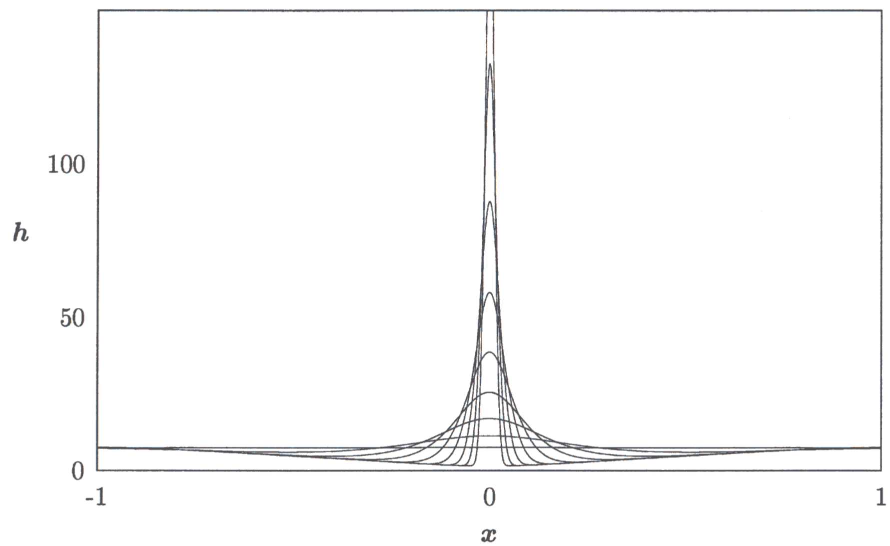
Dynamics of dissipation and blow-up for a critical-case thin film equation



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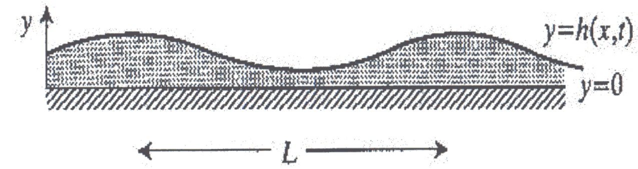
- Generalized thin film equations
- Blow-up of solutions above a finite critical mass
- Droplet solutions for self-similar dynamics:
  Infinite-time spreading and Finite-time blow-up
- Further studies of dynamics via numerical simulations

Generalized thin film PDEs: evolution equations for the height $h = h(x,t) \geq 0$ of thin layers of viscous fluids



$$\boxed{\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\left(h^m \frac{\partial h}{\partial x}\right) - \frac{\partial}{\partial x}\left(h^n \frac{\partial^3 h}{\partial x^3}\right)}$$

The thin film equation $\quad \dfrac{\partial h}{\partial t} = -\dfrac{\partial}{\partial x}\left(h^n \dfrac{\partial^3 h}{\partial x^3}\right) \qquad n \geq 0$

- Lubrication theory model for surface-tension driven spreading of viscous fluids ($n = 3$)
- 4th-order nonlinear diffusion equation

The porous medium equation $\quad \dfrac{\partial h}{\partial t} = \dfrac{\partial}{\partial x}\left(h^m \dfrac{\partial h}{\partial x}\right) \qquad m \geq 0$

- Lubrication theory model for gravity-driven diffusive spreading of viscous fluids ($m = 3$)
- The backward-in-time version $\quad \dfrac{\partial h}{\partial t} = -\dfrac{\partial}{\partial x}\left(h^m \dfrac{\partial h}{\partial x}\right)$

  is a 2nd-order **ill-posed** nonlinear problem

Dynamics of generalized thin film equations

$$\frac{\partial h}{\partial t} = \underbrace{-\frac{\partial}{\partial x}\left(h^m \frac{\partial h}{\partial x}\right)}_{\text{Destabilizing}} \underbrace{-\frac{\partial}{\partial x}\left(h^n \frac{\partial^3 h}{\partial x^3}\right)}_{\text{Stabilizing}}$$

A higher-order version of the problem of blow-up in $h_t = h^m + h_{xx}$

Competing influences **Near-instantaneous illposed break-down** vs **Capillary smoothing for all time**

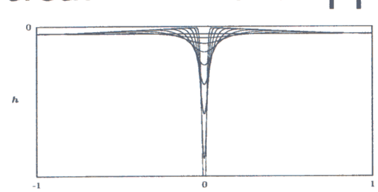
Resulting dynamics depends on relative strengths of terms as $h \to \infty$ [$(n > 0)$

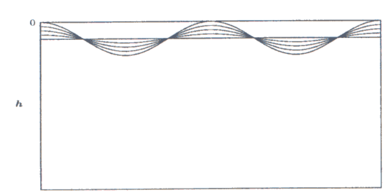
Bertozzi and Pugh, 1998, 2000]

$$\begin{cases} m > n+2 & \text{Supercritical – blow-up can occur, } h \to \infty \\ m = n+2 & \boxed{\text{Critical case}} \text{ – depends on mass} \\ m < n+2 & \text{Subcritical – solutions remain bounded } \forall t \end{cases}$$

Physical example: $m = n = 3$ – Liquid dripping(?) from a wet ceiling.

$3 < 3 + 2 \longrightarrow$ Subcritical $\longrightarrow$ No dripping





A critical-case thin film equation: $n = 1, m = n + 2 = 3$

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right) - \frac{\partial}{\partial x}\left(h \frac{\partial^3 h}{\partial x^3}\right)$$

Periodic boundary conditions on interval $-1 \leq x \leq 1$

Properties

1. Mass is conserved

$$M = \int_{-1}^{1} h \, dx \qquad \frac{dM}{dt} = 0$$

2. Write PDE as a generalized Reynolds equation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(h \frac{\partial p}{\partial x}\right) = 0$$

with Pressure is defined as

$$p = \tfrac{1}{3} h^3 + \frac{\partial^2 h}{\partial x^2}$$

3. Energy is defined as

$$\mathcal{E} = \int_{-1}^{1} \tfrac{1}{2} h_x^2 - \tfrac{1}{12} h^4 \, dx$$

Energy is dissipated by the PDE

$$\frac{d\mathcal{E}}{dt} = -\int_{-1}^{1} h \, p_x^2 \, dx \leq 0$$

Further properties (I): Proof of finite-time blow-up

Evolution of the second moment for the Cauchy problem:

$$\frac{d}{dt}\left(\int x^2 h\,dx\right) = -\tfrac{1}{2}\int h^4\,dx + 3\int h_x^2\,dx = 6\mathcal{E}$$

Energy is monotone decreasing, $\mathcal{E}(t) \le \mathcal{E}_0$, so

$$\frac{d}{dt}\left(\int x^2 h\,dx\right) \le 6\mathcal{E}_0$$

If the initial energy is negative, $\mathcal{E}_0 < 0$, then the second moment becomes **negative** in finite-time.

This is impossible since $h \ge 0$ $\quad \rightarrow \quad \int x^2 h\,dx \ge 0$

Resolution of the conflict:

The solution $h(x,t)$ ceases to exist at an earlier time.

If $\mathcal{E}_0 < 0$ then $h(x,t)$ blows-up in finite-time.

[Bernoff 1998, Bertozzi and Pugh, 2000]

An upper bound on the blow-up time: $t_c \le \dfrac{1}{6|\mathcal{E}_0|}\displaystyle\int x^2 h_0\,dx$

Further properties (II): Critical mass for blow-up

Consider the Cauchy problem on $-\infty < x < \infty$

$$\mathcal{E} = \tfrac{1}{2}\int h_x^2\,dx - \tfrac{1}{12}\boxed{\int h^4\,dx}$$

Use Sz.-Nagy's integral inequality **[Sz.-Nagy, 1941]**

$$\boxed{\int h^4\,dx} \le \frac{9}{4\pi^2}\left(\int h\,dx\right)^2 \int h_x^2\,dx$$

To yield

$$\mathcal{E} \ge \tfrac{1}{12}\left[6 - \frac{9}{4\pi^2}\left(\int h\,dx\right)^2\right]\int h_x^2\,dx$$

So, if

$$\int h\,dx < \boxed{M_c \equiv 2\pi\sqrt{\tfrac{2}{3}}}$$

then the energy is bounded from below, $\mathcal{E}(t) > 0$, and the H^1 norm and the maximum of the solution can be bounded:

no blow-up!

First-type Similarity Solutions via dimensional analysis

Rescale $x = \mathrm{L}\hat{x}$ $\quad t = \mathrm{T}\hat{t}$ $\quad h = \mathrm{H}\hat{h}$ $\quad$ to yield

$$\left[\frac{\mathrm{H}}{\mathrm{T}}\right]\frac{\partial \hat{h}}{\partial \hat{t}} = -\left[\frac{\mathrm{H}^4}{\mathrm{L}^2}\right]\frac{\partial}{\partial \hat{x}}\left(\hat{h}^3\frac{\partial \hat{h}}{\partial \hat{x}}\right) - \left[\frac{\mathrm{H}^2}{\mathrm{L}^4}\right]\frac{\partial}{\partial \hat{x}}\left(\hat{h}\frac{\partial^3 \hat{h}}{\partial \hat{x}^3}\right)$$

Make the PDE scale-invariant:

- Balance the spatial operators : $\quad \mathrm{H} = 1/\mathrm{L}$
- and the time-derivative : $\quad \mathrm{T} = \mathrm{L}^5$

Invariant quantities $= \left\{\dfrac{\text{Length}}{\text{Time}^{1/5}}, \text{Time}^{1/5}\text{Height}\right\}$

Similarity variables

$$\eta = \frac{x - x_c}{\tau} \qquad \tau = \left[5\sigma(t_c - t)\right]^{1/5}$$

x_c, t_c: translational shifts in spatial, temporal coordinates

$$h(x,t) = \frac{1}{\tau}H(\eta, s) \qquad s = -\frac{1}{\sigma}\ln\tau$$

Reformulation in similarity variables

$$\boxed{\mathrm{H} = 1/\mathrm{L} \qquad\qquad \mathrm{T} = \mathrm{L}^5}$$

Two classes of self-similar solutions:

(i) Infinite-time spreading solutions

$$\text{as } \mathrm{T} \to \infty \qquad \underbrace{\mathrm{L} \to \infty}_{\text{defocusing}} \qquad \underbrace{\mathrm{H} \to 0}_{\text{dissipation}}$$

(ii) Finite-time blow-up solutions

$$\text{as } \mathrm{T} \to 0 \qquad \underbrace{\mathrm{L} \to 0}_{\text{focusing}} \qquad \underbrace{\mathrm{H} \to \infty}_{\text{blow-up}}$$

Similarity PDE for $H(\eta, s)$

$$h(x,t) = \frac{1}{\tau} H(\eta, s) \qquad s = -\frac{1}{\sigma} \ln \tau \qquad \tau = [5\sigma(t_c - t)]^{1/5}$$

$$\boxed{\frac{\partial H}{\partial s} = -\frac{\partial}{\partial \eta}\left(H \frac{\partial}{\partial \eta}\left[\tfrac{1}{2}\sigma\eta^2 + \tfrac{1}{3}H^3 + H_{\eta\eta}\right]\right)}$$

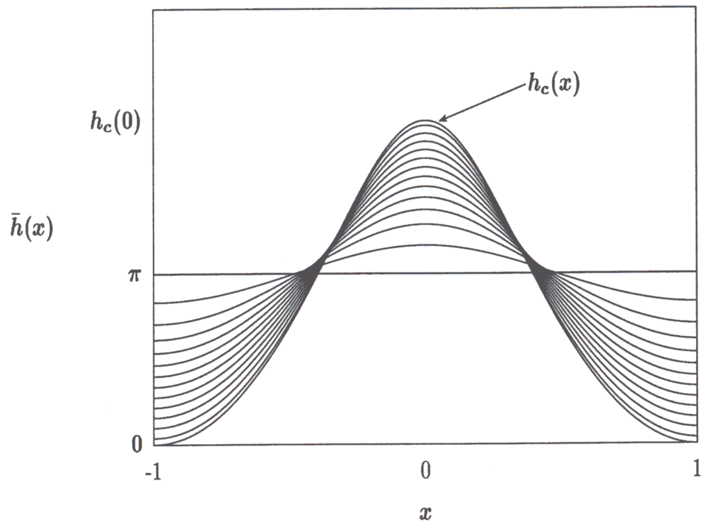
Three cases

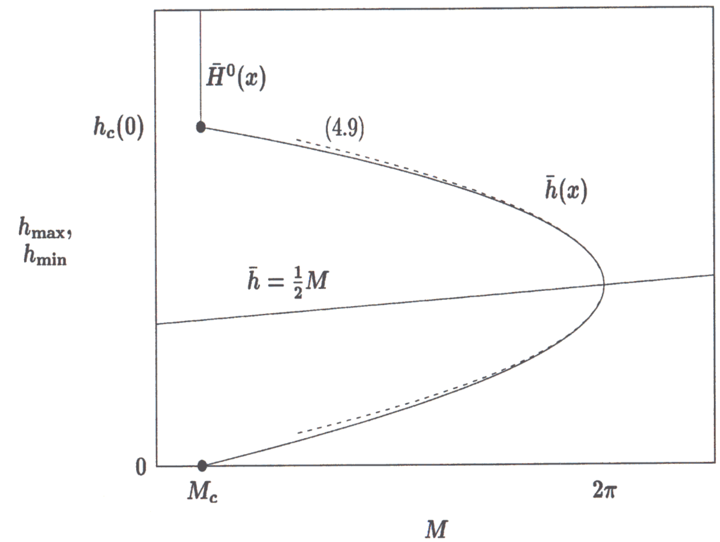
(i)	$\sigma = -1$	infinite-time dissipation	for $t > t_c$
(ii)	$\sigma = +1$	finite-time blow-up	for $t < t_c$
(iii)	$\sigma = 0$	near-equilibrium dynamics	for all t
		$\{s \to t,\ \eta \to x,\ H \to h\}$	

Generalized equilibria: similarity solutions and steady states

(i)	$\bar{H}^{-}(\eta)$	infinite-time spreading similarity soln
(ii)	$\bar{H}^{+}(\eta)$	finite-time blow-up similarity soln
(iii)	$\bar{H}^{0}(x)$	steady states

Positive periodic steady states ($\sigma = 0$) [Laugesen and Pugh 2000]





One parameter branches of solutions $\bar{h}(x)$ bifurcate from the trivial branch, $\bar{h} =$ const and terminate at the compactly-supported solution $h_c(x)$.

Compactly-supported equilibrium solution ($\sigma = 0$): $h_c(x) \geq 0$ on $-1 \leq x \leq 1$

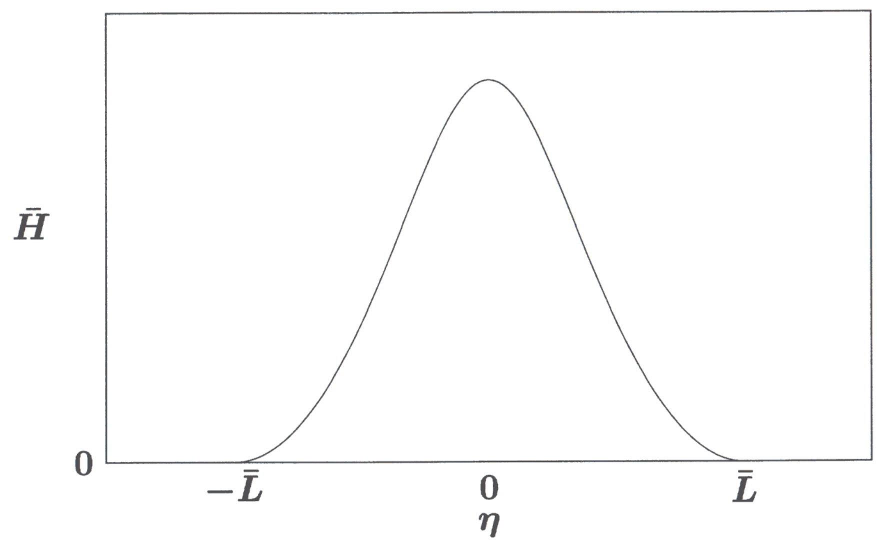
Mass

$$2\int_0^1 \frac{\sqrt{6}y\,dy}{\sqrt{y(1-y^3)}} = \boxed{2\pi\sqrt{2/3} = M_c} \quad (!)$$

One-parameter, scale-invariant family of "**droplet solutions**"

$$\boxed{\bar{H}^0(x) = \frac{1}{L}h_c(x/L)} \qquad L \leq 1$$

Droplet solutions: $\bar{H}^{\sigma}(\eta)$ for $\sigma = 0, \pm 1$



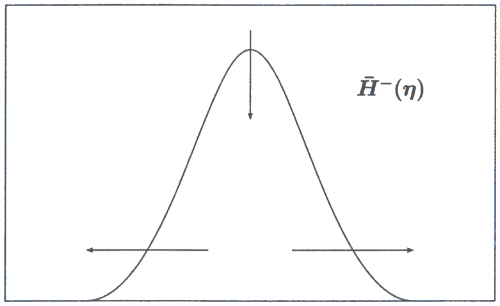
$$\boxed{\bar{H}'' + \tfrac{1}{3}\bar{H}^3 + \tfrac{1}{2}\sigma\eta^2 = \bar{P}} \qquad -\bar{L} \le \eta \le \bar{L}$$

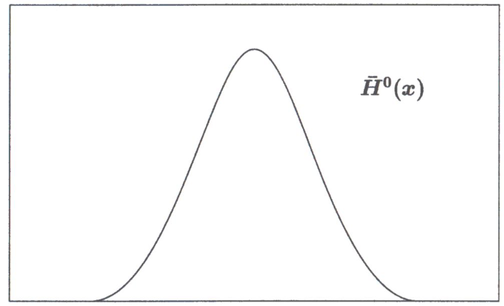
- Seek finite-mass, non-negative,compact, symmetric solutions.
- Compatibility condition for $\bar{P}$ $\quad \boxed{\bar{P} = \frac{1}{2\bar{L}} \int_{-\bar{L}}^{\bar{L}} \tfrac{1}{3}\bar{H}^3 + \tfrac{1}{2}\sigma\eta^2 \, d\eta}$
- A second-order nonlocal problem.

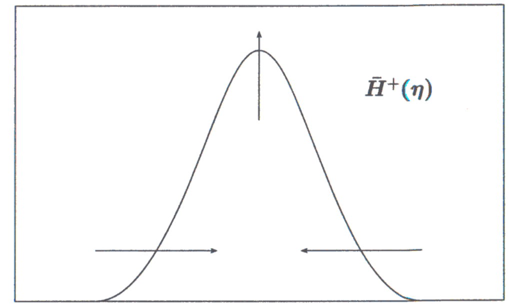
- Alternatively, can be written as a third-order ODE

$$\boxed{\bar{H}''' + \bar{H}^2\bar{H}' + \sigma\eta = 0} \qquad 0 \le \eta \le \bar{L}$$

$$\bar{H}'(0) = 0 \qquad \bar{H}(\bar{L}) = \bar{H}'(\bar{L}) = 0$$







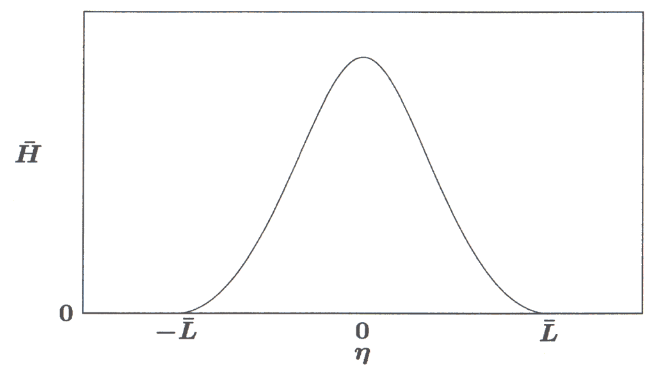
	$\bar{H}^{-}(\eta)$	$\bar{H}^{0}(x)$	$\bar{H}^{+}(\eta)$
σ	-1	0	$+1$
Critical Time	Infinite	—	Finite
Dynamics	Spreading	Steady-State	Blow-up
Mass	$0 \leq M < M_c$	$M = M_c$	$M > M_c$ (*)
Energy	$\mathcal{E} > 0$	$\mathcal{E} = 0$	$\mathcal{E} < 0$
Set of Solutions	Single branch, 1-parameter(M) family	Unique solution, Scale-invariant $x \to x/L$	Multiple branches, 1-parameter(M) families
Stability	Stable	Marginally Stable	1st Branch Stable, rest Unstable

Properties of Self-Similar Droplet Solutions

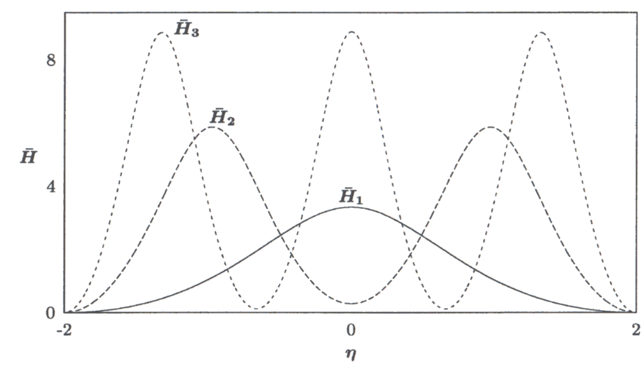
1. **Single Spreading vs. Multiple Blow-up Solutions**

- For each $0 < M < M_c$ there is a unique single-bump $\bar{H}^-(\eta)$
- For fixed $\bar{L}$ there are infinitely many multi-bump $\bar{H}^+(\eta)$

$\bar{H}^-(\eta)$

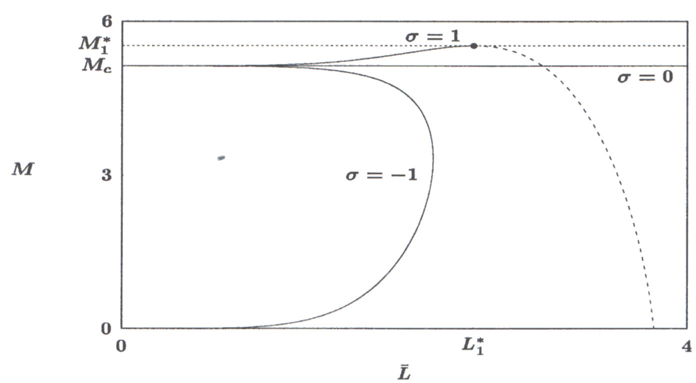


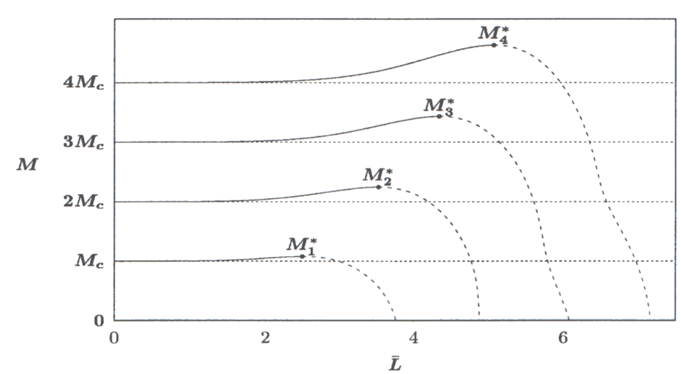
$\bar{H}^+(\eta)$



2. **Mass-dependent continuous branches of solutions**

- $\bar{H}^{\pm}(\eta) \to \bar{H}^0(x)$ for $M \to M_c$ and $\bar{L} \to 0$
- Discrete branches of multi-bump $\bar{H}_n^+(\eta)$ for $nM_c < M < M_{u,n}$





Spreading similarity solutions (I) ($\sigma = -1$)

Claim: There are only single-bump spreading droplet solutions.

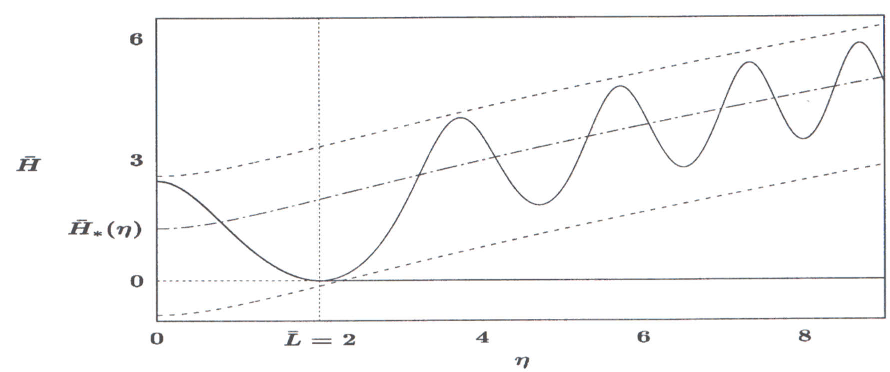
1. For $\sigma = 0$, the phase plane for $\boxed{\bar{H}'' + \frac{1}{3}\bar{H}^3 - \bar{P} = 0}$ has an elliptic fixed point at $\boxed{\bar{H}_* = (3\bar{P})^{1/3}}$ and a conserved quantity for all the periodic solutions

$$K = \tfrac{1}{2}\bar{H}_\eta^2 + \tfrac{1}{12}\bar{H}^4 - \tfrac{1}{3}\bar{H}_*^3\bar{H} \qquad \rightarrow \qquad \frac{dK}{d\eta} = 0$$

2. For $\sigma = -1$, $\boxed{\bar{H}'' + \frac{1}{3}\bar{H}^3 - (\bar{P} + \frac{1}{2}\eta^2) = 0}$, and define the elliptic "pseudo-fixed point" as $\boxed{\bar{H}_* = (3[\bar{P} + \frac{1}{2}\eta^2])^{1/3}}$, then the solutions have

$$\frac{dK}{d\eta} = -\eta\bar{H} \leq 0$$

That is, the solutions oscillate about $\bar{H}_*$, but as $\eta \nearrow$, the amplitude of the oscillations decrease. Since $\bar{H}_*(\eta) \nearrow$, there can be only a single minimum.



Spreading similarity solutions (II) ($\sigma = -1$)

$$\bar{H}_{\eta\eta\eta} + \bar{H}^2 \bar{H}_\eta - \eta = 0$$

Rescale by interval of support, $|\eta| \leq \bar{L}$: $\quad \eta = \bar{L} z$

Two distinguished limits for $\bar{L} \to 0$:

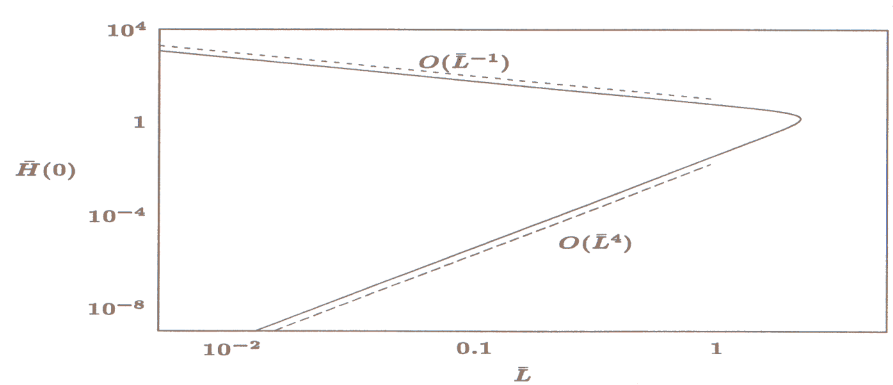
1. **Small mass**: $\quad \bar{H}(\eta) = \bar{L}^4 \mathcal{H}(z) \quad \longrightarrow \quad \mathcal{H}''' - z = -\bar{L}^{10}\mathcal{H}^2\mathcal{H}'$

   Source-type similarity solution of $n = 1$ thin film eqn

$$\bar{H}(\eta) = \frac{1}{24}\left(\bar{L}^2 - \eta^2\right)_+^2 + O(\bar{L}^{14})$$

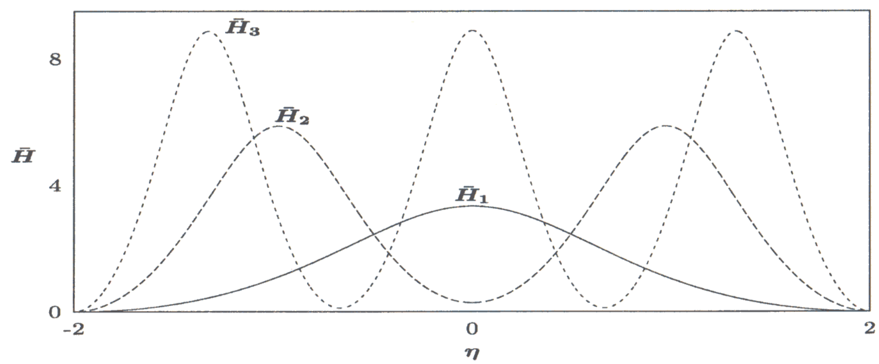
2. **Finite mass**: $\quad \bar{H}(\eta) = \mathcal{H}(z)/\bar{L} \quad \longrightarrow \quad \mathcal{H}''' + \mathcal{H}^2\mathcal{H}' = \bar{L}^5 z$

   Near-equilibrium solution $\bar{H}(\eta) = \dfrac{1}{\bar{L}} h_c(\eta) + O(\bar{L}^4)$



Finite-time blow-up similarity solutions ($\sigma = 1$)

For $\sigma = 1$, the single-bump claim does not apply. In fact, there is an infinite sequence of branches of multi-bump solutions, $\bar{H}_1, \bar{H}_2, \bar{H}_3, \cdots$. (only $\bar{H}_1$ is stable)

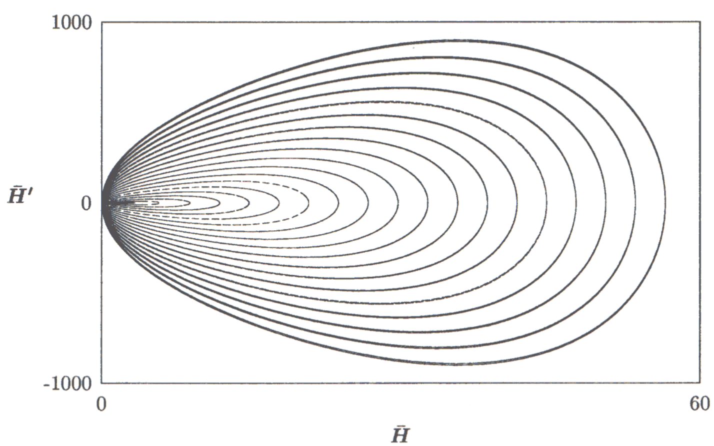


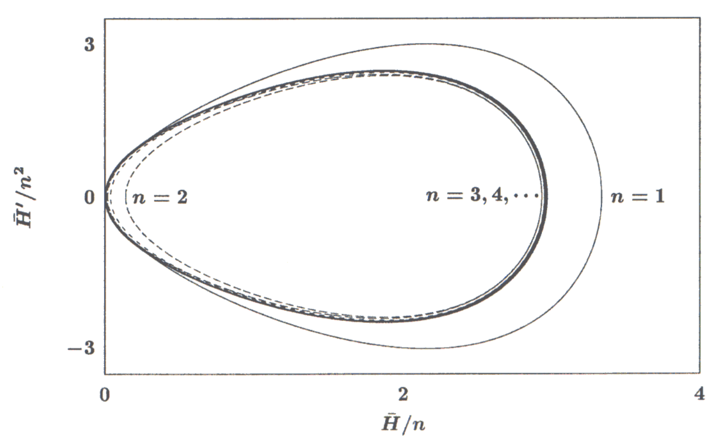
Asymptotics for n-bump blow-up solutions, $n \to \infty$

$$\bar{H}(\eta) = n\mathcal{H}(z) \qquad \eta = \frac{z}{n} \qquad \to \qquad \mathcal{H}''' + \mathcal{H}^2\mathcal{H}' = -\frac{1}{n^3}z$$

Nearly-steady-state periodic n-bump solutions

$$\boxed{\bar{H}_n(\eta) = n\bar{h}(n\eta)} + O(n^{-2})$$

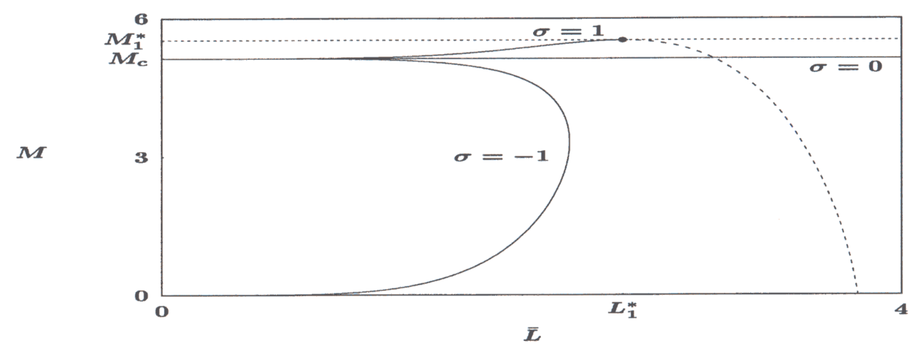




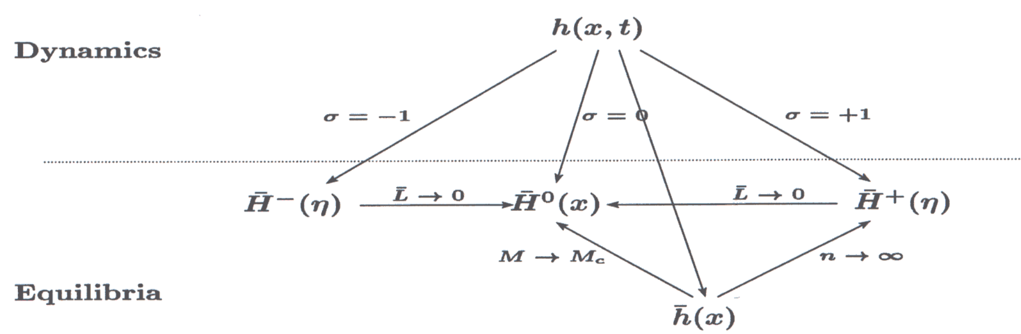
Connections between families of generalized equilibria

1. Near-steady-state limits

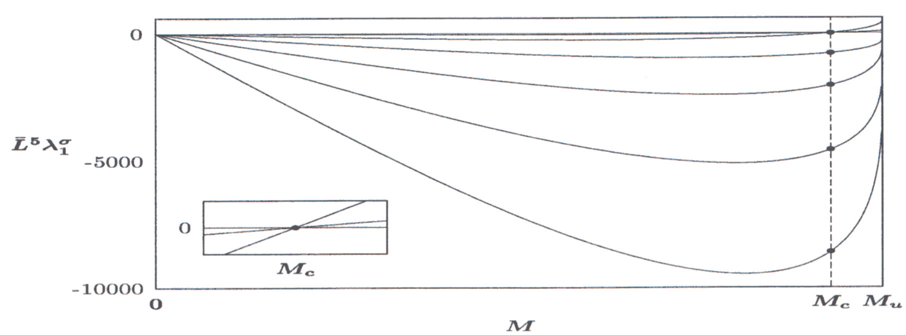
   The three classes of solutions $\sigma = 0, \sigma = \pm 1$ connect in the limit $L \to 0$, $M \to M_c$



2. Other connections



3. Linear stability



Insights from numerical simulations of the dynamics

1. Touch-down vs. Blow-up Singularities

- As blow-up is approached, $\boxed{\tau h(x,t) \longrightarrow \bar{H}^{+}(\eta)}$ with compact support, $\bar{H}^{+}(\eta) = 0$ for $\eta > \bar{L}$.
- Regularization needed for touch-down of thin film solutions, $h(x,t) \longrightarrow 0$

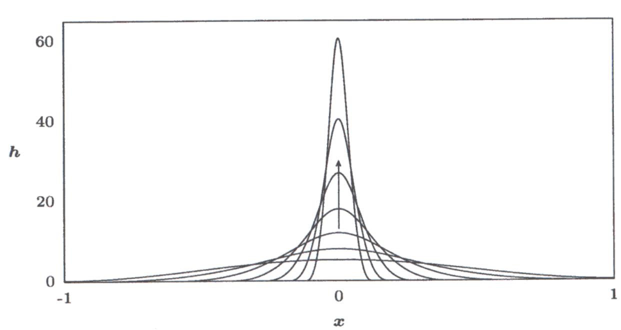
[Bernis and Friedman, 1990]

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\left(f_\epsilon(h)\frac{\partial p}{\partial x}\right) \qquad f_\epsilon(h) = \frac{h^4}{\epsilon^3 + h^3}$$

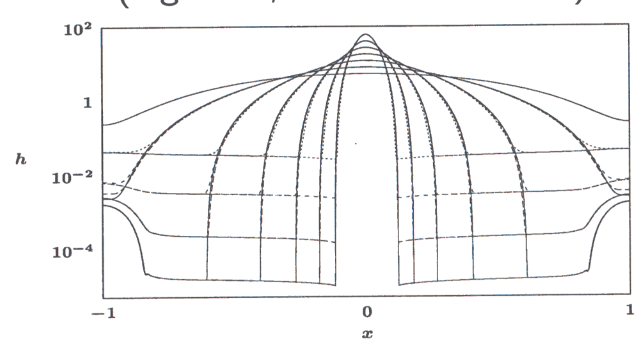
Two Routes to Blow-up

- Weak Blow-up: Touch-down first, $h(x,t) \longrightarrow 0$, then blow-up of a weak solution $h(x,t) \longrightarrow \infty$ with $\tau h \longrightarrow 0$ $(\eta > \bar{L})$ because $h \longrightarrow 0$

(regular scale)



(log-scale, $\epsilon = 10^{-k} \longrightarrow 0$)

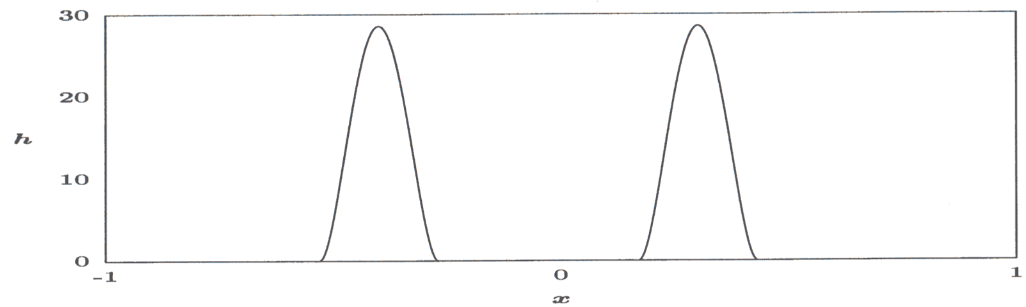


- Classical Blow-up: No touch-down, $h(x,t) > h_{\min}$ with $\tau h(x,t) \longrightarrow 0$ $(\eta > \bar{L})$ because $\tau \longrightarrow 0$ [Movie]

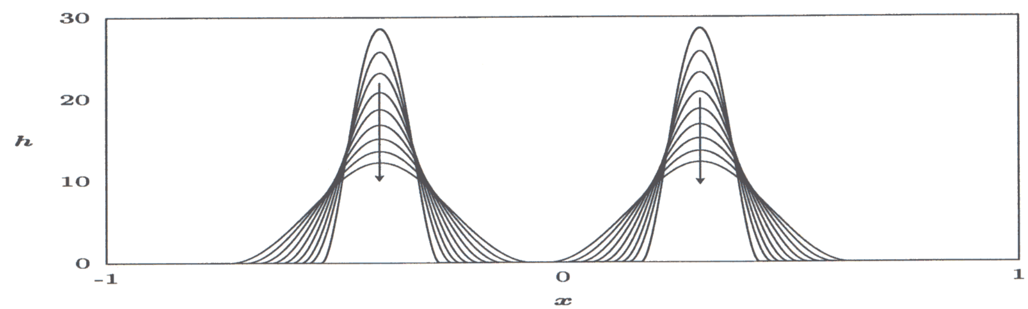
Insights from numerical simulations of the dynamics

2. Blow-up from the merger of subcritical solutions

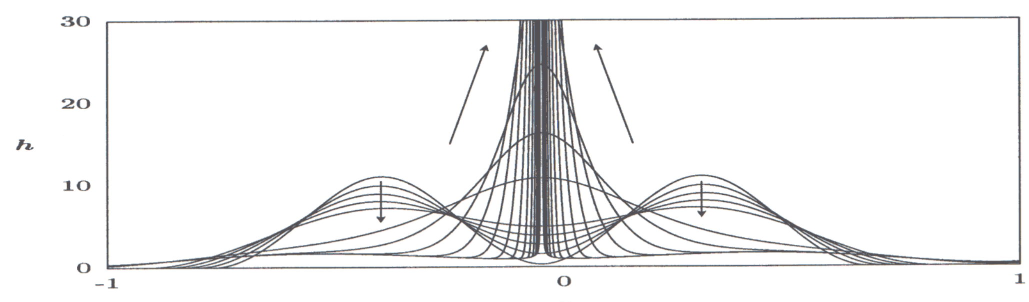
- Initial data: disjoint droplets with $M_1, M_2 < M_c$



- Early behavior: separate spreading via $\bar{H}^-(\eta)$'s



- Later behavior: merger, $M = M_1 + M_2 > M_c$, and blow-up via $\bar{H}^+(\eta)$!



How does the $\sigma = -1 \to 1$ transition happen?

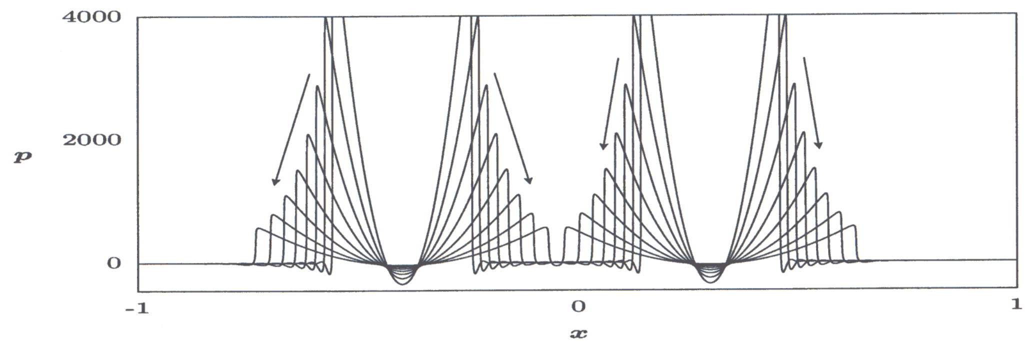
Insights from numerical simulations of the dynamics

2. Blow-up from merger of subcritical solutions (concl)

The pressure, $p = \frac{1}{3}h^3 + h_{xx}$

Spreading pressure waves with $p_{xx} > 0$

For $\bar{H}^{-}(\eta)$: $$p(x,t) = \frac{\frac{1}{3}\bar{H}^3 + \bar{H}''}{\tau^3} = \frac{1}{\tau^3}\left(\bar{P} + \tfrac{1}{2}\eta^2\right) \quad \tau \to \infty$$



Collision of pressure waves to produce a pressure maximum and $p_{xx} < 0$

For $\bar{H}^{+}(\eta)$: $$p(x,t) = \frac{\frac{1}{3}\bar{H}^3 + \bar{H}''}{\tau^3} = \frac{1}{\tau^3}\left(\bar{P} - \tfrac{1}{2}\eta^2\right) \quad \tau \to 0$$