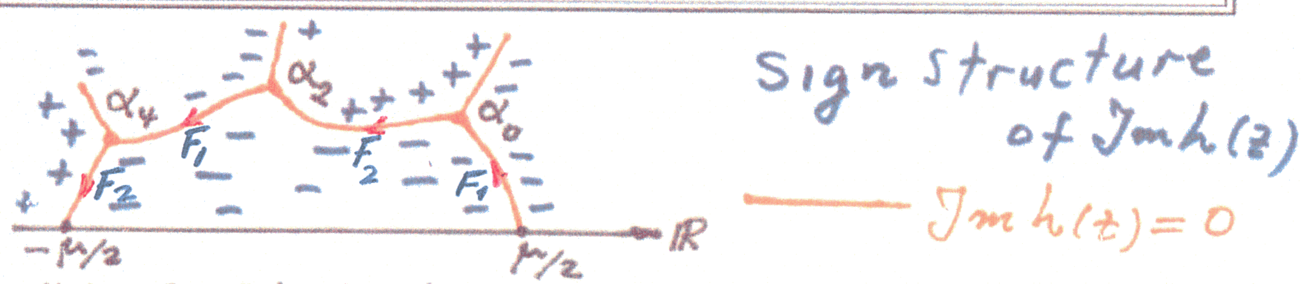
Let $h = 2g - f$ so as to deal with $h_+ \pm h_-$ and not with $g_+ + g_- - f$ and $g_+ - g_-$; index j labels arcs.

$$V^{(3)}|_{z\in\Sigma^+} = \begin{pmatrix} e^{\frac{i}{\varepsilon}(h_+ - h_-)} & 0 \\ -e^{\frac{i}{\varepsilon}(h_+ + h_-)} & e^{-\frac{i}{\varepsilon}(h_+ - h_-)} \end{pmatrix}$$

F1: $\begin{cases} h_+ + h_- = 2W_j, \\ (h'_+ + h'_- = 0) \end{cases}$ $\begin{cases} \Im h_- < 0, & \text{(right of contour)}, \\ \Im h_+ < 0, & \text{(left of contour)}, \end{cases}$

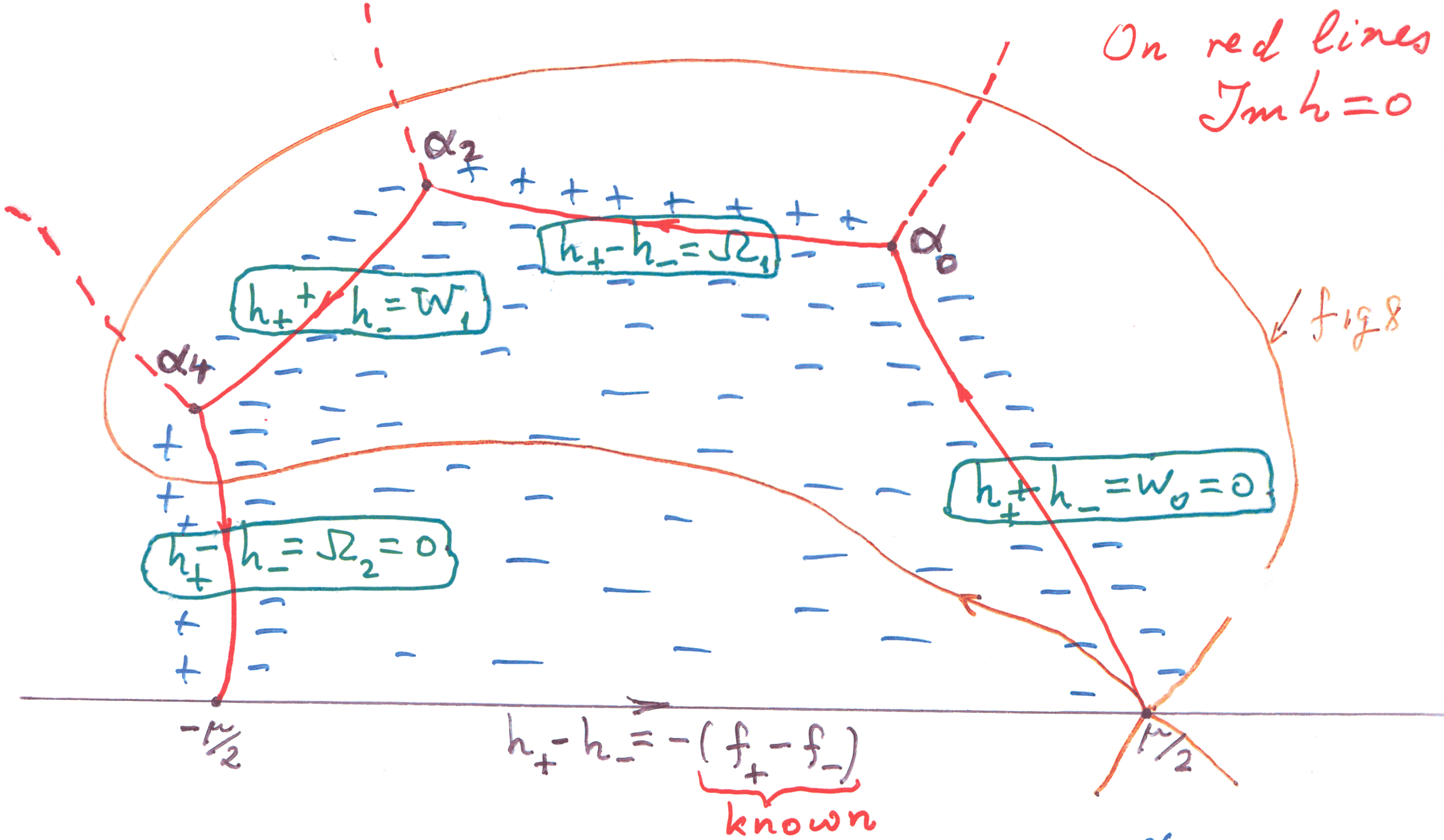
F2: $\begin{cases} h_+ - h_- = 2\Omega_j, \\ (h'_+ - h'_- = 0) \end{cases}$ $\begin{cases} \text{either } \Im h_- < 0, \\ \text{or } \Im h_+ < 0. \end{cases}$

W_j and Ω_j are real constants. On the contour $\Im h = 0$



Equalities for $h'_+ \pm h'_-$ pose a scalar RHP for h'. Solution,

$$\frac{R(z)}{2\pi i}\oint_{fig8} \frac{f'(\zeta)}{(\zeta - z)R(\zeta)} d\zeta, \; R(z) = \sqrt{\prod_{k=0}^{4N+1} (z - \alpha_k)}$$



$$h'(z) = \frac{R(z)}{2\pi i} \oint_{\text{fig 8}} \frac{f'(\zeta)}{(\zeta - z) R(\zeta)}$$

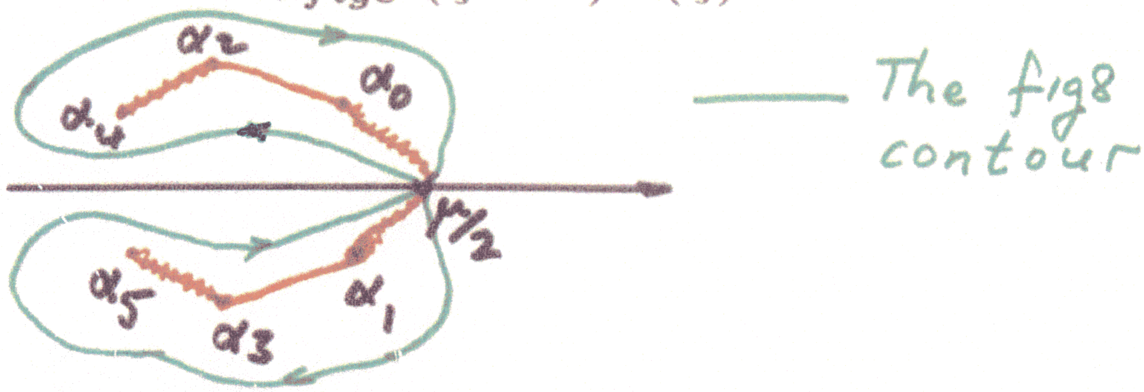
$$R(z) = \sqrt{(z-\alpha_0)(z-\bar{\alpha}_0)(z-\alpha_2)(z-\bar{\alpha}_2)(z-\alpha_4)(z-\bar{\alpha}_4)}$$

z : inside fig 8

$\alpha_0, \alpha_2, \alpha_4$
Real Const: Ω_1, W_1 } TBD

The Moment and the Integral Conditions

$$h'(z) = \frac{R(z)}{2\pi i} \oint_{fig8} \frac{f'(\zeta)}{(\zeta - z)R(\zeta)} d\zeta, \quad z \text{ inside } fig8.$$



Moment Conditions (analyticity of g at infinity)

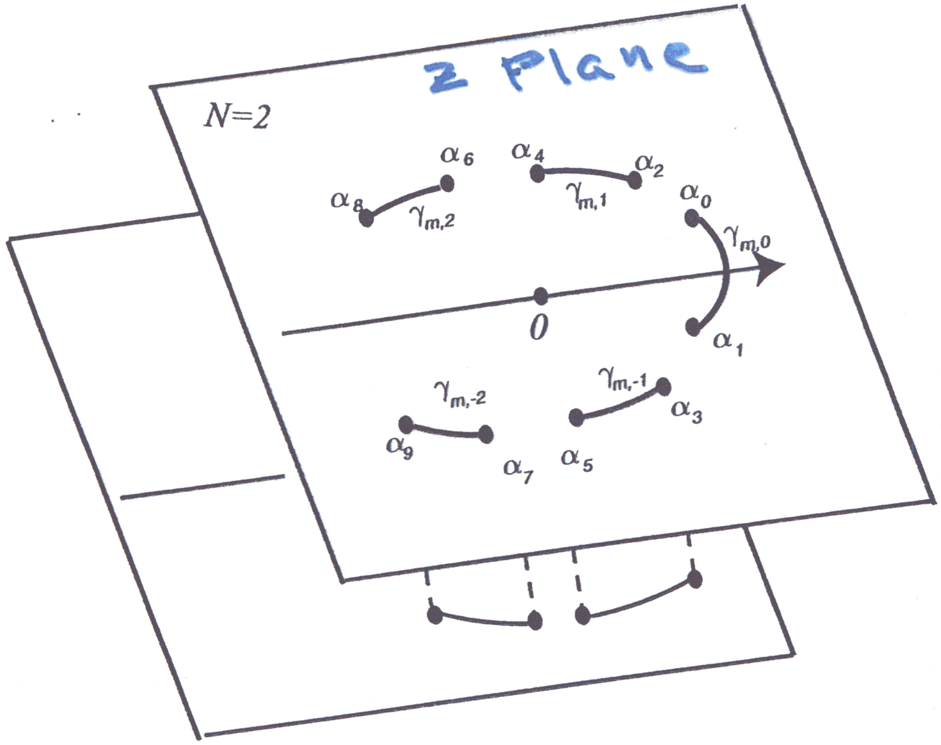
$$\text{Moment conditions } M_k : \quad \oint_{fig8} \frac{\zeta^k f'(\zeta)}{R(\zeta)} d\zeta = 0,$$

with $k = 0, 1, \cdots, 2N+1$. We obtain them by expanding $(\zeta - z)^{-1})$ in the integral in powers of z^{-1}.

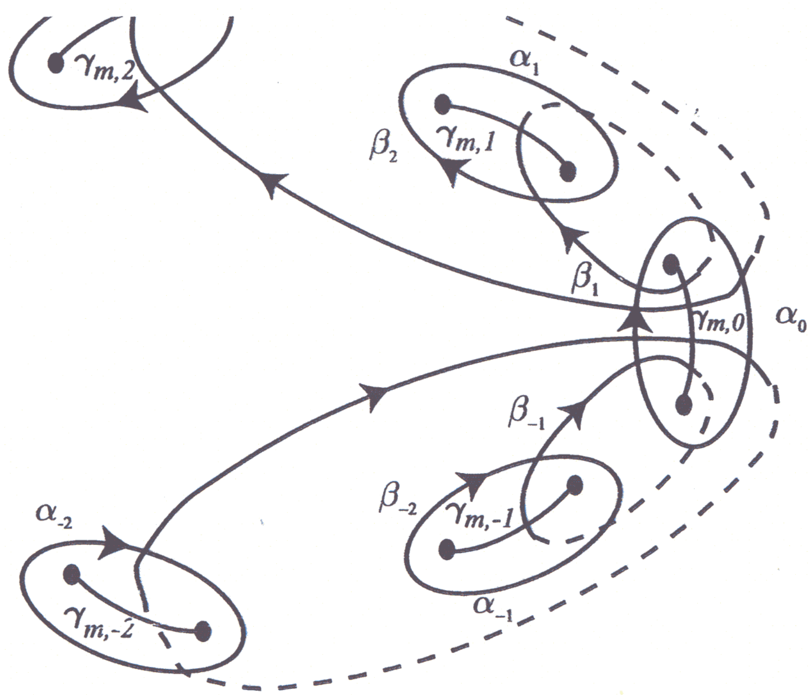
Integral conditions ($\Im h(\alpha_{2i}) = 0, \quad i = 0, 1, 2, \cdots, N$)

$$I_i : \Im \int_{\frac{\mu}{2}}^{\alpha_i} h'(\zeta) d\zeta = 0, \quad i = 1, 2, \cdots, N.$$

Constants: $W_j = W_j(\alpha)$; $\Omega_j = \Omega_j(\alpha)$;
$W_0 = 0$ $\qquad \Omega_{N+1} = 0$ (Norm.)



Riemann surface $\mathcal{R}(x,t)$

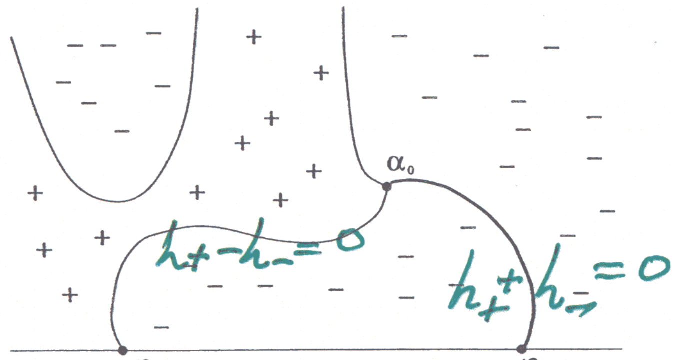


Basic cycles

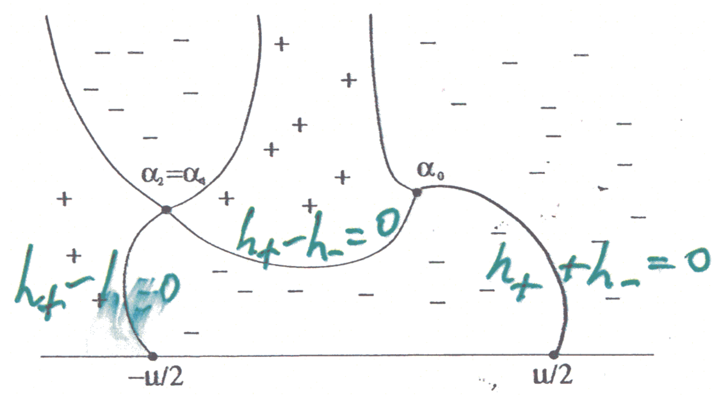
MECHANISM OF BREAKING

Contours: $\Im h = 0$

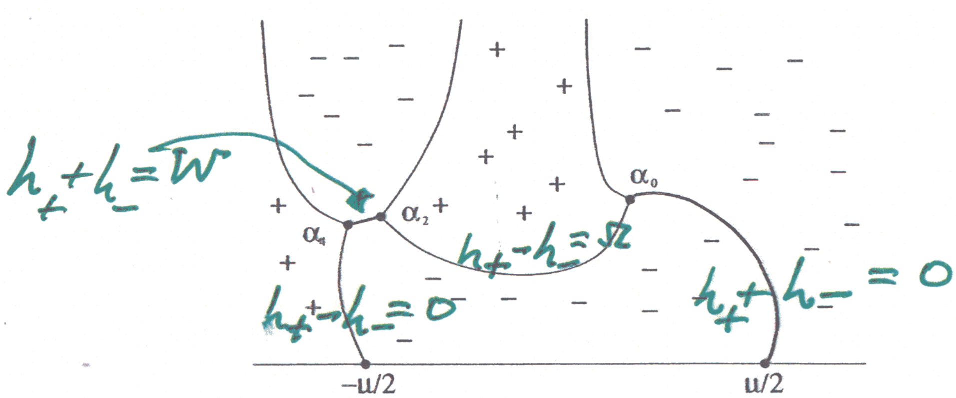
$\pm$: sign $\Im h$



Zero level curves of $\Im h$, pre-break



Zero level curves of $\Im h$, breaking point



Zero level curves of $\Im h$, post-break

The points $\alpha = (\alpha_j)_{j=0}^{4N+1}$

- The constant jump matrices on the green contour can be expressed as functions of the α ONLY.
- $\alpha = \alpha(x, t)$ (dependence in the large space-time scale)
- The theta function solving the problem has form $F(\alpha; \frac{x}{\varepsilon}, \frac{t}{\varepsilon})$, i.e. it varies (oscillates) in the small space-time scale in x and t in a way that is governed by α, the latter being modulated in the large space-time scale.
- The number of fully nonlinear oscillatory phases in the waveform is $2N$, where $N = N(x, t)$.
- The theta function solution of the model RHP (all green contour) provides the leading asymptotic behavior of NLS.
- The main calculational task for having the leading behavior is the calculation of N and α.
- Error Estimate

Breaking and Loss of Connection

The required zero and sign structure for a given N holds while x, t change as long as:

- All α_j remain distinct;
- The ratio $\frac{h'(z)}{R(z)} \neq 0$ for any $z \in \gamma$;
- The contour γ stays away from singularities of $f(z)$.

Breaking occurs at some x, t at which one of the above conditions is violated. Generically, it occurs on curves in the (x, t) plane that we call **breaking curves**, across which there is a jump in the genus N. A set of α satisfying the first two of the above conditions is called **nondegenerate**. Degeneracy can occur as the result of:

1. Collision between different α_{2k} in the upper half-plane (and the corresponding complex conjugates in the lower half-plane): death of a pair decreases the genus
2. Collision between the contour γ, which is a zero level curve of $\Im h(z)$, and any other branch of zero level curve of $\Im h(z)$: birth of a pair increases the genus.

The two events can be viewed as time-reverses of each other. In the case that we treat, the genus $2N = 2$ and we have $\alpha_0 \neq \alpha_2 = \alpha_4$. The Jacobian $|\frac{\partial F}{\partial \alpha}|$ becomes zero at this point. To establish the evolution through a breaking curve, we reparametrize the α and we obtain a nonzero Jacobian.

Off breaking curves: Evolution Theorem

Let $\alpha = (\alpha_0, \alpha_2, \alpha_4, \cdots \alpha_{4N})$ with distinct α_{2k} be a solution of the modulation system with genus $2N$ at some point (x_0, t_0). Then the solution $\alpha(x,t)$ can be continued uniquely with the same genus into a neighborhood of (x_0, t_0) and $\alpha(x,t)$ is a smooth function of x and t.

The proof is based on the implicit function theorem and the following expression for the Jacobian of the modulation system. The Jacobian $\partial MI/\partial\alpha$ is given by

$$|\frac{\partial MI}{\partial \alpha}| = \det D \prod_{j=0}^{2N} |\frac{h'(\alpha_{2j})}{2R(\alpha_{2j})}|^2 \prod_{j<l}(\alpha_l - \alpha_j)$$

where the determinant is

$$\begin{pmatrix} \int_{\hat{\gamma}_{m,1}} \frac{dz_1}{R(z_1)} & \int_{\hat{\gamma}_{c,1}} \frac{dz_2}{R(z_2)} & \cdots & \int_{\hat{\gamma}_{c,N}} \frac{dz_{2N}}{R(z_{2N})} \\ \int_{\hat{\gamma}_{m,1}} \frac{z_1 dz_1}{R(z_1)} & \int_{\hat{\gamma}_{c,1}} \frac{z_2 dz_2}{R(z_2)} & \cdots & \int_{\hat{\gamma}_{c,N}} \frac{z_{2N} dz_{2N}}{R(z_{2N})} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \int_{\hat{\gamma}_{m,1}} \frac{z_1^{2N-1} dz_1}{R(z_1)} & \int_{\hat{\gamma}_{c,1}} \frac{z_2^{2N-1} dz_2}{R(z_2)} & \cdots & \int_{\hat{\gamma}_{c,N}} \frac{z_{2N}^{2N-1} dz_{2N}}{R(z_{2N})} \end{pmatrix}$$

The Jacobian is nonzero as long as all α_j are distinct and $\frac{h'(z)}{R(z)}|_{z-\alpha_j} \neq 0$.

Passage through breaking curve, Change of Genus: Degeneracy Theorem

Suppose

$$\frac{\Im h'(z_0)}{R(z_0)} = 0, \quad \left(\text{by symmetry also: } \quad \frac{\Im h'(\bar{z}_0)}{R(\bar{z}_0)} = 0\right),$$

for some point $z_0 \in \gamma$. Then:

1. Replacing $R(z)$ with $\tilde{R} = R(z)(z - z_0)(z - \bar{z}_0)$ (the multiplicities of z_0 and $\bar{z}_0$ are thus increased by two) does not change the functions $h'(z)$ and $h(z)$, i.e. $h'(z; \tilde{R}) = h'(z; R)$ and $h(z; \tilde{R}) = h(z; R)$.

2. If the original α satisfy the MI conditions with genus $2N$, then the new α corresponding to $\tilde{R}$, also satisfy the MI conditions with genus $2(N+1)$.

3. Conversely, if a degenerate $\alpha = \alpha_0, \alpha_2, \cdots \alpha_{4N}$ with $\alpha_{2k} = \alpha_{2k+2} = z_0$ satisfy the MI conditions with genus $2N$, then the α that is obtained by removing the degenerate pair and its complex conjugate satisfies the MI conditions for genus $2(N-1)$. Furthermore, after the removal, $h'/R = 0$ at the site z_0 of the removed pair.

Equations for the α's (modulation equations, N=1)

$$R(z) = \sqrt{\prod_{j=0}^{2} (z-\alpha_{2j})(z-\overline{\alpha}_{2j})}, \quad \alpha_{2j} = a_{2j} + ib_{2j},$$

$$Q_i(z) = \left(\frac{(z-\alpha_{2j})(z-\alpha_{2k})}{(z-\overline{\alpha}_{2j})(z-\overline{\alpha}_{2k})} \right)$$

Moment conditions

$$\mathrm{M}_0 : \quad \int_{\mathbb{R}} \frac{sgn\zeta}{|R(\zeta)|} d\zeta = 0$$

$$\mathrm{M}_1 : \quad \int_{\mathbb{R}} \frac{(\zeta - a_{2j})sgn\zeta}{|R(\zeta)|} d\zeta = 8t$$

$$\mathrm{M}_2 : \quad \int_{\mathbb{R}} \frac{(\zeta - a_{2j})(\zeta - a_{2k})sgn\zeta}{|R(\zeta)|} d\zeta = 2x + 8ta_{2i}$$

$$\mathrm{M}_3 : \quad \int_{\mathbb{R}} \left[1 - \frac{(\zeta - a_0)(\zeta - a_2)(\zeta - a_4)sgn\zeta}{|R(\zeta)|} \right] d\zeta$$

$$= \quad 4t(b_0^2 + b_2^2 + b_4^2) + 2$$

Integral conditions for α_{2i} (True also for subscripts 2j, 2k)

$$\Im \left[\pi i(|a_{2i}| - 1) - 8ti \int_{a_{2i}}^{\alpha_{2i}} \sqrt{b^2 - y^2} Q_i(z) dy \right]$$

$$+ \Im \left[\int_{a_{2i}}^{\alpha_{2i}} \int_{\mathbb{R}} \frac{Q_i(z)\sqrt{b^2 - y^2}\, sgn\zeta}{(\zeta - z)Q_i(\zeta)\sqrt{(\zeta - a)^2 + b^2}} d\zeta dz \right] = 0$$

Summary of Work Required

- Solve System of equations for the α_{2i}, $i = 1, 2, \cdots, 4N$

CATCH: For what value of N?

ANSWER: For the value of N for which there is a connection from $\mu/2$ to $-\mu/2$ by a zero-level curve of $\Im h$ satisfying the above sign structure

We may start the procedure at $t = 0$ where $N = 0$ and evolve in time.

Derivation of inequalities

$$\mathsf{M}_1: \qquad b_j b_k < \frac{\pi}{64t^2};\ j \neq k \ \ (\text{two smaller } b < \frac{\sqrt{\pi}}{8t})$$

$$\mathsf{M}_3: \qquad a_0 + a_2 - a_4 < 2 + 4t(b_0^2 + b_2^2 + b_4^2)$$

$$\mathsf{I}_j: \qquad |a_{2j}| = 1 + 2tb_{2j}^2\nu_j + O(b_{2j})$$

where $|\nu_j| < 1;\ \ j = 1, 2, 3.$

1^{st} *Inequality* : Positive Integrand, Cauchy-Schwartz

2^{nd} *Inequality* : Positive Integrand, Area argument

3^{rd} *Relation* : Bound on double integral.

Theorem (pure radiation $\mu \geq 2$, for simplicity $\mu = 2$)

All three a_{2j} and all three b_{2j} are bounded for $t \leq \infty$. As $t \to \infty$, the genus 2 region is $|x| < 4t$ and for $x > 0$ and,

$$a_0 \to 1, \quad a_4 \to -1, \quad a_2 \sim -x/4t,$$

$$b_0 \sim e^{-8t+2x},\ b_4 \sim e^{-8t-2x},\ b_2 \sim \sqrt{\frac{1}{2t}(1 - \frac{x}{4t})}$$

Directions

- Long-time asymptotics in the presence of solitons.

  Does $N(t) \to \infty$ as $t \to \tau \leq \infty$?

- Eigenvalues of the ZS problem: Facing the lack of a Sommerfeld eigenvalue condition.

- Non analyticity

  Initial Data

  Scattering Data

  Chaotic behavior ?