Mortality risk and real optimal asset allocation for pension funds

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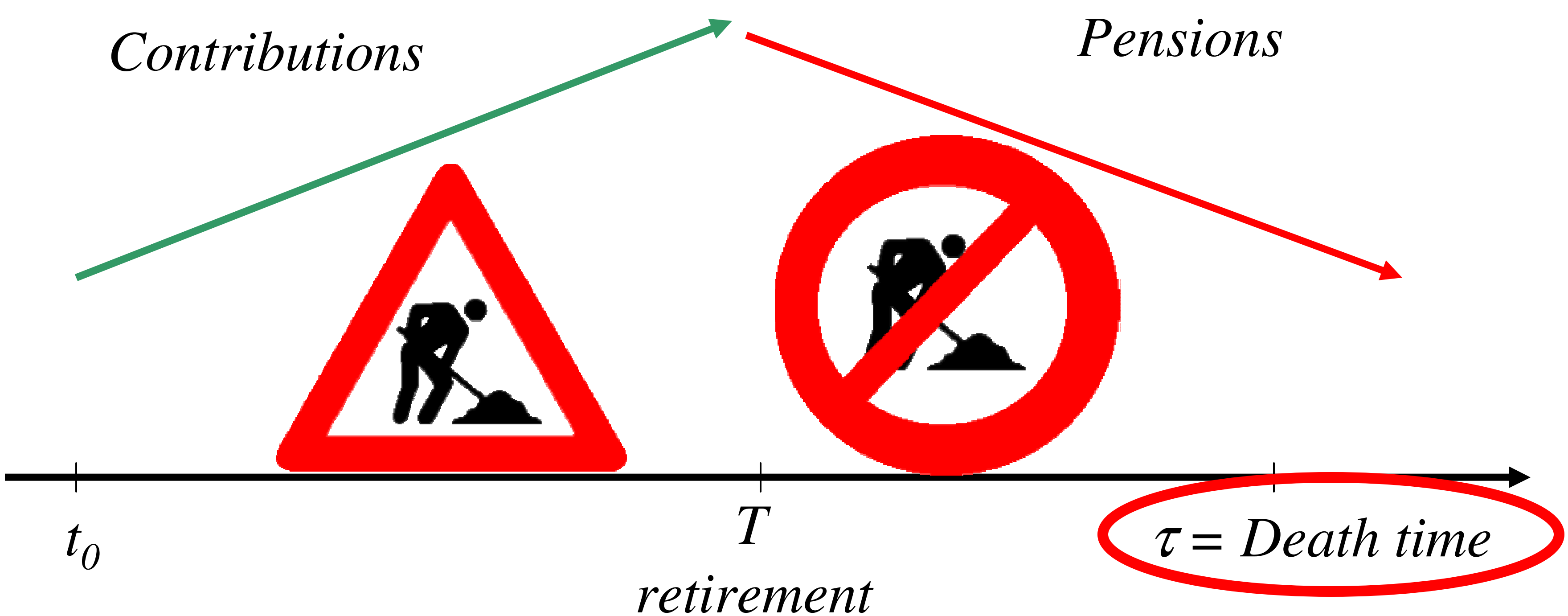
Why do we care about pension funds?

Year	**People over 60 (% of the world pop.)**
2000	10%
2025	15%
2050	21%

Peculiarities of a Pension Fund

- There exists a background risk (contributions and pensions)
- The final date (on an individual - fully funded - system) is uncertain

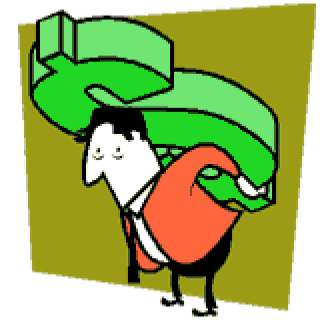


(Nominal) Financial Market

- There are *s* state variables: $dX=f(X,t)dt+g(X,t)'dW$;
- there are *n* risky assets: $dS=\mu(S,X,t)dt+\Sigma(S,X,t)'dW$;
- there is *1* riskless asset: $dG=r(X,t)Gdt$.

THE MARKET IS COMPLETE ($\exists\ \Sigma^{-1}$).

Inflation



- X contains the consumption price process

$$dP = P\mu_\pi(X,t)dt + P\sigma_\pi(X,t)'dW, \quad P(t_0)=1.$$

- Remark: dP/P = realized rate of inflation. Complete market dP/P inferred ONLY from price level itself.

Contributions and Pensions (1)

- *L* is the cumulated process of contributions and pensions:

$$dL=\mu_L(L,X,t)dt+\Lambda(L,X,t)'dW.$$

- Remark: *L* can be spanned.

Contributions and Pensions (2)

- *L* is different before and after *T*

$$\mu_L(L,X,t)=\mu_A(L,X,t)\ \text{¨}_{t<T}+\ \mu_D(L,X,t)(1-\text{¨}_{t<T})$$

$$\Lambda_L(L,X,t)=\Lambda_A(L,X,t)\ \text{¨}_{t<T}+\Lambda_D(L,X,t)(1-\text{¨}_{t<T})$$

Contributions and Pensions (3)

- Contributions and Pensions are linked: it must be profitable to underwrite a pension plan.

~ t0 — ,τ  ↲]

= ~ t0 — ,τ  ↲]

The Fund's objective function

- HARA preferences: $U(R)=(R+\Delta)^{1-\beta}/(1-\beta)$
- R = Fund's real wealth
- Δ = Prospective Mathematical Reserve

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Other features of the model

- Fund's profits are shared
  (according to a deterministic rule)
- Mortality density function is not specified
- All the computations are made in real terms

Results (1): the optimal portfolio

- The optimal portfolio is the sum of four componets:
- $w_1=\Sigma^{-1}(R_N\,\sigma_\pi - \mu_L+\Lambda'\sigma_\pi)$
- $w_2=\beta^{-1}(R_N + P\Delta)(\Sigma'\Sigma)^{-1}(\mu - Sr - \Sigma'\sigma_\pi)$
- $w_3=(\beta F)^{-1}(R_N + P\Delta)\Sigma^{-1}\Omega F_z$
- $w_4= - P\Sigma^{-1}\Omega\Delta_z$

Where z are ALL the state variables and Ω their diffusion. F is a “suitable” function.

Results (2): deterministic τ

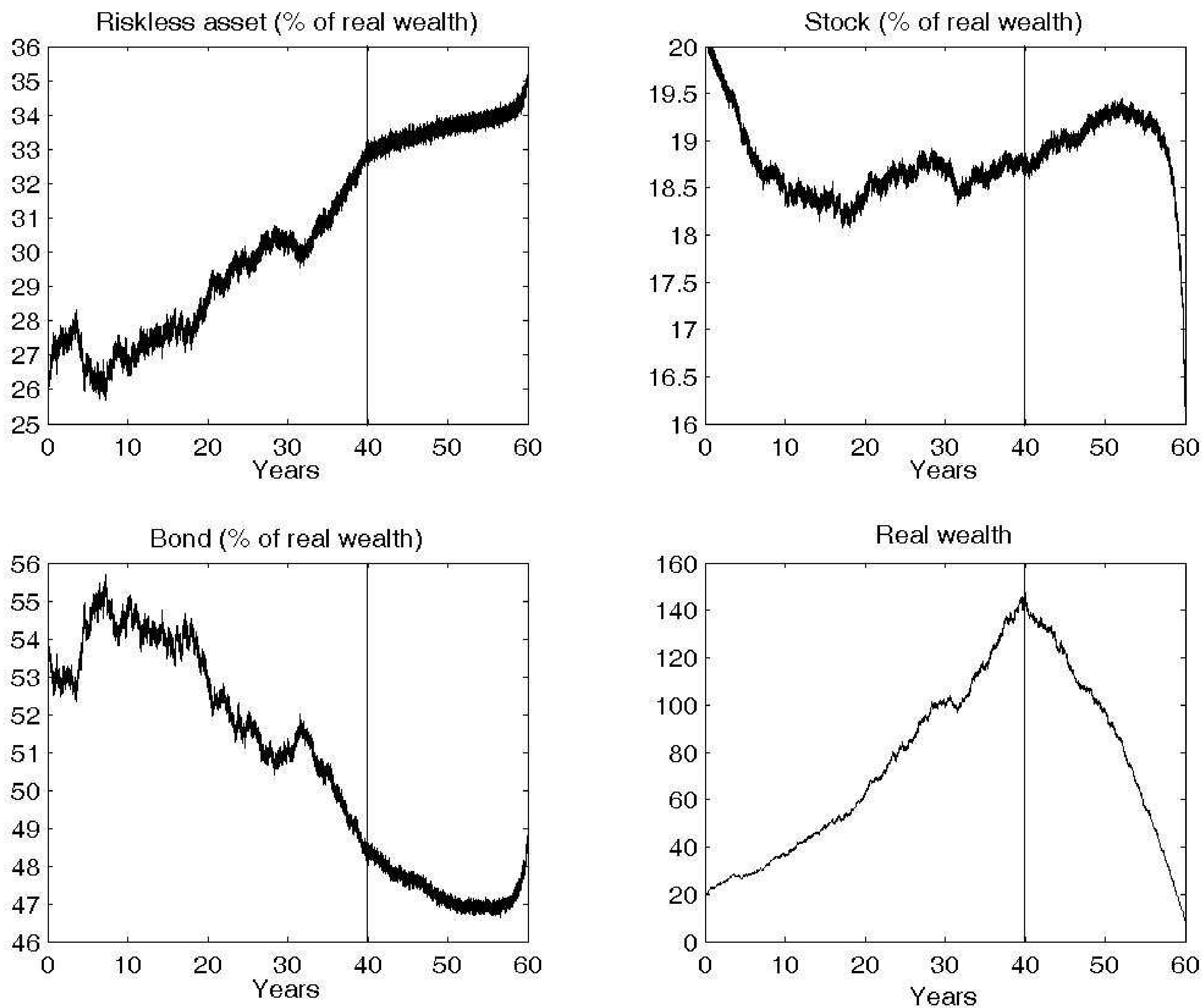
- When the death time is deterministic ($\tau = H$) the function F is given by h^{β} where h is the value of a *ZCB* whose return is given by the sum of:
- the real riskless interest rate ( )
- the square of the market price of risk
- $w_3 = (h)^{-1}(R_N + P\Delta)\Sigma^{-1}\Omega h_z$

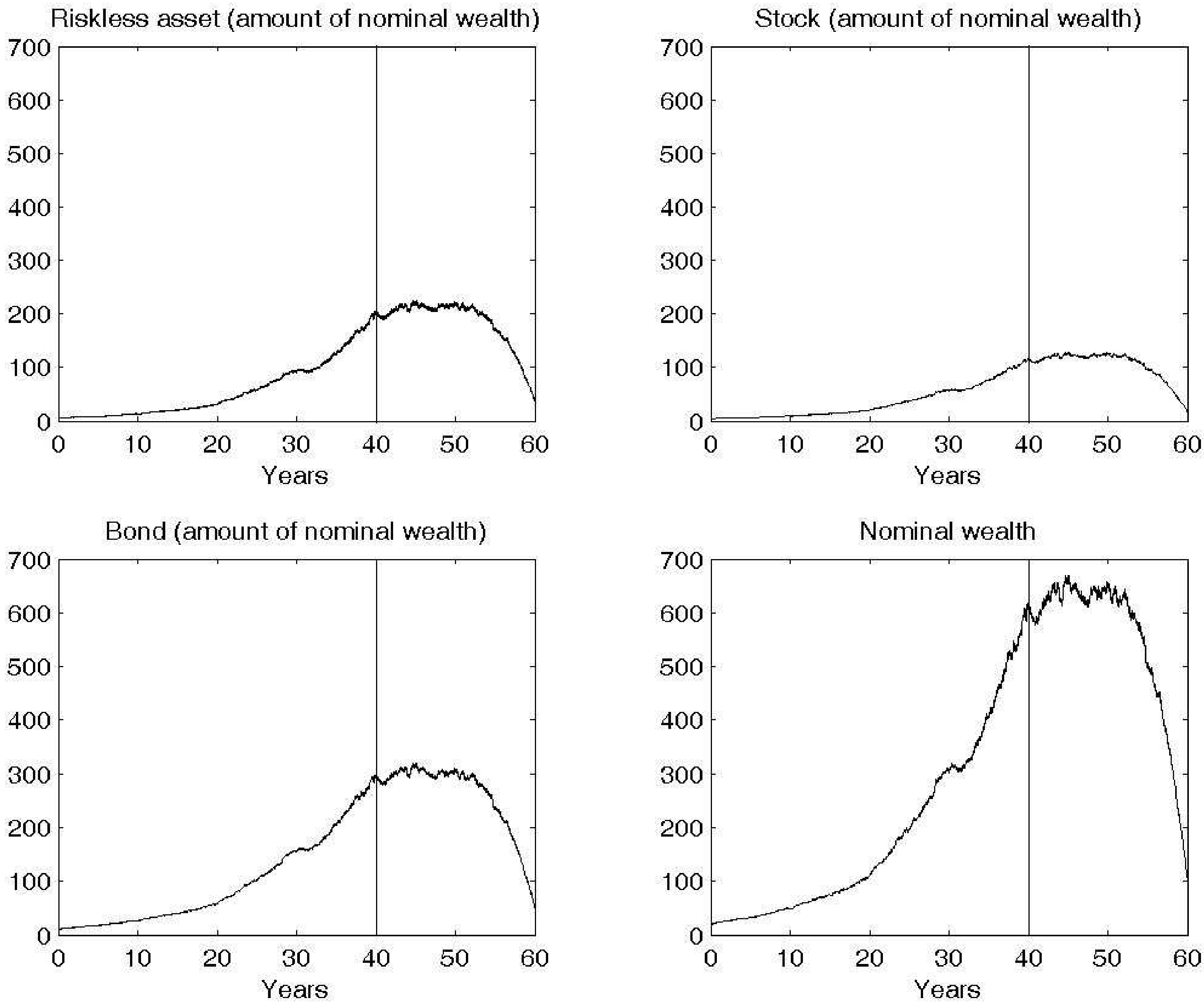
Results (3): an explicit solution

- If neither the market price of risk nor the real riskless interest rate depend on the state variables:
- $w_1 = \Sigma^{-1}(R_N \sigma_\pi - \mu_L + \Lambda' \sigma_\pi)$
- $w_2 = \beta^{-1}(R_N + P\Delta)(\Sigma'\Sigma)^{-1}(\mu - Sr - \Sigma' \sigma_\pi)$
- $w_3 = 0$
- $w_4 = -P\Sigma^{-1}\Omega\Delta_z$

Numerical simulation

- $dr = \eta\,(r^* - r)\,dt - \sigma_r\,dW_r$
- $dP = P\,(r+m_\pi)\,dt + P\sigma_{\pi r}\,dW_r + P\sigma_{\pi S}\,dW_S$
- $dG = G\,r\,dt$
- $dS = S\,(r+m_S)\,dt + S\sigma_{Sr}\,dW_r + S\sigma_S\,dW_S$
- $dB = B\,(r+a_K\sigma_r\chi)\,dt + Ba_K\sigma_r\,dW_r$
- $dL = (u\ \text{¨}_{t<T} - v\,(1-\ \text{¨}_{t<T}))\,dt$
- *Mortality law = Gompertz-Makeham*


Riskless asset (% of real wealth)
Stock (% of real wealth)
Bond (% of real wealth)
Real wealth
Years


Riskless asset (amount of nominal wealth)
Stock (amount of nominal wealth)
Bond (amount of nominal wealth)
Nominal wealth
Years
0
100
200
300
400
500
600
700
10
20
30
40
50
60