THE DYNAMICS OF MODULATED WAVE TRAINS

(in reaction-diffusion equations)

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* What's a 'wave train'?
  - Why do they appear in families?
  - Examples (cGL, FH-N)

* Modulated wave trains
  - (Weak) shocks / defects ⟷ 'SINKS'

* The derivation of Burgers equation as PHASE DIFFUSION EQUATION
  - Stability of the wave train
  - 'Ansatz'
  - Solvability

* VALIDITY

* Shocks & defects
  - In Burgers eq
  - In the full system ⟷ group speed

WHAT'S A WAVE TRAIN?

R-d system : $u_t = D u_{xx} + F(u)$

$u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^N$

Note : $x \in \mathbb{R}$

D = diffusion matrix

$F(u)$: Reaction / nonlinearities

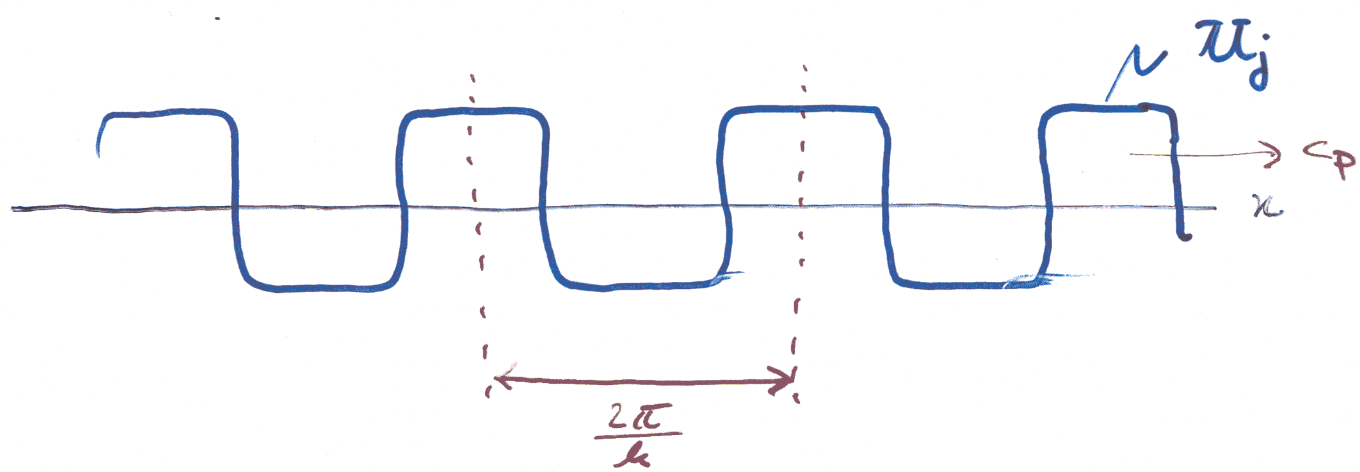
$u(x,t) = u_0(\omega t - kx) = u_0(\phi)$

$\boxed{\text{a wave train}}$

$-k(x - \frac{\omega}{k} t)$

2π-per in ϕ

$\boxed{c_p = \frac{\omega}{k} \quad \text{"phase speed"}}$



CLAIM : In general there is a 1-parameter family of wave trains

$$\mathcal{U}(x,t) = u_0(\omega(k)t - kx;\, k)$$

WHY ? Eq for $u(\phi)$ $(\phi = \omega t - kx)$

$$\boxed{k^2 D u_{\phi\phi} - \omega u_\phi + F(u) = 0}$$

+ per. b.c. on $(0, 2\pi)$

a 2N-dim ODE

[redacted]

Let's vary k : $\frac{\partial}{\partial k}$ evaluated at u_0, ω_0, k_0

$\omega|_{k=k_0}$

$$u_k = \frac{\partial u}{\partial k}\Big|_{k=k_0}$$

$$\omega_k = \frac{\partial \omega}{\partial k}\Big|_{k=k_0}$$

$$\mathcal{L}_0 u_k = \left[k_0^2 D \partial^2_\phi - \omega_0 \partial_\phi + F'(u_0)\right] u_k$$

$$= \omega_k \partial_\phi u_0 - 2k_0 \partial^2_\phi u_0$$

Operator on $L_2(0,2\pi)$ associated to lin. stab. of $u_0(\phi)$

ASSUME : generalized kernel of $\mathcal{L}_0$ is 1-d

$\Rightarrow$ spanned by $\partial_\phi u_0$

translation invariance

Def Adjoint operator $\mathcal{L}_{ad}$ ($\mathcal{L}_0$ in general not self-adj.)

Kernel of $\mathcal{L}_{ad}$ spanned by u_{ad}

normalization : $\langle \partial_\phi u_0 , u_{ad} \rangle_{L_2} = 1$

Now $\mathcal{L}_0 u_k = b$ ("inhomogeneous")

$\Rightarrow$ $\boxed{\text{solvability criterion} : \langle b, u_{ad} \rangle = 0}$

(since : $\langle b, u_{ad} \rangle = \langle \mathcal{L}_0 u_k , u_{ad} \rangle$
$= \langle u_k , \mathcal{L}_{ad} u_{ad} \rangle$)

existence $\frac{\partial u}{\partial k}$

$\Rightarrow$ $\boxed{\langle w_k \partial_\phi u_0 - 2 k_0 D \partial^2_\phi u_0 , u_{ad} \rangle = 0}$

$$w_k = 2 k_0 \langle D \partial^2_\phi u_0 , u_{ad} \rangle$$

$\boxed{w_k = \frac{\partial \omega}{\partial k} = c_g : \text{GROUP SPEED}}$

Thus : for this w_k , $\mathcal{L} u_k = b$ can be solved
(uniquely up to transl)

$c \, \partial_\phi u_0$

With this assumption on $\mathcal{L}_0$ there is 1- parameter fam.

Not.n $\boxed{\omega = \omega_{nl}(k) \text{ "nonlinear dispersion"}}$

EXAMPLE : complex Ginzburg – Landau

$$\boxed{A_t = (1+i\alpha)A_{xx} + A - (1+i\beta)|A|^2 A}$$

$$A = u_1 + i\,u_2 \to \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^2$$

'near equilibrium'

α, β : parameters

↳ 'the underlying system'

Wave trains : $A_0(x,t) = R\, e^{i(\omega t - kx)}$

$$\to \begin{cases} R^2 + k^2 = 1 \implies R = R(k) = \sqrt{1-k^2} \\ \omega = \omega(k) = \beta - (\alpha-\beta)k^2 \end{cases}$$

$$\boxed{\omega_{nl}(k) = \beta - (\alpha-\beta)k^2}$$

Note $\omega_{nl} \equiv 0$ in RGL – case ($\alpha = \beta = 0$)

Note $A_0(\omega t - kx; k)$ is stable if

$$1 + \alpha\beta > 0$$

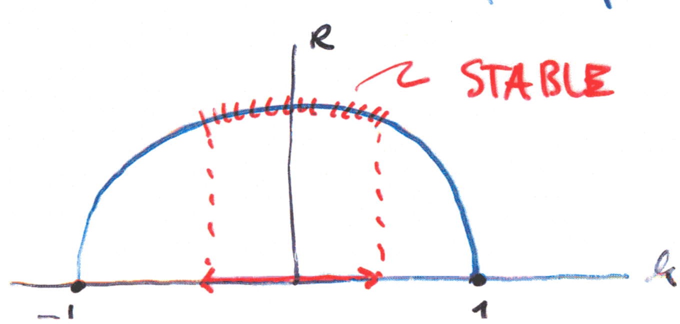
&

$$k^2 < \frac{1+\alpha\beta}{3 + \alpha\beta + 2\beta^2}$$

Benjamin – Feir

Eckhaus,

Newell, ...



FitzHugh – Nagumo $\longleftrightarrow$ far-from-equilibrium

$$\begin{cases} u_t = u_{xx} + u(1-u)(u-a) + w \\ w_t = \varepsilon(u - bw) \end{cases}$$

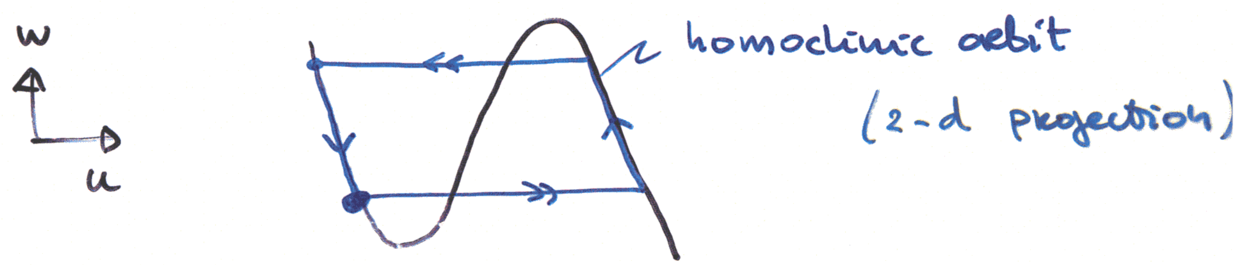
$\rightarrow$ $\exists$ traveling LOCALIZED pulse (& it is stable [Jones])

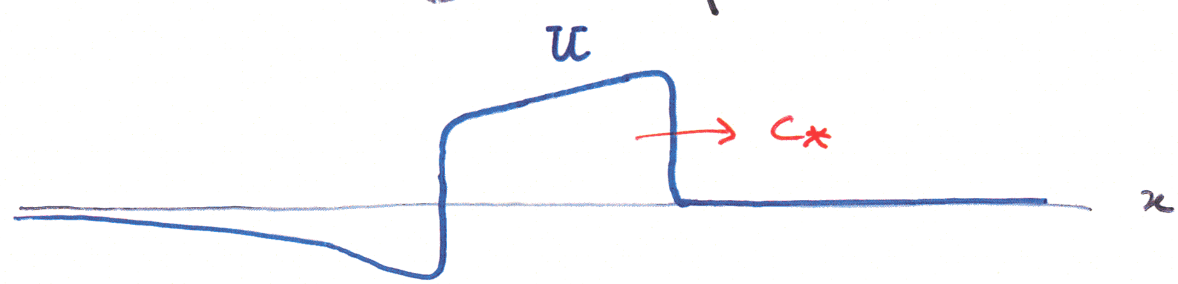
$\updownarrow$

well-defined c_*

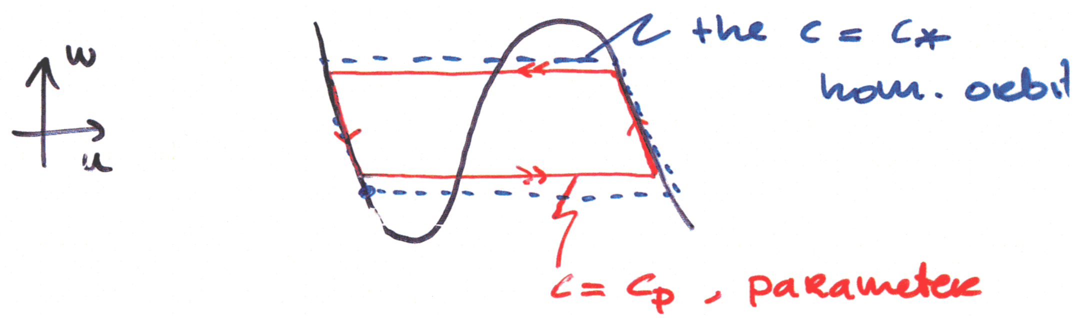
[Hastings] For c near c_* there exist a family of traveling wave trains

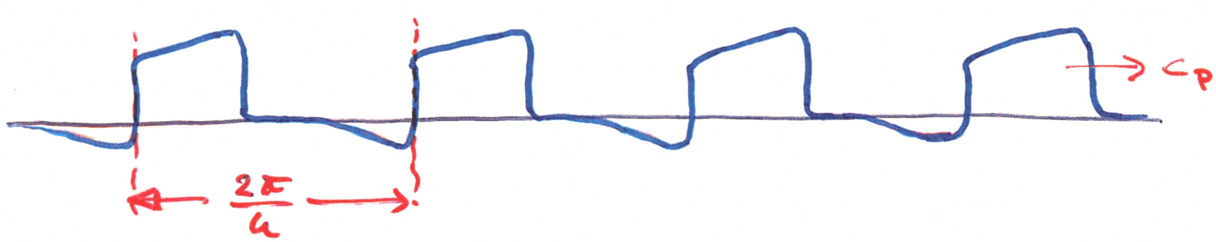
PULSE





WAVE TRAIN





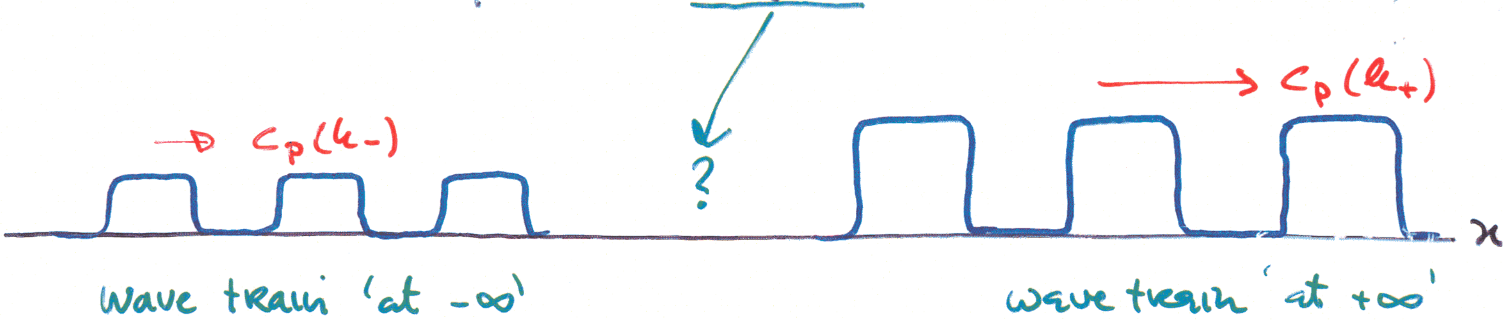
Stability: Eszter & Gardner

MODULATED WAVE TRAINS

$\Updownarrow$

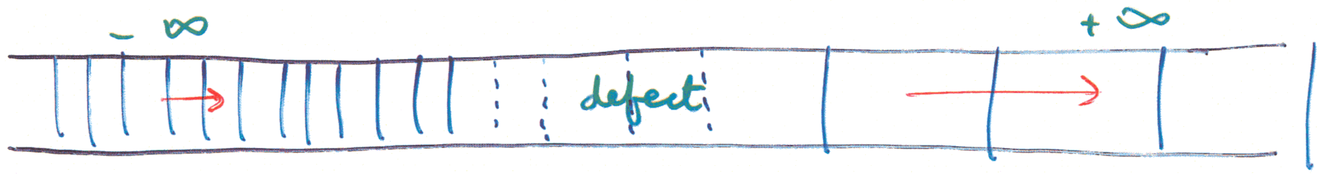
DYNAMICS (of full PDE) IN, OR NEAR, THE FAMILY OF TRAVELING WAVE TRAINS

For instance: Q/ Can 'defects' exist (& be stable)?



" heteroclinic orbit between wave train

From above / semi-2-D



ANSWER: Yes, but it depends on the $\boxed{\text{GROUP SPEEDS}}$

i.e. 'Defects' or 'weak shocks' exist & are stable

if $c_- = \dfrac{\partial \omega_{nl}}{\partial k}(-\infty) > c_+ = \dfrac{\partial \omega_{nl}}{\partial k}(+\infty)$

&

defect travels with speed $c^* \in (c_-, c_+)$

$\Updownarrow$

SINKS

"information is transported into the defect"

THE DERIVATION OF BURGERS EQUATION

THE DYNAMICS OF MODULATED WAVE TRAINS

I STABILITY OF A WAVE TRAIN

$$\boxed{U_t = D U_{xx} + F(U)}$$

pick a $k_0, \omega_0 = \omega_{nl}(k$

in $\phi = \omega_0 t - k_0 x$ frame

(that travels along with speed c_p)

$$\boxed{U_t = k_0^2 D U_{\phi\phi} - \omega_0 U_\phi + F(U)}$$

linearize around $u_0(\phi; k_0)$: $U = u_0 + u$

$$\boxed{u_t = k_0^2 D u_{\phi\phi} - \omega_0 u_\phi + F'(\underbrace{u_0(\phi)}_{})\, u}$$

2π-periodic in ϕ

$\Updownarrow$

FLOQUET THEORY

$$u \longleftrightarrow e^{-\frac{\nu}{k_0}\phi} \underbrace{v(\phi)}_{2\pi\text{-periodic}} e^{\lambda t}$$

$\nu \in i\mathbb{R}$: Floquet exponent

$\lambda_{lin}(\nu)\, v$

$\parallel$

$$\boxed{\mathcal{L}_\nu v = \left[k_0^2 D \left(\partial_\phi - \frac{\nu}{k_0}\right)^2 - \omega_0 \left(\partial_\phi - \frac{\nu}{k_0}\right) + F'(u_0(\phi)) \right] v}$$

NOTE : $\boxed{\mathcal{L}_0 = \mathcal{L}_0 \longleftrightarrow \text{existence pb.}}$

Assumption on w_0 $\Rightarrow$ $\exists$ (analytic) curve of eigenvalues $\lambda = \lambda_{lin}(\nu)$

$\Updownarrow$

near $\lambda = 0$

(gen) kernel is spanned by $\partial_\phi u_0$

same argument as in ex. th.

$$\boxed{\lambda = \lambda_{lin}(\nu) \quad (\nu \in i\mathbb{R}) : \text{the LINEAR dispersion}}$$ relation

Derivative with respect to ν evaluated at $\nu = 0$

(note: $v|_{\nu=0} = \partial_\phi u_0$)

$$\mathcal{L}_0 v_\nu = \left(\lambda'_{lin}(0) - \frac{w_0}{k_0} \right) \partial_\phi u_0 + 2k_0 D \partial_{\phi\phi} u_0$$

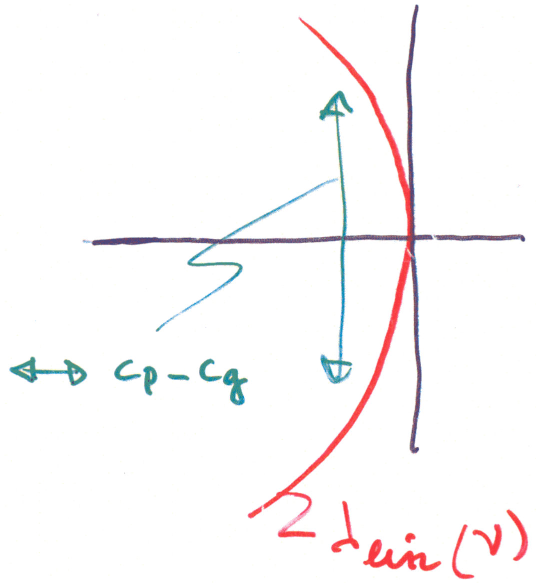
$$\left(\text{Ex} \quad \mathcal{L}_0 u_k = \omega'_{nl}(k_0)\, \partial_\phi u_0 - 2k_0 D \partial_{\phi\phi} u_0 \right)$$

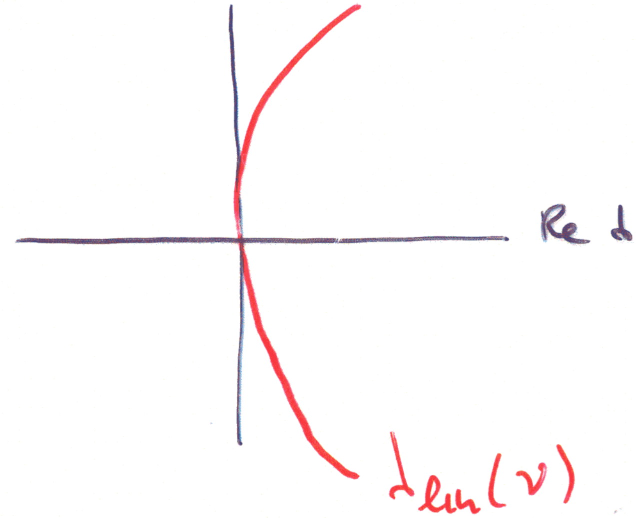
SOLVABILITY : $\langle \text{inhomogeneous}\ u_{ad} \rangle_{L_2} = 0$

$$\underbrace{\lambda'_{lin}(0) - \frac{w_0}{k_0}}_{=\, c_p(k_0)} = \underbrace{-2k_0 \langle D \partial_{\phi\phi} u_0, u_{ad} \rangle}_{=\, -\omega'_{nl}(k_0)}$$

$$\Rightarrow \boxed{\lambda'_{lin}(0) = c_p(k_0) - c_g(k_0)}$$

& similar relations between $\lambda''_{lin}(0)$





STABLE : $\lambda''_{lin}(0) > 0$

UNSTABLE : $\lambda''_{lin}(0) < 0$

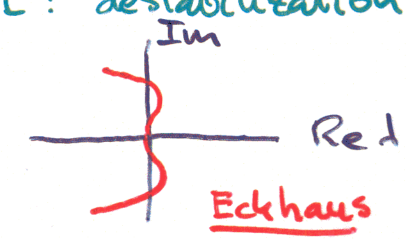
$\Updownarrow$

(at least "locally")

$\Updownarrow$

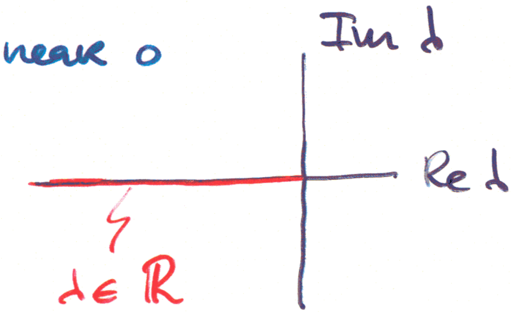
λ's near 0

NOTE : destabilization



'dissipative' / STABLE

NOTE : $c_p = c_g$

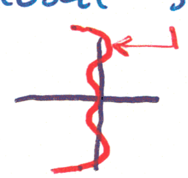


Assumption $\quad \omega''_{nl}(k_0) \neq 0$ & $\lambda''_{lin}(0) > 0$

Ex: cGL, not RGL (later)

Assumption $\quad$ For ANY $\nu \in i\mathbb{R}$, $\nu \notin ik_0\mathbb{Z}$, we have

$\mathrm{Re}\,\lambda < 0$ $\quad$ ("global stability")



NOTE : 'In practice' very hard ⟷ van der Ploeg (in Greiner–Meinhardt

for a given model

II: MODULATED WAVES: THE ANSATZ

Wave train: $U = u_0\left(\omega_{nl}(k)t - kx - \Phi\ ;k\right)$, k par.

phase, a priori a free constant

Assume Φ is allowed to evolve slowly in x & t,

so that $U(x,t)$ remains in / near the fam. of wave trs.

in a small nbhd of original k

Introduce $\delta \ll 1$: $k \to k+\delta$

'ARTIFICIAL'

$$\Rightarrow\ u_0\left(\omega(k)t - kx\ ;k\right) \longrightarrow$$

a nearby wave tra

$$u_0\left(\omega(k+\delta)t - (k+\delta)x\ ; k+\delta\right)$$

$$= u_0\left(\left[\omega(k)t + \delta\,\omega'(k) + O(\delta^2)\right]t - [k+\delta]x\ ;\ \cdot\right)$$

$$= u_0\Big(\underbrace{[\omega(k)t - kx]}_{\delta=0} - \delta\underbrace{[x - c_g(k)t]}_{X\ \text{slow space}} + \underbrace{\delta^2 t}_{T\ \text{slow time}}[\cdots]\Big)$$

$$\Phi = \Phi(X,T)$$

$$= \Phi\left(\delta(x-c_g t), \delta^2 t\right)$$

"natural (slow) x & t scales of phase"

Note : $\Phi(X,T)$ induces change in k ("vice versa")

$$kx + \Phi = kx + \Phi_0 + X\Phi_X + O(\delta^2) \quad \leftarrow X^2, T$$

$$= \underbrace{(k + \delta\Phi_X)}_{\text{change in } k} x + \Phi_0 - c_g t \delta + O(\delta^2)$$

$\uparrow$ ~~related~~ induced change in $\omega_{nl}(k)$

⇒ ANSATZ

$X = \delta(x - c_g t)$, $T = \delta^2 t$

$$\boxed{u(x,t) = u_0\left(\omega_{nl}(k)t - kx - \Phi(X,T);\, k + \delta\Phi_X\right) + \delta^2 \underbrace{u_2(\omega_{nl}t - kx;\, X, T)}_{\text{"Remainder"}}}$$

SUBSTITUTE THIS ANSATZ INTO $u_t = Du_{xx} + F(u)$

Note : $\frac{\partial}{\partial t} u = \frac{\partial}{\partial t}\left[u_0(\phi - \Phi(X,T);\, k + \delta\Phi_X(X,T)) + \delta^2 u_2(\phi, X, T)\right]$

$$= \partial_\phi u_0\left[\omega + \delta c_g \Phi_X - \delta^2 \Phi_T\right] + \partial_k u_0\left[-\delta^2 c_g \Phi_{XX}\right]$$

$$+ \delta^2 \partial_\phi u_2\,[\omega]$$

$$+ O(\delta^3)$$

→ ETCETERA ←

III THE SOLVABILITY CONDITION

$$0 = -\mathcal{U}_t + D\,\mathcal{U}_{xx} + F(\mathcal{U})$$

$$= \cdots$$

Existence pb.

$$= -\omega\, \partial_\phi u_0 + k^2 D\, \partial^2_\phi u_0 + F(u_0)$$

Exist. pb ($\exists$ fam.)

$$+ \delta \left[\mathcal{L}_0\, \partial_k u_0 + 2k D\, \partial^2_\phi u_0 - c_g\, \partial_\phi u_0 \right]$$

"correct speed"

$$+ \delta^2 \Big[\mathcal{L}_0 u_2 + \Phi_T [\partial_\phi u_0] + \Phi_{XX} [\ldots] + (\Phi_X)^2 [\ldots] \Big] + O(\delta^3)$$

$$\Rightarrow \boxed{\mathcal{L}_0 u_2 = -\Phi_T[\ldots] - \Phi_{XX}[\ldots] - (\Phi_X)^2[\ldots]}$$

$\mathcal{L}_0$ has nontrivial kernel $\partial_\phi u_0$ $\Longleftrightarrow$

$\mathcal{L}_{ad}$ has kernel spanned by u_{ad}

& $\langle \partial_\phi u_0, u_{ad} \rangle_{L_2} = 1$

$\mathcal{L}_0 u_2 = b \Longrightarrow$ only solvable if

$\langle b, u_{ad} \rangle_{L_2} = 0$

$$\Phi_T = \langle -[\cdots], u_{ad} \rangle \, \Phi_{XX} + \langle -[\cdots], u_{ad} \rangle (\Phi_X)^2$$

$$\Big\downarrow \text{ identities for } d'_{lin}(0), d''_{lin}(0) \text{ \& } \omega'_{nl}(k), \omega''_{nl}(k)$$

$$\boxed{\Phi_T = \tfrac{1}{2} d''_{lin}(0)\, \Phi_{XX} - \tfrac{1}{2} \omega''_{nl}(k) (\Phi_X)^2}$$

PHASE DIFFUSION EQUATION

Def : $q(X,T) = \Phi_X(X,T)$

"wave number modulation"

$$\Rightarrow \quad \boxed{q_T = \tfrac{1}{2} d''_{lin}(0)\, q_{XX} - \omega''_{nl}(k)\, q\, q_X}$$

BURGERS EQUATION (!)

Recall : we assumed $d''_{lin}(0) > 0 \leftrightarrow$ STABILITY

&

$\omega''_{nl}(k) \neq 0$

Th (For each $C_1>0$, $L>0$ & $R \in [0,2)$ $\exists\ \delta_0, C_0>0$ st the following holds:)

Let $q(X,T)$ be a solution of

$$q_T = \frac{1}{2} d''(0)\, q_{XX} - \omega''(k)\, q\, q_X$$

s.t. $\sup_{[0,T_0]} \| q(\cdot,T) \|_{\mathcal{X}} \leq C_1$

define $\Phi(X,T) = \int_0^X q(Y,T)\,dY$ &

$$U_{app.}(x,t) = u_0\big(\omega t - kx - \Phi(X,T)\ ;\ k + \delta\, q(X,T)\big)$$

Then there exists a sol. U of $U_t = D U_{xx} + F(U)$

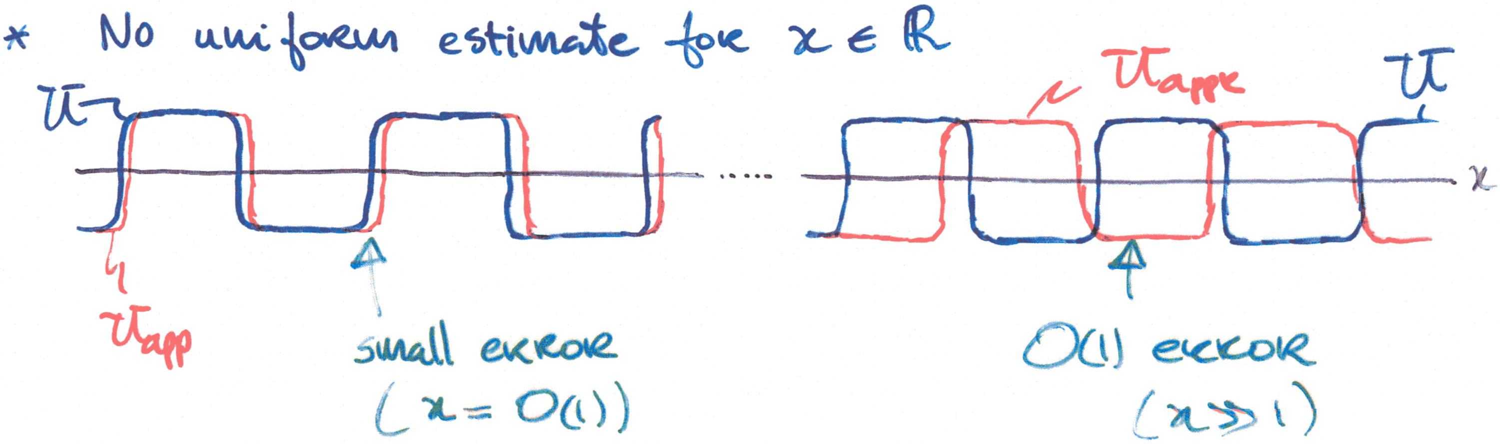
s.t. $\sup_{t\in[0,T_0/\delta^2],\ |x|<L/\delta^R} \left| U(x,t) - U_{appr}(x,t) \right| \leq C_0\, \delta^{3-R}$

for each $\delta \in (0,\delta_0)$

Note * Validity on t-scale Burgers & beyond its x-scale ($R = 3/2$)

* "gronwall" $\rightarrow$ $O(\frac{1}{\delta})$ t-scale, useless

* No uniform estimate for $x \in \mathbb{R}$



Burgers $\quad q_T = \frac{1}{2} d'' q_{XX} - \omega'' q q_X$

Stationary fronts ("we're already traveling with c_g") (*)

$\rightarrow$ if K s.t. sign K = sign ω'' then

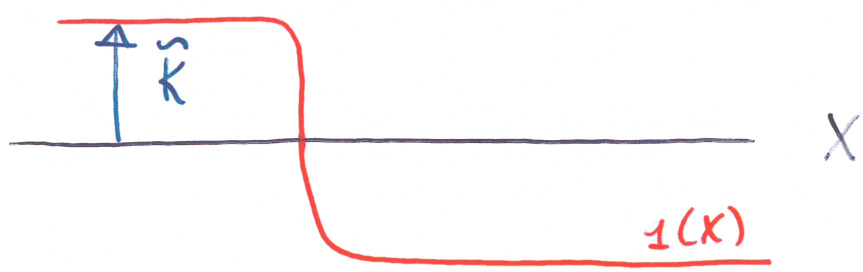
$$q(X) = \alpha \tanh \beta X$$

def $\tilde{K} > 0$

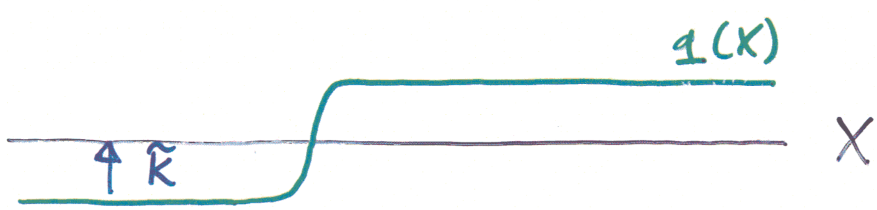
$$\text{with } \alpha = -\operatorname{sign}\omega'' \sqrt{\frac{K}{\omega''}}$$

$$\beta = \frac{1}{d''}\sqrt{K\omega''}$$

$\omega'' > 0$



$\omega'' < 0$



$\boxed{\text{All fronts } (\exists \text{ families}) \text{ are (nonl) stable}}$

$\Updownarrow$

$\tilde{K} > 0$

(*) Traveling front $\Longleftrightarrow$ stationary front

for Burgers eq 'centered' around $\tilde{k}_0 = k_0 + O(\varepsilon)$

slightly different c_g

$$\omega'' \gtrless 0 \Rightarrow g \xrightarrow{X \to \pm\infty} \mp \tilde{K}, \pm \tilde{K}$$

$$\Rightarrow \Phi \xrightarrow{X \to \pm\infty} \mp \hat{K} X, \pm \hat{K} X$$

$(+\Phi_0 t \ldots)$

$$\Rightarrow \omega'' > 0 \Rightarrow U \xrightarrow[\underbrace{-\Phi}]{X \to \pm\infty} u_0(\omega t - kx \pm \tilde{K} X; k \mp \delta \hat{K})$$

$$\omega'' < 0 \Rightarrow U \xrightarrow{X \to \pm\infty} u_0(\omega t - kx \mp \hat{K} X; k \pm \delta \hat{K})$$

$\delta(x - c_g t)$

$\Rightarrow \omega'' \gtrless 0$: shock / defect connects wave train

$$\text{"}\underline{\text{at } -\infty}\text{"}: u_0(\underbrace{[\omega \pm \delta c_g \hat{K}]}t - [k \pm \delta \hat{K}]x; k \pm \delta \hat{K})$$

$\parallel$ (approx)

$$u_0(\underbrace{\omega(k \pm \delta \hat{K})}t - (k \pm \delta \hat{K})x; k \pm \delta \hat{K})$$

$$\text{to "}\underline{\text{at } +\infty}\text{"}: u_0(\omega(k \mp \delta \hat{K})t - (k \mp \delta \hat{K})x; k \mp \delta \hat{K})$$

<u>THUS</u> $\omega'' \gtrless 0 \Rightarrow c_g = \omega'$ increasing / decreasing as fct. of k

$$\Rightarrow \boxed{c_g(-\infty) > c_g(k) > c_g(+\infty)}$$

$$\omega'' \gtrless 0 : \left\{ \begin{array}{lcl} k + \delta \hat{K} & \longrightarrow & k - \delta \hat{K} \\ k - \delta \hat{K} & \longrightarrow & k + \delta \hat{K} \\ -\infty & & +\infty \end{array} \right.$$

$\hat{K}$ par $\hat{K} > 0$

$\Longrightarrow$ Burgers equation gives nearly exact

approximation (on finite t & x - domains !)

of SHOCK / DEFECTS with !!

$$\boxed{c_{\text{group at } -\infty} > c_{\text{defect}} > c_{\text{group at } +\infty}}$$

$\Updownarrow$

SINKS

$k_\pm = k \mp \delta \hat{K}$

$k_\pm = k \pm \delta \hat{K}$

(Rankine - Hugoniot)

Note : $c_g(k) = c_{\text{defect}} = \dfrac{\omega(k_+) - \omega(k_-)}{k_+ - k_-} = \ldots$

where $k_\pm = k \Big|_{\text{at } \pm\infty}$

CAN WE PROVE THE EXISTENCE & STABILITY OF SUCH SHOCKS / DEFECTS FOR THE FULL EQUATION (& FOR $T \geq 0$, $X \in \mathbb{R}$) ?

UNBOUNDED !

NOTE : BURGERS GIVES DYNAMICS OF ALL POSSIBLE PHASE EVOLUTIONS (bounded T & X)

$\Updownarrow$

NOW : EX & STAB OF A SPECIAL KIND OF BEHAVIOR

TRAVELING TIME-PERIODIC SOLUTIONS

$$\mathcal{U}(x,t) = u_*(\underbrace{x - c_* t}_{\text{trav. coord } \xi}, \underbrace{\omega_* t}_{\tau})$$

τ, 2π-per.

u_* must be bi-asymptotic to wave trains

$$u_*(x - c_* t, \omega_* t) \xrightarrow{x \to \pm\infty} u_0(\omega_\pm t - k_\pm x ; k_\pm)$$

$x = \xi + c_* t$

More precisely :

$$\left\| u_*(\xi, \cdot) - u_0\left(\frac{\omega_\pm - c_* k_\pm}{\omega_*} \cdot - k_\pm \xi ; k_\pm\right)\right\|_{H^1(0,2\pi)} \xrightarrow{\xi \to \pm\infty} 0$$

$\tau = \omega_* t$

$\Rightarrow$ JUMP CONDITIONS :

$$\boxed{u_0 \text{ must have frequency } 1 \text{ in } \tau \ !}$$

$$\frac{\omega_+ - c_* k_+}{\omega_*} = 1 = \frac{\omega_- - c_* k_-}{\omega_*}$$

$$\Rightarrow \quad \boxed{c_* = \frac{\omega_+ - \omega_-}{k_+ - k_-} \quad (\longleftrightarrow \text{Burgers})}$$

$$\omega_* = \omega_\pm - c_* k_\pm$$

Th (Assumptions on d'', ω'', stab.)

For all wave numbers k_- & k_+ close to k_0

s.t. $\boxed{c_g^- = \omega'(k_-) > c_g^+ = \omega'(k_+)}$

there exists a $\left\{\begin{matrix}\text{weak viscous shock} \\ \text{defect} \\ \text{sink}\end{matrix}\right\}$ solution

$$\mathcal{U}(x,t) = u_*(x - c_* t, \omega_* t) \quad \text{with}$$

$$c_* = \frac{\omega(k_+) - \omega(k_-)}{k_+ - k_-} \in (c_g^+, c_g^-)$$

$$\omega_* = \omega(k_\pm) - k_\pm c_*$$

u_* is unique, up to translations in x & t

MOREOVER

Th $u_*(x - c_* t, \omega_* t)$ is nonlinearly, asymptotically stable w.r.t. perturbations in X (~ locally H^1)

Proof: SPATIAL DYNAMICS & CENTER MANIFOLDS

↑ Kirchgässner

$$\begin{cases} \mathcal{U}_\xi = \mathcal{V} \\ \mathcal{V}_\xi = -D^{-1}\left[-\omega_* \partial_\tau \mathcal{U} + c_* \mathcal{V} + F(\mathcal{U})\right] \end{cases} \longleftrightarrow \vec{\mathcal{U}}_\xi = \mathcal{L}(\mathcal{U}) + N(\mathcal{U})$$

ill-posed dyn. syst

APPLICATIONS

* COMPLEX GINZBURG-LANDAU

$$A_t = (1+i\alpha) A_{xx} + A - (1+i\beta)|A|^2 A$$

& $A_0(\omega t - kx; k) = R(k)\, e^{i(kx-\omega t)}$

Consider $k_0 = 0 \to A_0 = 1\, e^{-i\omega_0 t}$

→ Burgers : $\boxed{q_T = (1+\alpha\beta) q_{xx} + 2(\beta-\alpha) q q_x} + O(\delta)$

$\Updownarrow$

$d''_{lin}(0) > 0$

$\Updownarrow$

$\boxed{1+\alpha\beta > 0}$

$\omega''_{nl}(k_0=0) = 0$

if $\alpha = \beta = 0$

$\Updownarrow$

Real GL.

Literature : in essence already in Howard & Kopell ('77)
(no validity)
& Manneville & Pomeau ('79)

MOST LITERATURE FOCUSES ON $1+\alpha\beta \simeq 0$ (& $\alpha=\beta=$

$\Updownarrow$

the transition to instability

$\boxed{\text{Ex. analysis in ODE!}}$

$\Updownarrow$

'Rigorous shocks' : $\alpha = \beta = 0$ Bricmont & Kupiainen ('9

$\alpha = 0$ Kapitula ('91)

near the Eckhaus instability

Note $\quad 1+\alpha\beta \simeq 0 \quad \Longrightarrow$

* $\beta \neq \alpha$ ($\Leftrightarrow$ $\omega'' \neq 0$) Burgers transforms into

(rescale X)

$$q_T = (\gamma_1 q_{XXX} + 2(\beta-\alpha) q q_X) + \delta\left[\gamma_2 q_{XXXX} + \gamma_3 q_{XX} + \gamma_4 (q^2)_{XX}\right]$$

'a perturbed KdV – eq'

different Xd

Bernoff, van Harten ('88) ('94)

* $\omega'' = 0$ (as in real GL)

$$q_T = \gamma_2 q_{XXXX} + \gamma_3 q_{XX} + \gamma_4 (q^2)_{XX}$$

(rescale T)

Kuramoto – Shivasinsky eq

Kramer & Zimmerman '85

a more general setting than rGL

= FORMAL ANALYSIS =

VALIDITY ALONG THE LINES OF OUR APPROACH TO BURGERS

OTHER APPLICATIONS

* Shocks & defects for FitzHugh-Nagumo
&
other well-studied systems
(Gray-Scott, Gierer-M.)

* $\boxed{\text{Systems near equilibrium}}$

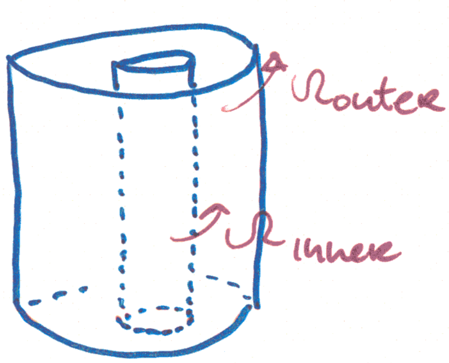
$\downarrow\uparrow$ Mielke, Schneider, Coullet & Eckmann, ...

Bif. described by cGL

$\downarrow\uparrow$ [DSSS]

Burgers & shocks in cGL

⇝ Shocks & defects in Taylor-Couette



GL bif to rings
(RGL)

or spirals
(cGL)

FINALLY : PULSE INTERACTIONS & LATTICE MODELS

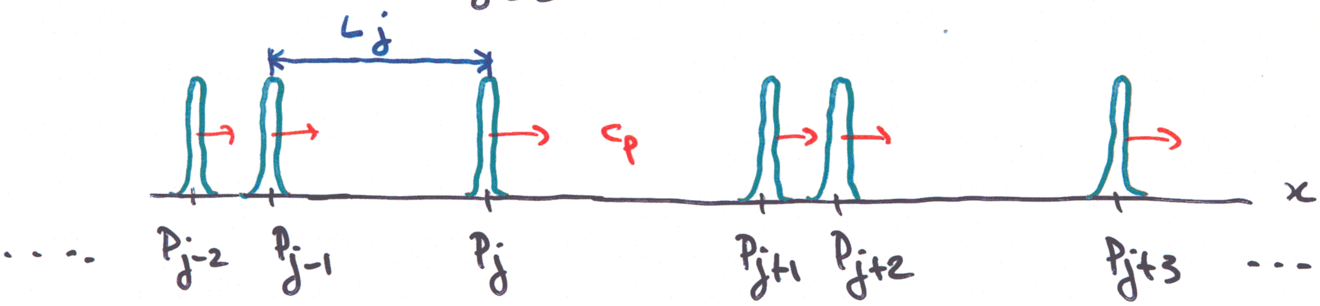
$\Updownarrow$

the long wave length limit

Assume : $\quad U(x,t) = h(x - c_p t)$

stable localized traveling pulse

Set : $\quad U(x,t) \approx \sum_{j \in \mathbb{Z}} h(x - c_p t + P_j(t))$



$\xrightarrow[\text{cond.}]{\text{some}}$ $\quad \boxed{\dot{P}_j = a \, e^{-b(P_j - P_{j-1})}}$

"lattice equation"

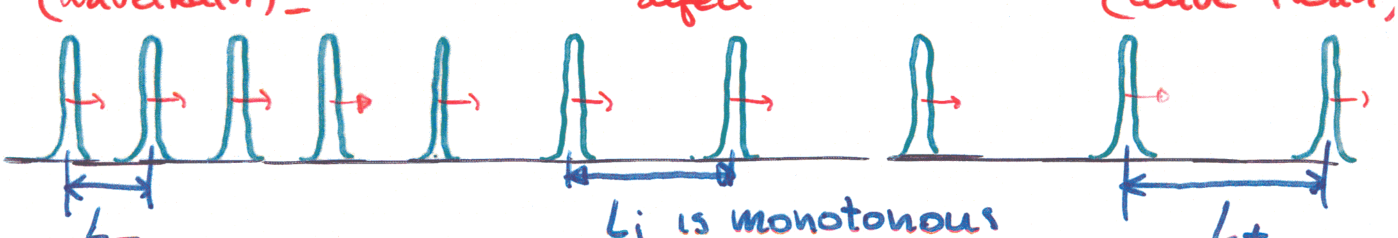
Ei, Sandstede,.

c_g condition; Note $L_+ - L_-$ NC SMALL

Th (Assumptions)

For any $L_+ > L_- > 0$ there exists a shock / defect solution with $L_j \longrightarrow L_\pm$ as $\xi \to \pm\infty$

(& NOT if $L_+ < L_-$!)



CONCLUSION / RECAPITULATION

$$\boxed{\text{(Slow) dynamics of modulated wave trains "generically" captured by BURGERS eq}}$$

= at finite (& LONG) spatial & temporal scales =

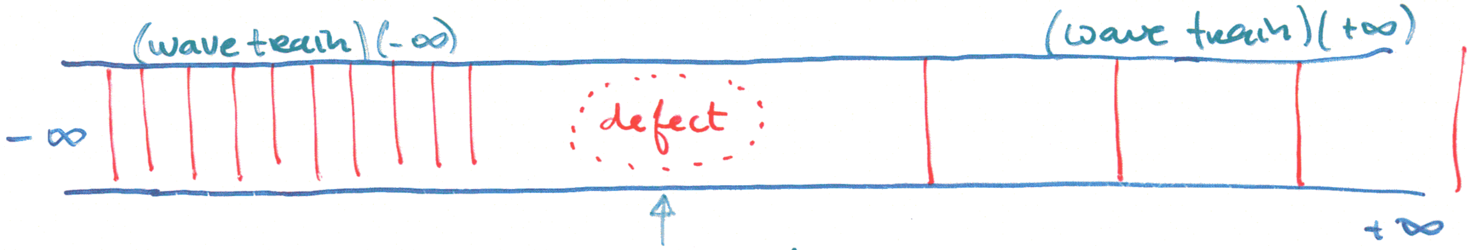
&

Existence and stability of DEFECTS OF SINK-TYPE

(for $x \in \mathbb{R}$, $t \in \mathbb{R}^+$) in full equation

(approximated by the (viscous) shock waves of Burgers approx.)

SINKS



TRAVELS WITH c_* s.t.

$$c_{group}(+\infty) < c_* < c_{group}(-\infty)$$