# *Fluctuations of Eigenvalues and Orthogonal Polynomials*

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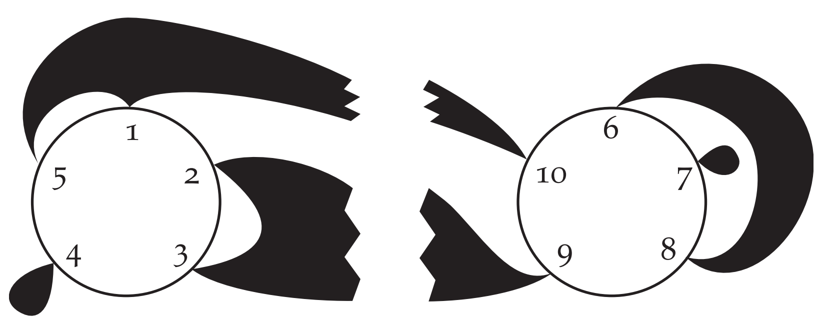


*notation and genus expansions – Mingo and Nica, Inter. Math. Res. Notices (2004)*

— $X_N \in \mathrm{SGRM}(N, 1/N)$ = self-adjoint Gaussian random matrices = $N \times N$ matrices with independent Gaussian entries normalized so that $\mathcal{E}(\mathrm{Tr}(X^2)) = N$

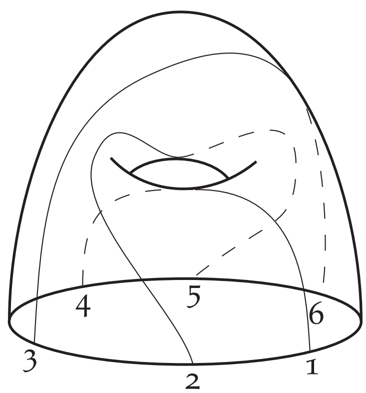
— $\pi \in S_n$ has genus $g$ where

$$\#(\gamma) + \#(\gamma \circ \pi^{-1}) + \#(\pi) = n + 2(1 - g)$$

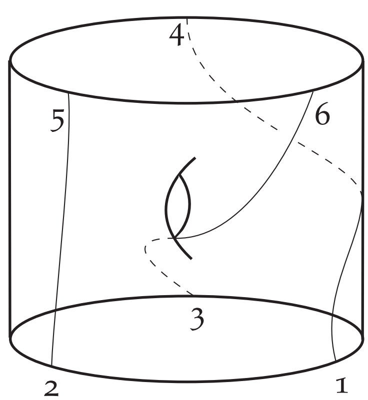
$\gamma = (1, 2, 3, \ldots, n)$; $\#(\pi)$ = no. of orbits

— genus expansions

$$\mathcal{E}(\mathrm{Tr}(X^k)) = \sum_{\pi \in \mathcal{P}_2[k]} N^{-2g+1}$$




$$\mathcal{E}\Big(\mathrm{Tr}(X^p)\mathrm{Tr}(X^q)\Big) - \mathcal{E}(\mathrm{Tr}(X^p))\mathcal{E}(\mathrm{Tr}(X^q)) = \sum_{\substack{\pi \in \mathcal{P}_2[p+q] \\ \text{connected}}} N^{-2g}$$



# *asymptotic Gaussianity – Johansson (1998)*

$$\mathcal{E}(\mathrm{Tr}(X^p)\mathrm{Tr}(X^q)\mathrm{Tr}(X^r)) - \mathcal{E}(\mathrm{Tr}(X^p)\mathrm{Tr}(X^q))\mathcal{E}(\mathrm{Tr}(X^r))$$

$$-\mathcal{E}(\mathrm{Tr}(X^p)\mathrm{Tr}(X^r))\mathcal{E}(\mathrm{Tr}(X^q)) - \mathcal{E}(\mathrm{Tr}(X^p))\mathcal{E}(\mathrm{Tr}(X^q)\mathrm{Tr}(X^r))$$

$$+2\mathcal{E}(\mathrm{Tr}(X^p))\mathcal{E}(\mathrm{Tr}(X^q))\mathcal{E}(\mathrm{Tr}(X^r)) = \sum_{\substack{\pi\in\mathcal{P}_2[p+q+r]\\ \text{connected}}} N^{-2g-1}$$

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$$\mathcal{E}(\mathrm{Tr}(X_N^{2k})) = \frac{1}{k+1}\binom{2k}{k}N + O(N^{-1})$$

$\therefore$ $\mathrm{Tr}\Big(X_N^{2k} - \frac{1}{k+1}\binom{2k}{k}I_N\Big)$ is asymptotically centered

$$\kappa_p\left(\mathrm{Tr}\Big(X_N^{2k} - \frac{1}{k+1}\binom{2k}{k}I_N\Big), \dots, \mathrm{Tr}\Big(X_n^{2k} - \frac{1}{k+1}\binom{2k}{k}I_N\Big)\right)$$

$$= \kappa_p\left(\mathrm{Tr}\Big(X_N^{2k}\Big), \dots, \mathrm{Tr}\Big(X_N^{2k}\Big)\right)$$

$$= \sum_{\substack{\pi\in\mathcal{P}_2[2kn]\\ \text{connected}}} N^{-2g(\pi)-p+2}$$

$\therefore$ $\mathrm{Tr}\Big(X_N^{2k} - \frac{1}{k+1}\binom{2k}{k}I_N\Big)$ is asymptotically Gaussian

— $T_n(\cos\vartheta) = \cos(n\vartheta)$, $C_n(x) = 2T_n(x/2)$, Chebyshev polynomials of the first kind

— $\{\mathrm{Tr}(C_n(X))\}_{n\geq 1}$ is asymptotically Gaussian and independent (*almost*)

# *complex Gaussian random matrices*

— $G : \Omega \to M_{M,N}(\mathbb{C})$, $G = (g_{i,j})$, $g_{i,j}$ is a complex Gaussian random variable with $\mathcal{E}(g_{i,j}) = 0$ and $\mathcal{E}(|g_{i,j}|^2) = 1/N$, $\{g_{i,j}\}$ is independent

— $X = G^*G$ is an $N \times N$ Wishart random matrix

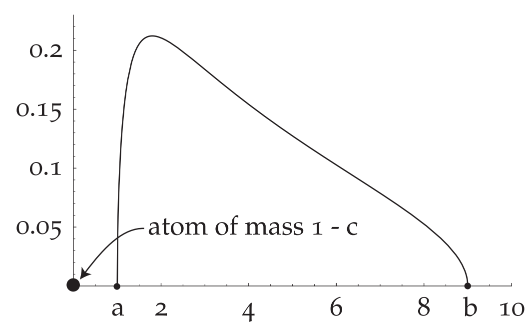
$$\mathcal{E}(\mathrm{Tr}(X^p)) = \sum_{\pi \in \mathfrak{S}_p} \left(\frac{M}{N}\right)^{\#(\pi)} N^{-2g(\pi)+1}$$

— if $\lim\limits_{N\to\infty} M/N = c > 0$, then the eigenvalue distribution of $X$ has a limiting distribution called the Marchenko-Pasteur distribution $\mu_c$

$$\frac{d\mu_c}{dt} = \frac{\sqrt{(b-t)(t-a)}}{2\pi t}$$

$$a = (\sqrt{c} - 1)^2$$

$$b = (\sqrt{c} + 1)^2$$




— $\{\Pi_n(x)\}_n$ — orthogonal polynomials for $\mu_c$

| | |
|---|---|
| $\Pi_0(x) = 1$ | $x^0 = \Pi_0(x)$ |
| $\Pi_1(x) = x - c$ | $x^1 = \Pi_1(x) + c\Pi_0(x)$ |
| $\Pi_2(x) = x^2 - (1+2c)x + c^2$ | $x^2 = \Pi_2 + (1+2c)\Pi_1 + (c+c^2)\Pi_0$ |

$$\int_a^b \Pi_m(x)\Pi_n(x)d\mu_c(x) = \delta_{m,n}c^m$$

$$x\Pi_n(x) = \Pi_{n+1}(x) + (1+c)\Pi_n(x) + c\Pi_{n-1}(x)$$

## *shifted Chebyshev polynomials — Haagerup-Thorbjørnsen (1999)*

— $U_n(\cos\vartheta) = \dfrac{\sin((n+1)\vartheta)}{\sin(\vartheta)}$ Chebyshev polynomials of the second kind

— $S_n(x) = U_n(x/2)$ monic & orthogonal for $\dfrac{\sqrt{4-t^2}}{2\pi}$

— $\Pi_n(x) =$
$\sqrt{c^n}\, S_n\left(\dfrac{x-(1+c)}{\sqrt{c}}\right) + \sqrt{c^{n-1}}\, S_{n-1}\left(\dfrac{x-(1+c)}{\sqrt{c}}\right)$

— when $t = (x-(1+c))/\sqrt{c}$,

$$c\sqrt{4-t^2}\,dt = \sqrt{(b-x)(x-a)}\,dx$$

— let $\Gamma_n(x) = \sqrt{c^n}\, C_n\left(\dfrac{x-(1+c)}{\sqrt{c}}\right)$ – orthogonal polynomials for the shifted arc-sine law

$$\Gamma_0(x) = 1$$

$$\Gamma_1(x) = x - (1+c)$$

$$\Gamma_2(x) = x^2 - 2(1+c)x1 + c^2$$

$$x\Gamma_n(x) = \Gamma_{n+1}(x) + (1+c)\Gamma_n(x) + c\Gamma_{n-1}(x)$$

# *asymptotic Gaussianity for the Wishart case — single matrix – Cabanal-Duvillard (2001)*

$$d_n = \begin{cases} -1 & n = 0 \\ 1 & n = 1 \\ (-1)^n(c-1) & n > 1 \end{cases} \qquad \Bigg| \qquad \widetilde{\Gamma}_n(x) = \Gamma_n(x) + d_n$$

— $\int_a^b \widetilde{\Gamma}_n(x)\, d\mu_c(x) = 0$

$\therefore$ $\lim_N \mathcal{E}(\mathrm{Tr}(\widetilde{\Gamma}_n(X)))$ can exist if $c' = \lim_N N\Big(\frac{M}{N} - c\Big)$ exists, in which case

$$\lim_N \mathcal{E}(\mathrm{Tr}(\widetilde{\Gamma}_n(X))) = (-1)^{n+1}c'$$

$\diamond$

Thm: $$\lim_N \kappa_2\Big(\mathrm{Tr}(\widetilde{\Gamma}_m(X)), \mathrm{Tr}(\widetilde{\Gamma}_n(X))\Big) = m\,\delta_{m,n}$$

$$\lim_N \kappa_p\Big(\mathrm{Tr}(\widetilde{\Gamma}_n(X)), \ldots, \mathrm{Tr}(\widetilde{\Gamma}_n(X))\Big) = 0$$

$\because$ $\kappa_p\Big(\mathrm{Tr}(\widetilde{\Gamma}_n(X)), \ldots, \mathrm{Tr}(\widetilde{\Gamma}_n(X))\Big) = O(N^{-p+2})$ by genus expansion, *(first part later)*

## *asymptotic independence and Gaussianity*

— for each $N$ let $X$ and $Y$ be *independent* Wishart distributed random matrices

— for $\vec{m} = (m_1, m_2, m_3, \ldots, m_r)$ and $\vec{n} = (n_1, n_2, n_3, \ldots, n_r)$ $r$-tuples of positive integers, let

$$S_{\vec{m},\vec{n}}(X, Y) = \Pi_{m_1}(X)\Pi_{n_1}(Y) \cdots \Pi_{m_r}(X)\Pi_{n_r}(Y)$$

— by the genus expansion (again)

$$\{\mathrm{Tr}(\widetilde{\Gamma}_i(X)), \mathrm{Tr}(\Gamma_j(Y)), \mathrm{Tr}(S_{\vec{m},\vec{n}}(X, Y))\}_{i,j,(\vec{m},\vec{n})}$$

are asymptotically Gaussian

**THM:** If we choose for each $(\vec{m}, \vec{n})$ one cyclic representative then $\{\mathrm{Tr}(\widetilde{\Gamma}_k(X)), \mathrm{Tr}(\widetilde{\Gamma}_j(Y)), S_{\vec{m},\vec{n}}(X, Y)\}_{k,j,(\vec{m},\vec{n})}$ are asymptotically independent and $\lim_N \kappa_2(S_{\vec{m},\vec{n}}(X, Y), S_{\vec{m},\vec{n}}(X, Y))$ is the number of cyclic equivalences of $(\vec{m}, \vec{n})$ with itself $\times\, c^{|\vec{m}|+|\vec{n}|}$.

$$\begin{array}{l|l}
\widetilde{\Gamma}_0(x) = 1 & x^0 = \widetilde{\Gamma}_0(x) \\
\widetilde{\Gamma}_1(x) = x - c & x^1 = \widetilde{\Gamma}_1(x) + c\,\widetilde{\Gamma}_0(x) \\
\widetilde{\Gamma}_2(x) = x^2 - 2(1+c)x & x^2 = \widetilde{\Gamma}_2(x) + 2(1+c)\widetilde{\Gamma}_1(x) \\
\qquad\qquad + c + c^2 & \qquad\qquad +(c+c^2)\widetilde{\Gamma}_0(x)
\end{array}$$

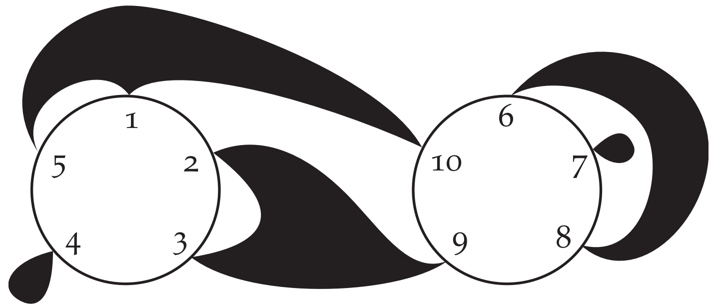
$$\widetilde{\Gamma}_n(x) = \sum_{k=0}^{n} q'_{n,k} x^k \qquad x^n = \sum_{k=0}^{n} q_{n,k} \widetilde{\Gamma}_k(x)$$

$$\widetilde{\Gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -c & 1 & 0 & 0 \\ c + c^2 & -2 - 2c & 1 & 0 \\ -c - c^3 & 3 + 3c + 3c^2 & -3 - 3c & 1 \end{pmatrix}$$

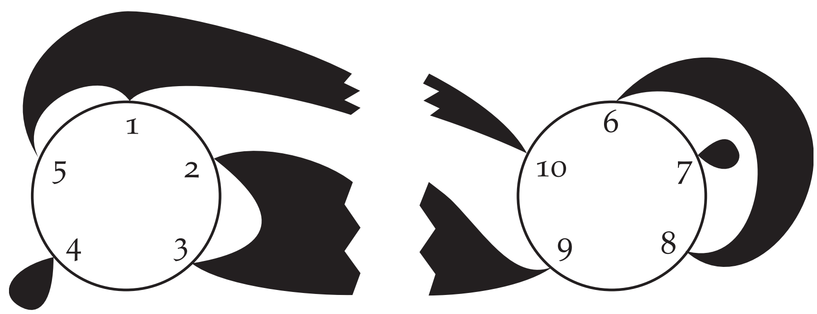
$$\widetilde{\Gamma}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ c & 1 & 0 & 0 \\ c + c^2 & 2 + 2c & 1 & 0 \\ c + 3c^2 + c^3 & 3 + 9c + 3c^2 & 3 + 3c & 1 \end{pmatrix}$$

— $q_{n,k} = \sum_\pi c^{|\pi|}$ where the sum is over the non-crossing circular half-permutations on $[n]$ with $k$ blocks sortant

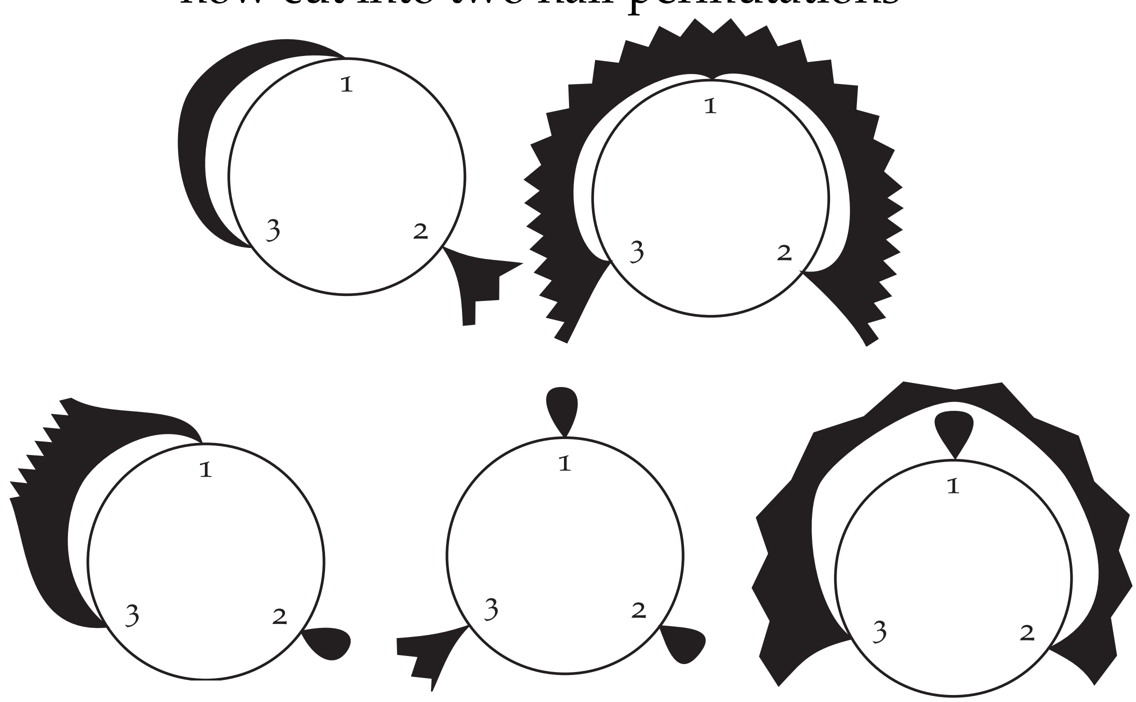
# *non-crossing half permutations*




a non-crossing annular permutation




now cut into two half permutations




15 half-permutations on [3] with one block sortant

— $S_{NC}(m,n)$ is the collection of non-crossing annular permutations on $[m,n]$; for a permutation $\pi$ we let $|\pi|$ be the number of cycles in $\pi$; for $A \subset S_{NC}(m,n)$ let $|A|_c = \sum_{\pi \in A} c^{|\pi|}$

$|S_{NC}(1)|_c = c$

$|S_{NC}(2)|_c = c + c^2$

$|S_{NC}(3)|_c = c + 3c^2 + c^3$

— $S_{NC}\binom{m}{k}, \binom{n}{j}$ is the collection of non-crossing $(m,n)$-annular permutations with $k$ blocks sortant on the first circle and $j$ blocks sortant on the second circle. *(empty if $k \neq j$)*

— $S_{NC}(\underline{k}, j)$ is the set of non-crossing $(k, j)$-annular permutations in which each of the $k$ points is a block sortant

— recall

$$\widetilde{\Gamma}_n(x) = \sum_{k=0}^{n} q'_{n,k} x^k \qquad x^n = \sum_{k=0}^{n} q_{n,k} \widetilde{\Gamma}_k(x)$$

$$q_{n,k} = \sum_{\pi} c^{|\pi|}$$

where the sum is over all half-permutations of $[n]$ with $k$ blocks sortant and $|\pi|$ is the number of complete blocks

$$\begin{aligned}
&\lim_{n} \kappa_2(\mathrm{Tr}(\widetilde{\Gamma}_m(X)), \mathrm{Tr}(\widetilde{\Gamma}_n(X))) \\
&= \sum_{k,j} q'_{m,k} q'_{n,j} \lim_{N} \kappa_2(\mathrm{Tr}(X^k), \mathrm{Tr}(X^j)) \\
&= \sum_{k,j} q'_{m,k} q'_{n,j} |S_{NC}(k,\, j)|_c \\
&= \sum_{\substack{k,j \\ r,s}} q'_{m,k}\; q'_{n,j} \left| S_{NC} \begin{pmatrix} k & j \\ r & s \end{pmatrix} \right|_c \\
&= \sum_{\substack{k,j \\ r,s}} q'_{m,k}\; q'_{n,j} q_{k,r} q_{j,s} |S_{NC}(\underline{r}, \underline{s})|_c \\
&= \sum_{r,s} \delta_{m,r}\; \delta_{n,s} |S_{NC}(\underline{r}, \underline{s})|_c \\
&= \sum_{r,s} \delta_{m,r}\; \delta_{n,s} \delta_{r,s} c^r r \\
&= \delta_{m,n} c^m m
\end{aligned}$$

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| | |
|---|---|
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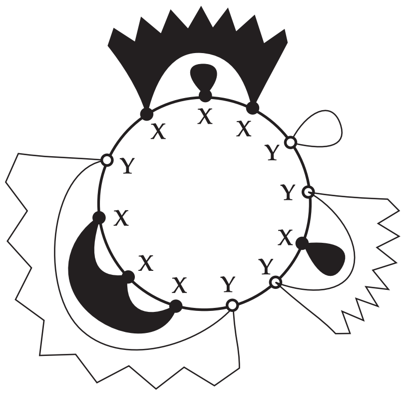
| $\Pi_1(x) = x - c$ | $x^1 = \Pi_1(x) + c\,\Pi_0(x)$ |
| $\Pi_2(x) = x^2 - (1+2c)x$ | $x^2 = \Pi_2(x) + (1+2c)\Pi_1(x)$ |
| $+\,c^2$ | $+(c+c^2)\Pi_0(x)$ |

$$\Pi_n(x) = \sum_{k=0}^{n} p'_{n,k} x^k \qquad x^n = \sum_{k=0}^{n} p_{n,k} \Pi_k(x)$$

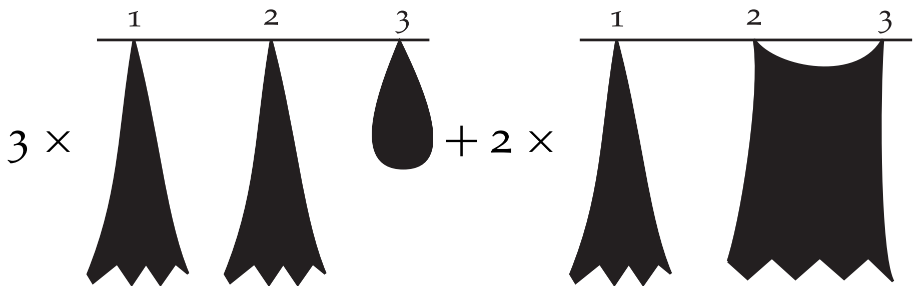
$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -c & 1 & 0 & 0 \\ c + c^2 & -2 - 2\,c & 1 & 0 \\ -c - c^3 & 3 + 3\,c + 3\,c^2 & -3 - 3\,c & 1 \end{pmatrix}$$

$$\Pi^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ c & 1 & 0 & 0 \\ c + c^2 & 1 + 2\,c & 1 & 0 \\ c + 3\,c^2 + c^3 & 1 + 5\,c + 3\,c^2 & 2 + 3\,c & 1 \end{pmatrix}$$

— $p_{n,k} = \sum_\pi c^{|\pi|}$ where the sum is over the non-crossing linear half-permutations on $[n]$ with k blocks sortant and $|\pi|$ is the number of complete blocks




$\mathrm{Tr}(X^3Y^2XY^2X^3Y)$




$$= 2 + 3\,c = p_{3,2}$$

$$\Pi_n(x) = \sum_{k=0}^{n} p'_{n,k} x^k \qquad x^n = \sum_{k=0}^{n} p_{n,k} \Pi_k(x)$$

$$p_{n+1,k} = c\; p_{n,k+1} + (1 + c)\, p_{n,k} + p_{n,k-1}$$

$$x\,\Pi_n(x) = \Pi_{n+1}(x) + (1 + c)\Pi_n(x) + c\,\Pi_{n-1}(x)$$