Viscous entrainment: creating an atomic-sized liquid spout

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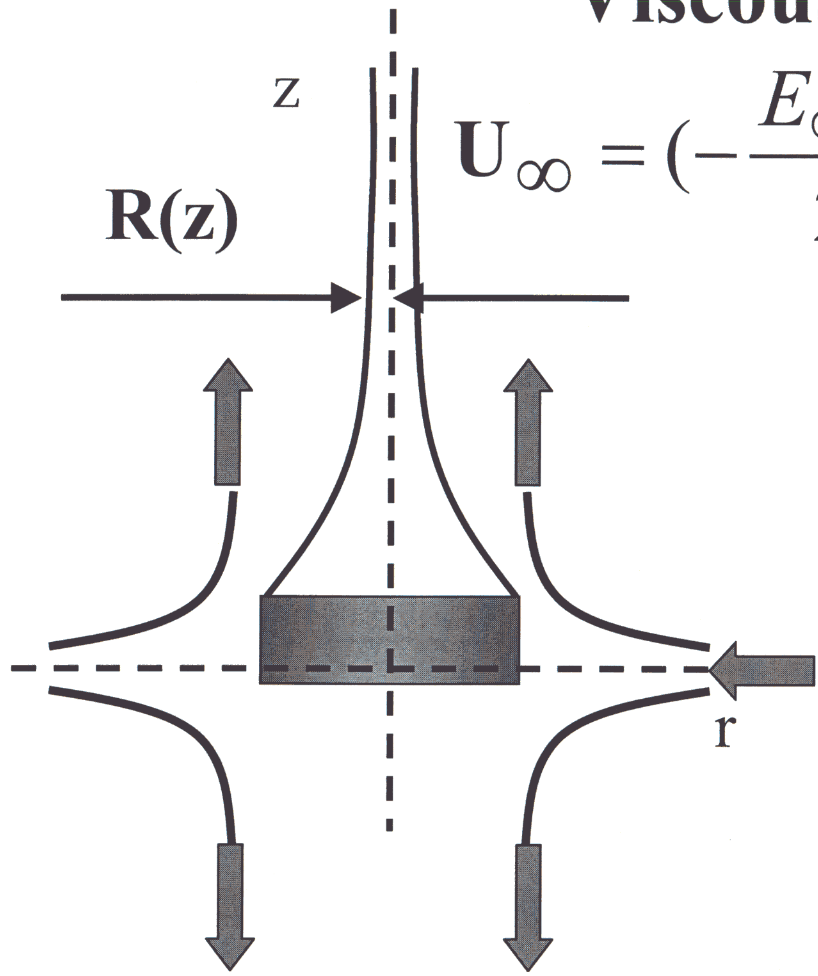
Viscous entrainment from a nozzle



$$U_\infty = (-\frac{E_\infty r}{2}, 0, E_\infty z)$$

<u>exterior viscous stresses</u>

against

<u>surface tension pressure</u>

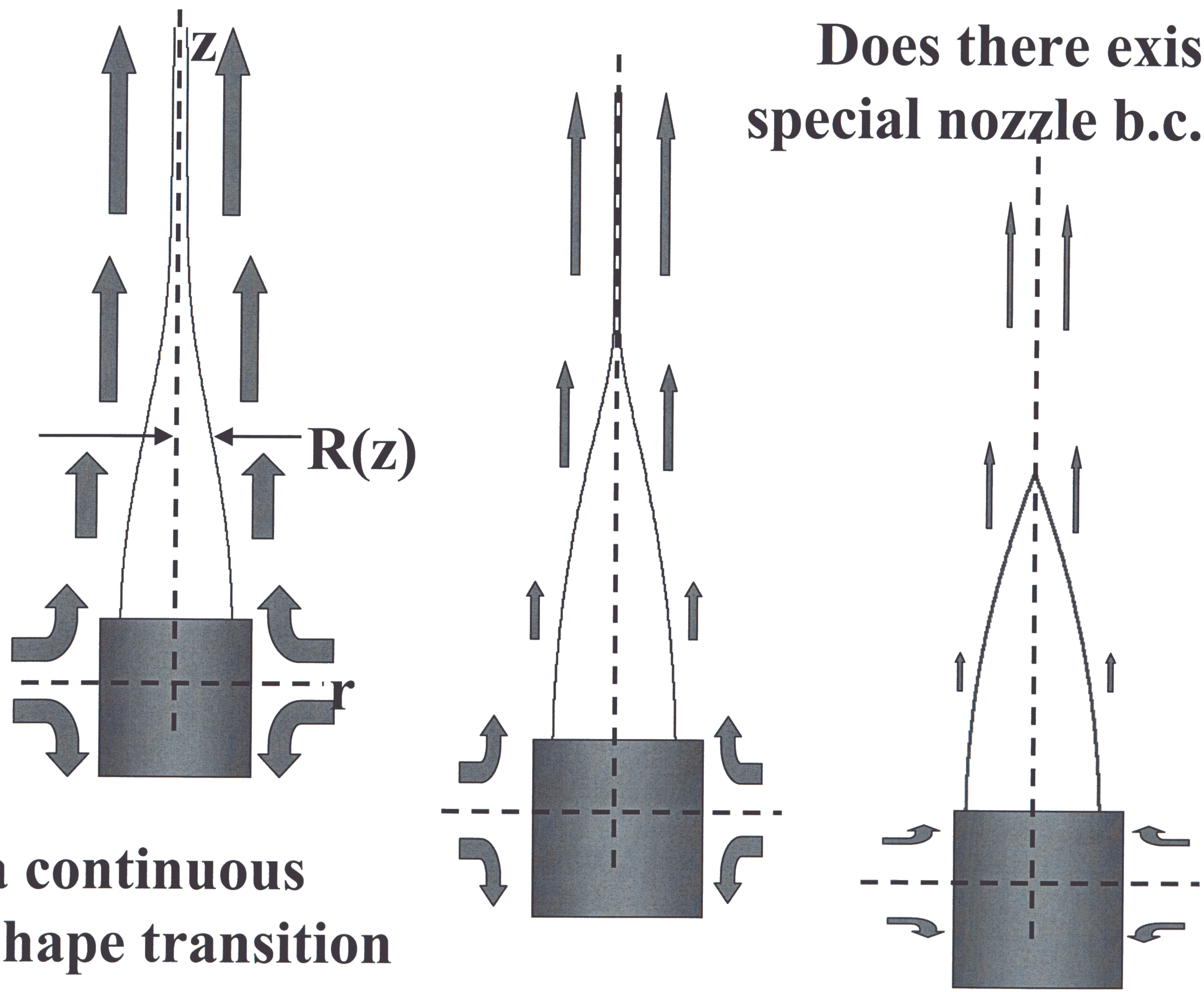
$$P_\gamma = \gamma\left(\frac{1}{R_{\text{radial}}} - \frac{1}{R_{\text{axial}}}\right)$$

sharpens **smoothes**

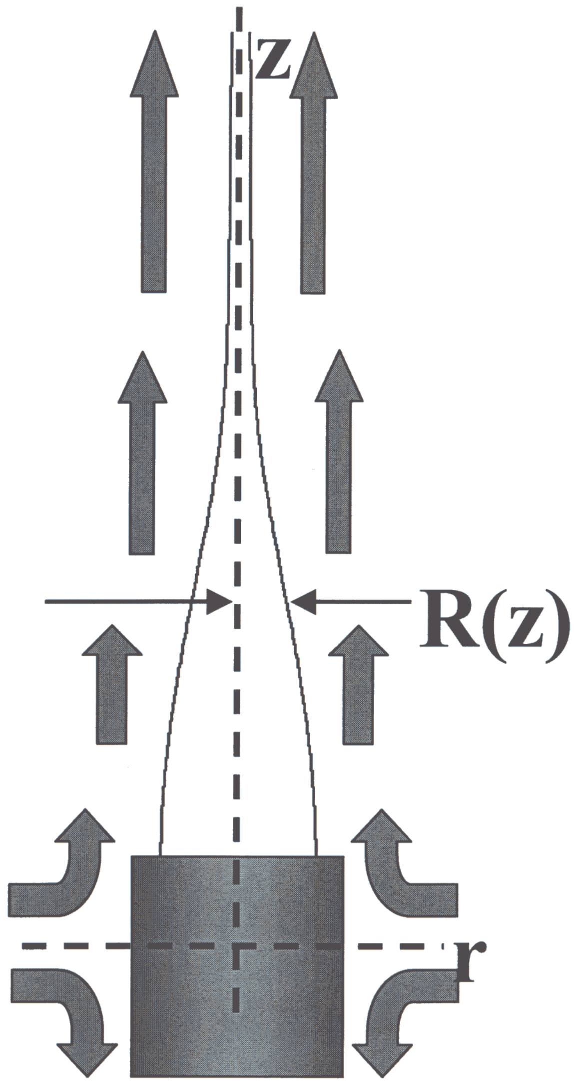
<u>& interior viscous stresses</u>

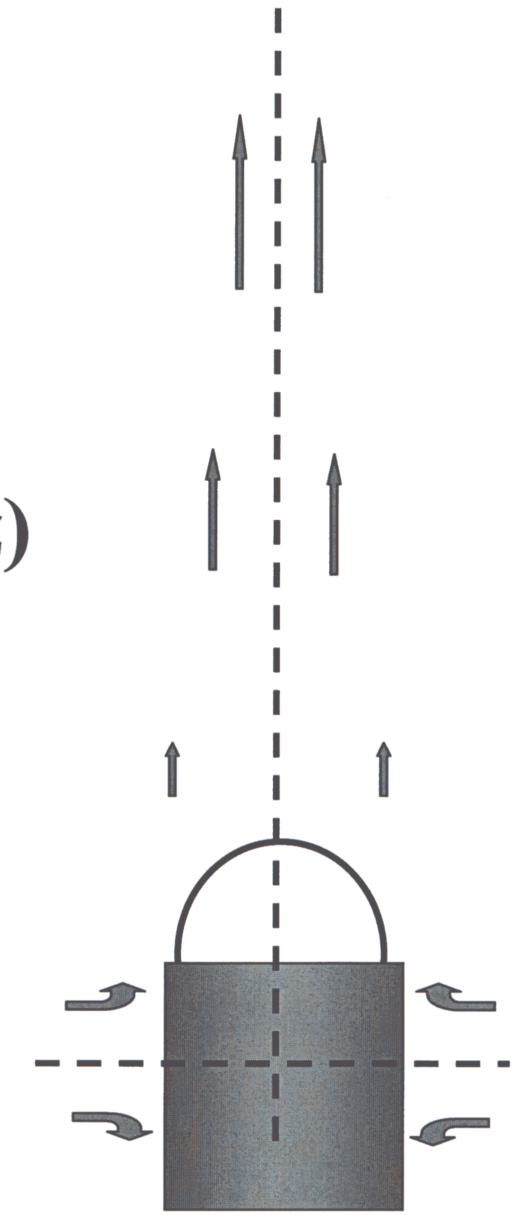
(spout far less viscous than exterior)

Does there exist special nozzle b.c.'s ?



a continuous shape transition

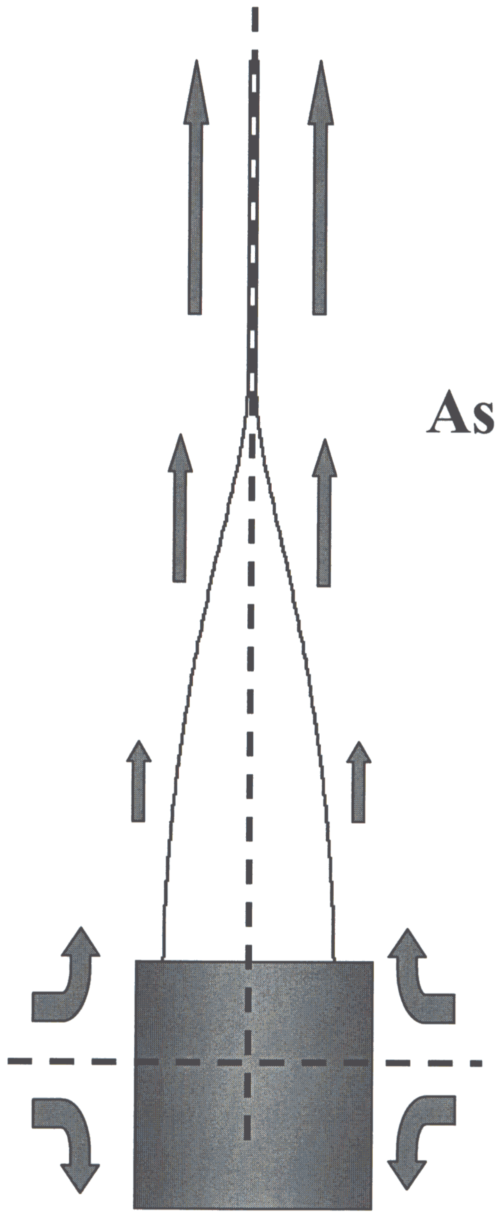




generic boundary
conditions
at the nozzle

↓

discontinuous
shape transition
at onset
of entrainment



As Qspout → 0 (or R(z) → 0)

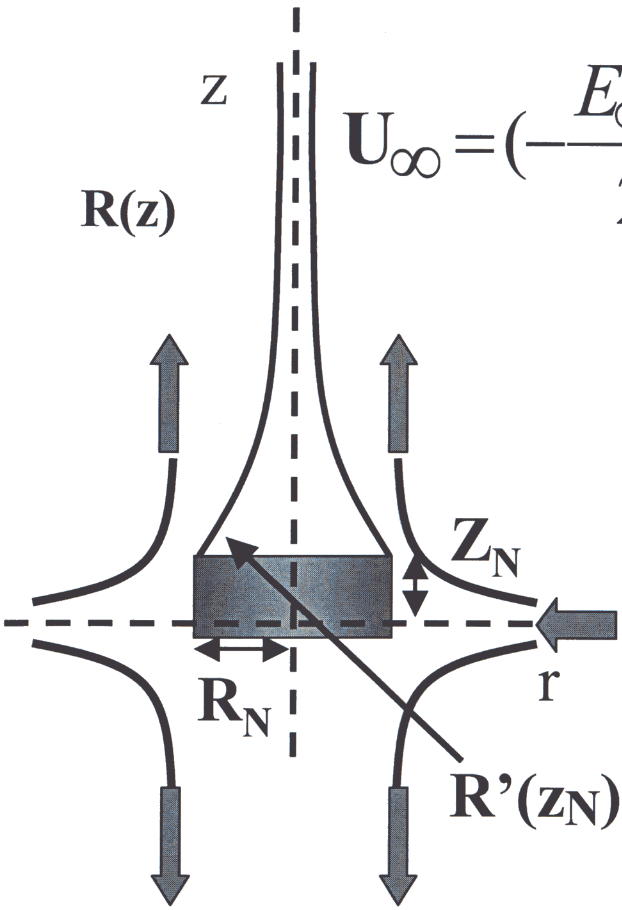
$$P_\gamma = \gamma\left(\frac{1}{R(z)} - \frac{1}{R_{\text{axial}}}\right) \rightarrow \textbf{infinite}$$

continuous shape transition ⟷ **steady-state singularity on fluid surface**

Is a continuous shape transition possible at the onset of viscous entrainment?

Nature of entrainment transition can be tuned by tuning the macroscopic boundary conditions

Long-wavelength model of viscous withdrawal



$$U_\infty = \left(-\frac{E_\infty r}{2}, 0, E_\infty z\right)$$

large reservoir (no global volume constraint)

Spout far less viscous than exterior
→ Spout long & slender
→ Free-slip surface
→ Correction flow in exterior due to spout surface is a line of volume sources/sinks

Steady-State Only

Two tunable nozzle b.c.'s

nozzle location	z_N
nozzle radius	R_N
contact angle	$R'(z_N)$

Taylor, G. I. Proc. Roy. Soc. 1932
Acrivos & Lo, JFM 1975

Nondimensionalize

Rescale radius R by R_N nozzle radius

Rescale downstream distance z by $R_N / 2\sqrt{\lambda}$

Rescale stress/pressure by $\mu_{ext} E_\infty$

viscous stresses in the exterior

$$\lambda = \frac{\mu_{spout}}{\mu_{exterior}}$$ **far less than 1**

dimensionless strain rate

$$Ca = \frac{2\mu_{ext} E_\infty R_N}{\gamma}$$

Governing equation

$$Q_{\text{spout}} = Ca\left(z\ R^2(z)\right) - R^4(z)\frac{dP}{dz}$$

unknown volume flux in spout	interior flow induced by exterior flow	interior flow driven by pressure gradient

$$P(z) = \frac{1}{R} - 4\lambda\frac{d^2R}{dz^2} + Ca\frac{z}{R}\frac{dR}{dz}$$

surface tension	interior viscous stresses

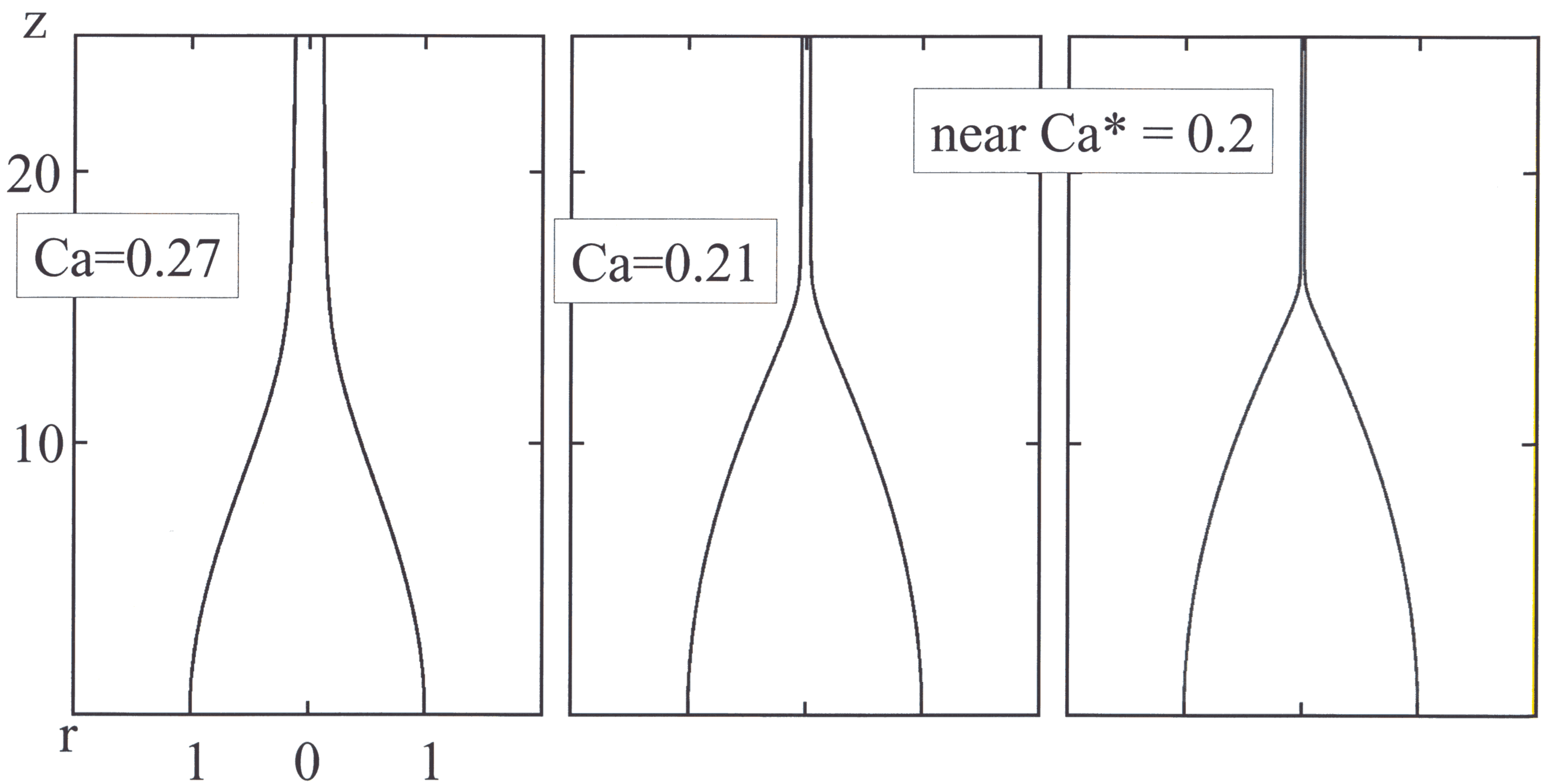
Boundary conditions

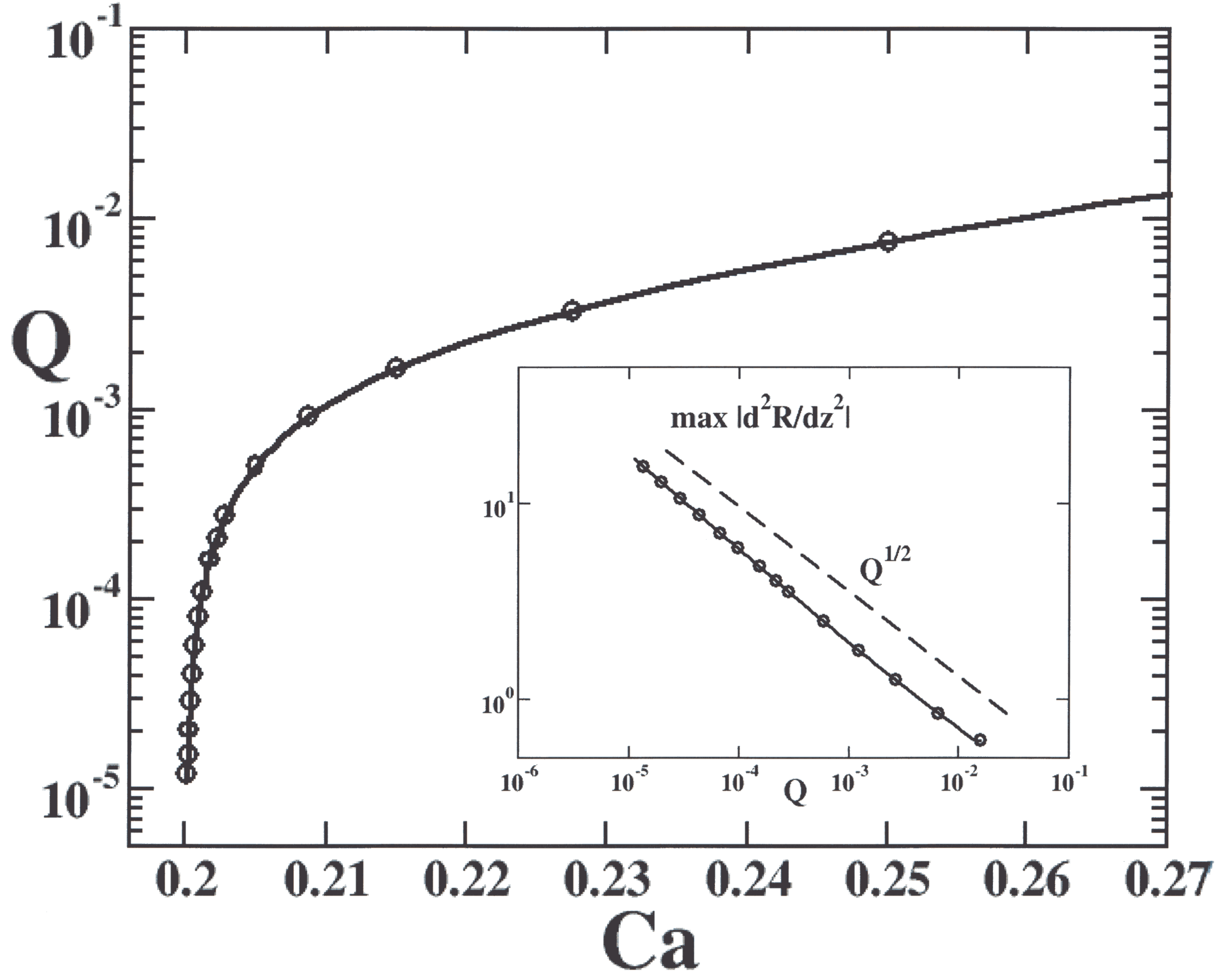
at nozzle: $R(z_N) = 1 \qquad \left.\frac{dR}{dz}\right|_{z_N} = \cot(\theta_{contact})$

far downstream: $R(z) \to \sqrt{\frac{2Q_{spout}}{zCa}} \qquad z \to \infty$

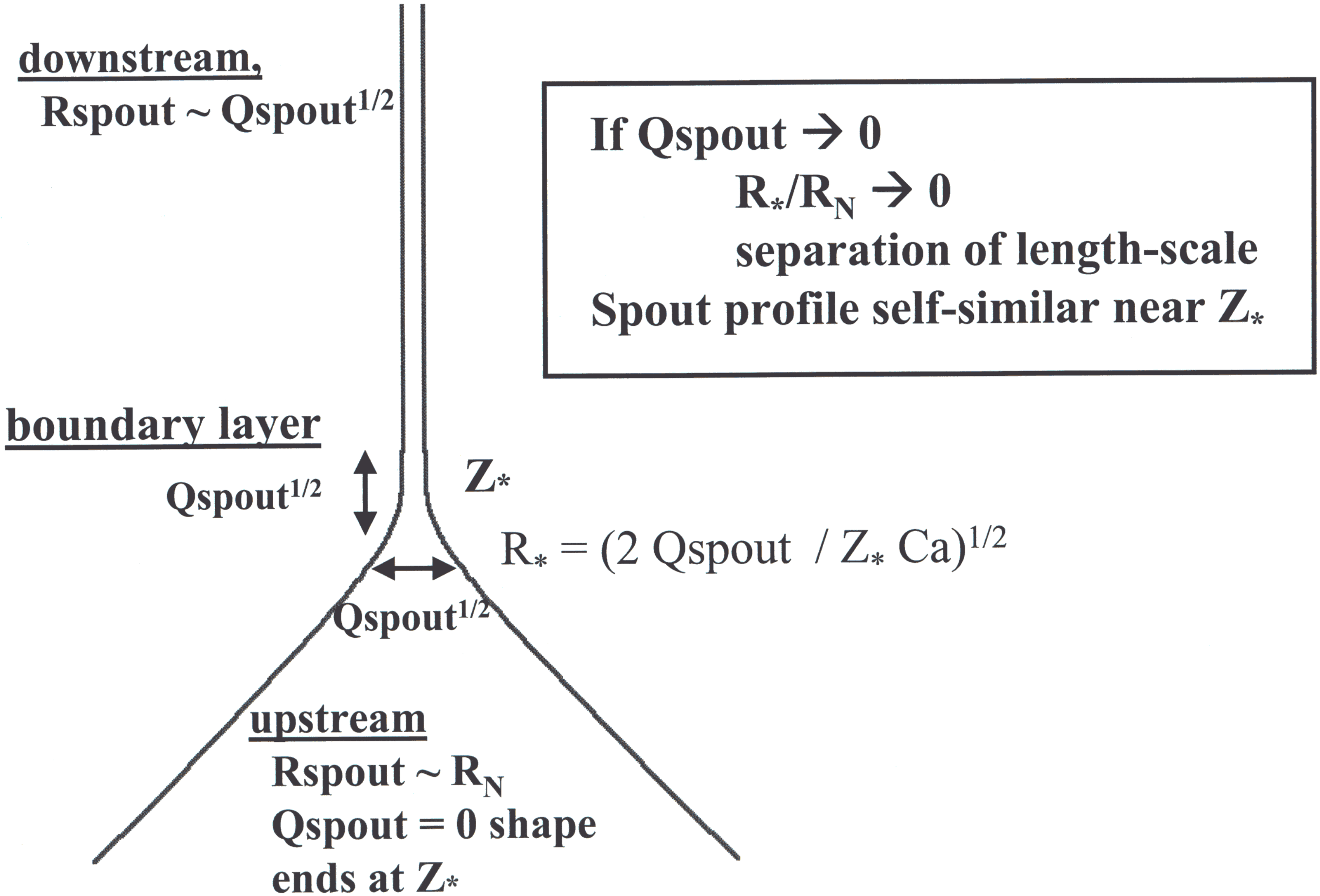
Numerical solution of long-wavelength equation: decreasing exterior withdrawal rate

$Ca \rightarrow Ca_*$ λ=0.00625



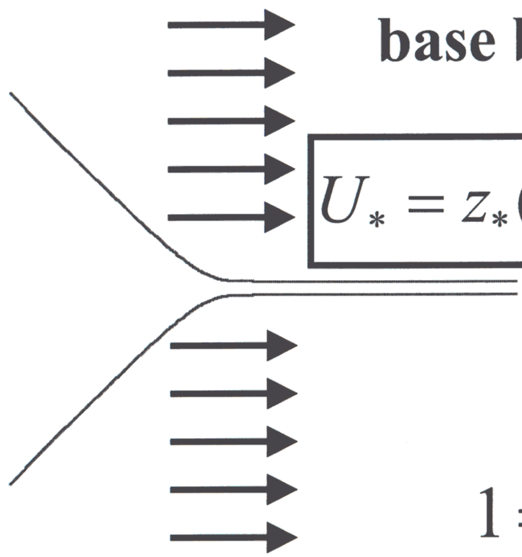


Q
Ca
10^-1
10^-2
10^-3
10^-4
10^-5
0.2
0.21
0.22
0.23
0.24
0.25
0.26
0.27
max |d^2R/dz^2|
Q^1/2
10^1
10^0
10^-6
10^-5
10^-4
10^-3
10^-2
10^-1


downstream,
Rspout ~ Qspout$^{1/2}$
If Qspout → 0
R_*/R_N → 0
separation of length-scale
Spout profile self-similar near Z_*
boundary layer
Qspout$^{1/2}$
Z_*
$R_* = (2\ Qspout\ /\ Z_*\ Ca)^{1/2}$
Qspout$^{1/2}$
upstream
Rspout ~ R_N
Qspout = 0 shape
ends at Z_*

Self-similar dynamics ⇔ Simplified problem

entrainment of a liquid cylinder from a conical base by a uniform flow



$$\boxed{U_* = z_*(Ca)Ca} \qquad \rho(\eta) = \frac{1}{\sqrt{Qspout}} R(\eta = \frac{z - z*}{\sqrt{Qspout}})$$

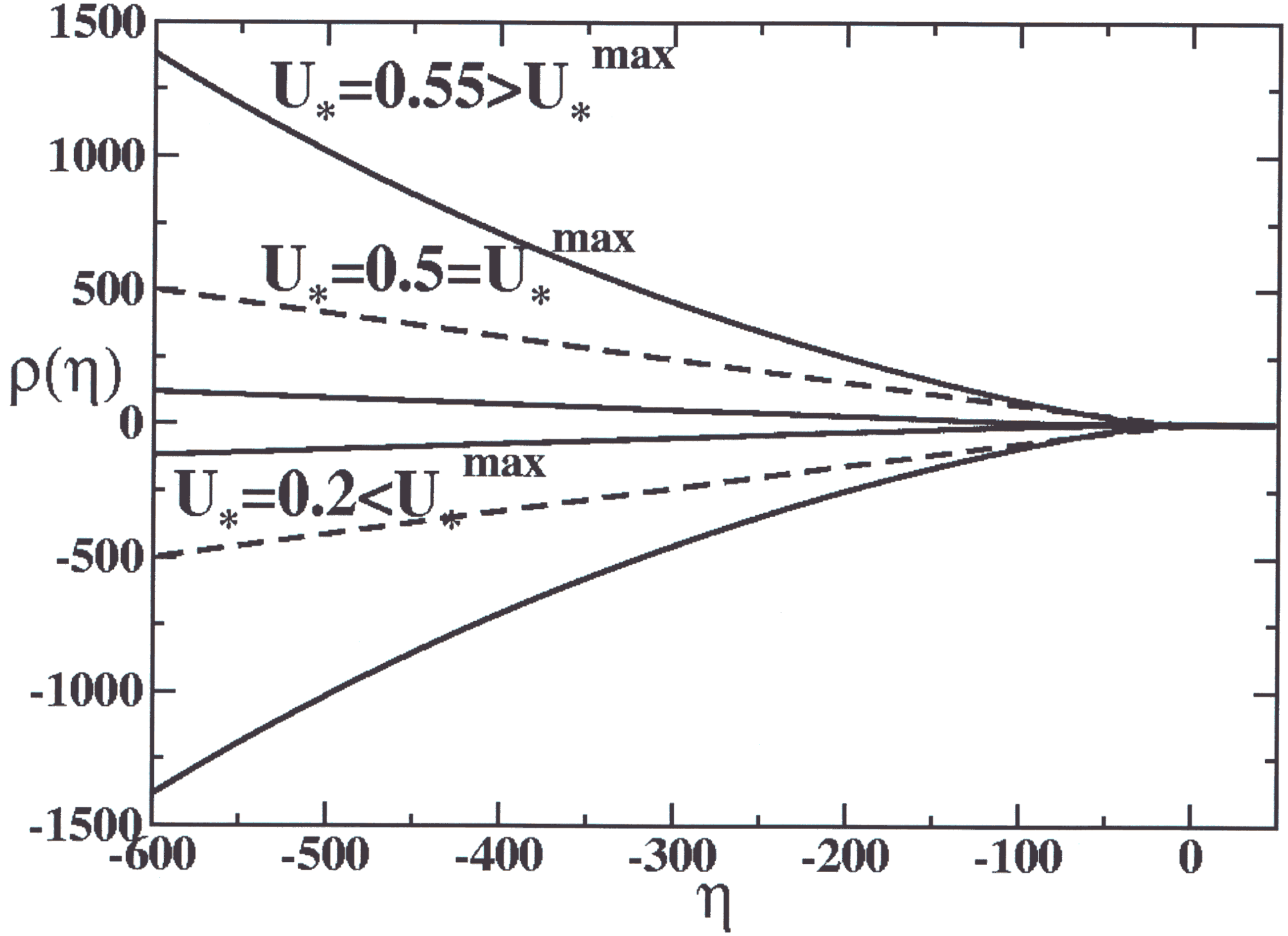
$$1 = U*\rho^2 - \rho^4 \frac{d}{d\eta}\left[\frac{1}{\rho} - 4\lambda \frac{d^2\rho}{d\eta^2} + \frac{U*}{\rho}\frac{d\rho}{d\eta}\right]$$

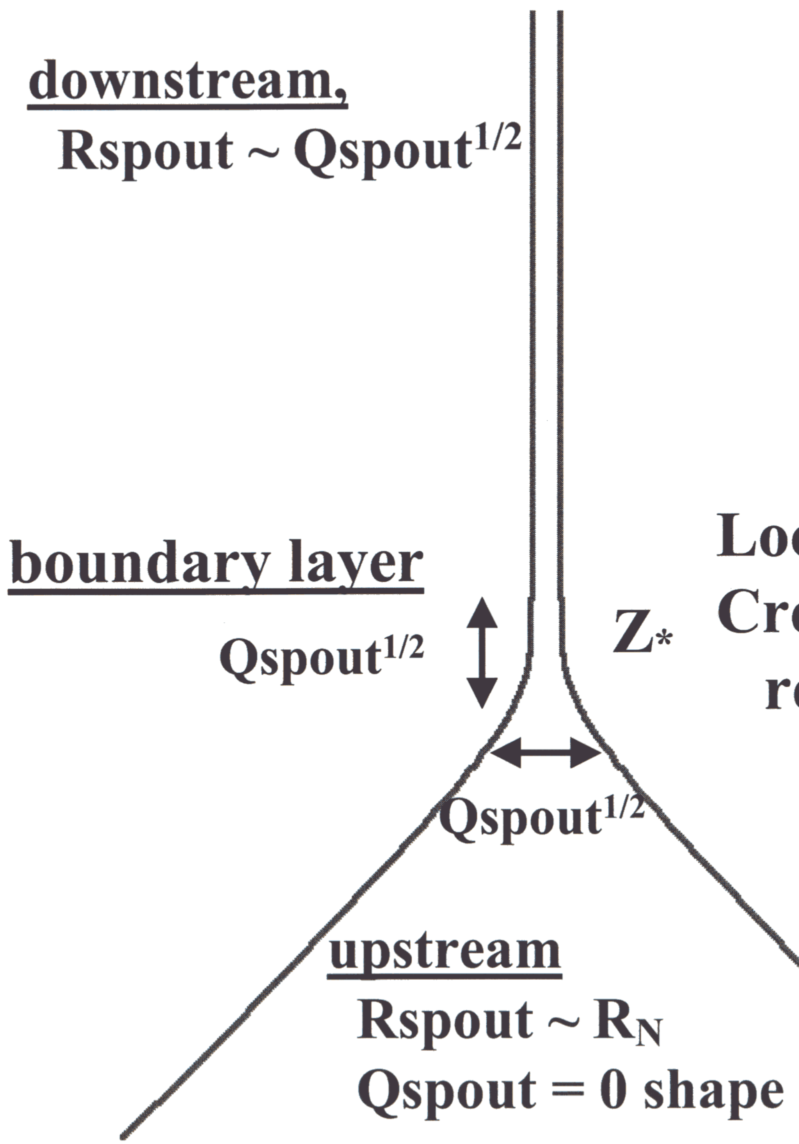
boundary conditions for similarity solution

upstream of z_* $\quad \rho(\eta) \to s\eta \qquad \eta \to -\infty$

downstream of z_* $\quad \rho(\eta) \to \sqrt{\frac{1}{U*}} \qquad \eta \to \infty$

A continuous family of similarity solutions exists for $U_* < Umax$, or $S < Smax$





Local condition:

Creating a singular spout requires a shallow cone base profile

Global condition:

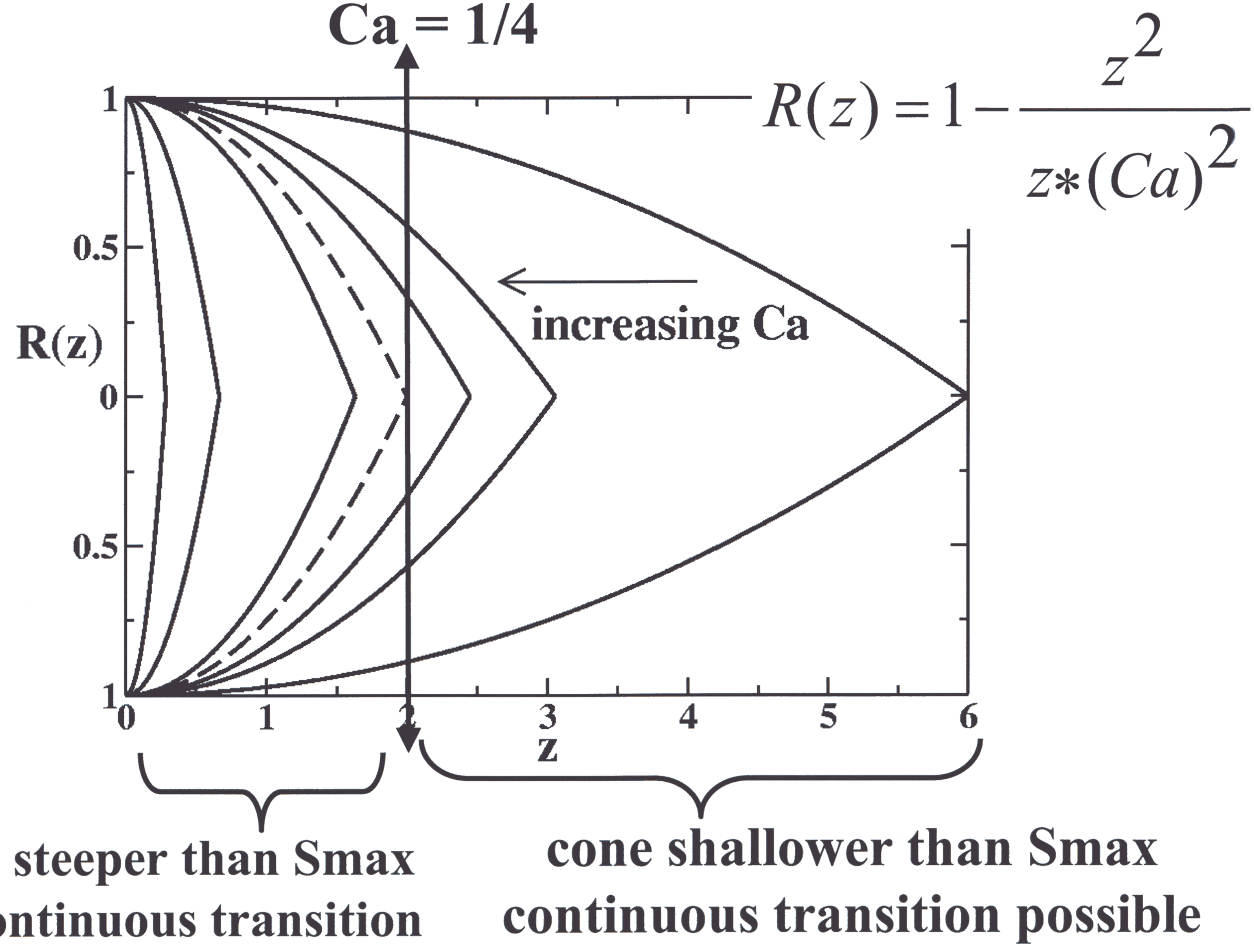
What nozzle boundary conditions gives rise to a shallow cone?

Governing equation supports a family of exact solutions for Qspout=0; valid for arbitrary λ

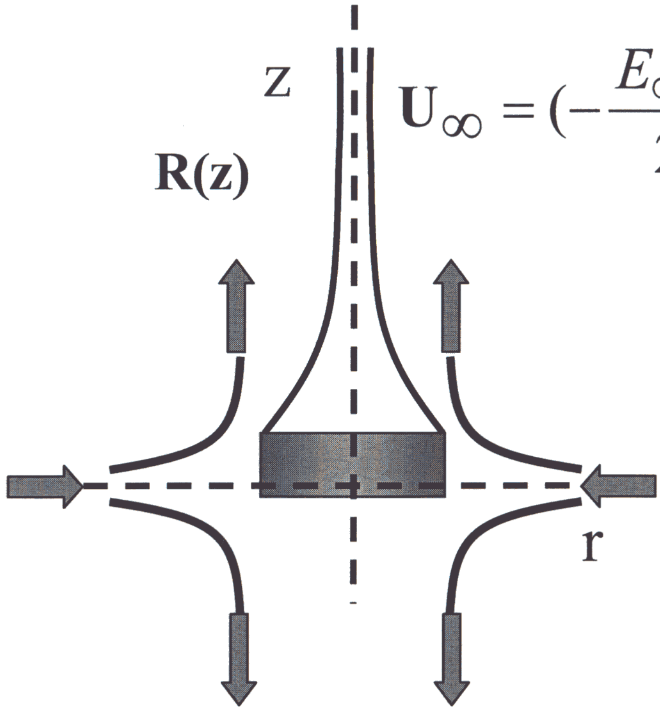
$$R(z) = 1 - \frac{z^2}{z*(Ca)^2} \qquad z*(Ca) = 2\sqrt{\frac{1}{2Ca} - 1}$$

$$Qspout = Caz\ R^2(z) - R^4(z)\frac{dP}{dz} \qquad P(z) = \frac{1}{R} - 4\lambda\frac{d^2R}{dz^2} + Ca\frac{z}{R}\frac{dR}{dz}$$

Solutions correspond to drops with conical tips



Strategy for creating singular (near singular) spouts

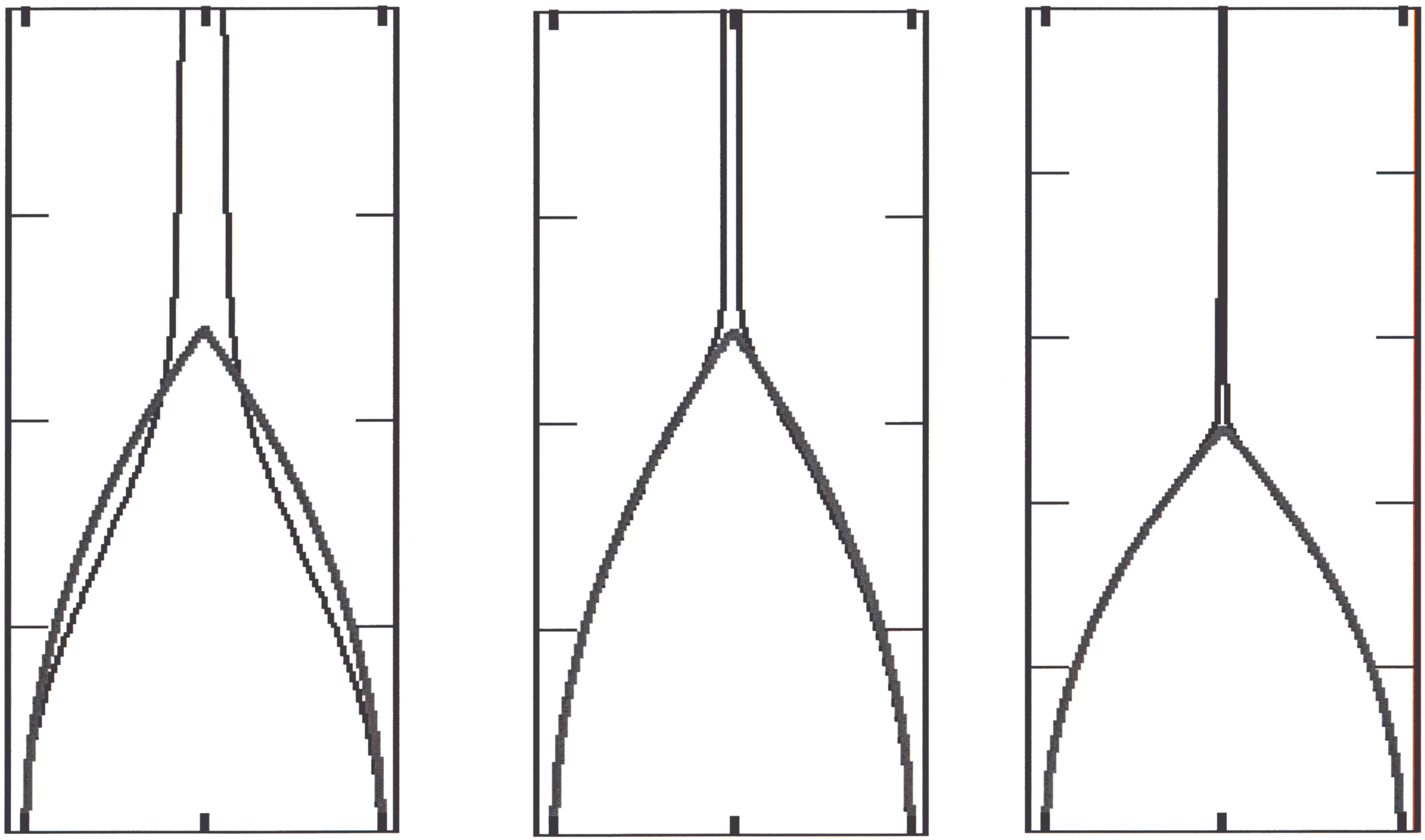


$$U_\infty = (-\frac{E_\infty r}{2}, 0, E_\infty z)$$

Let R(z=0) = 1, fix z_N

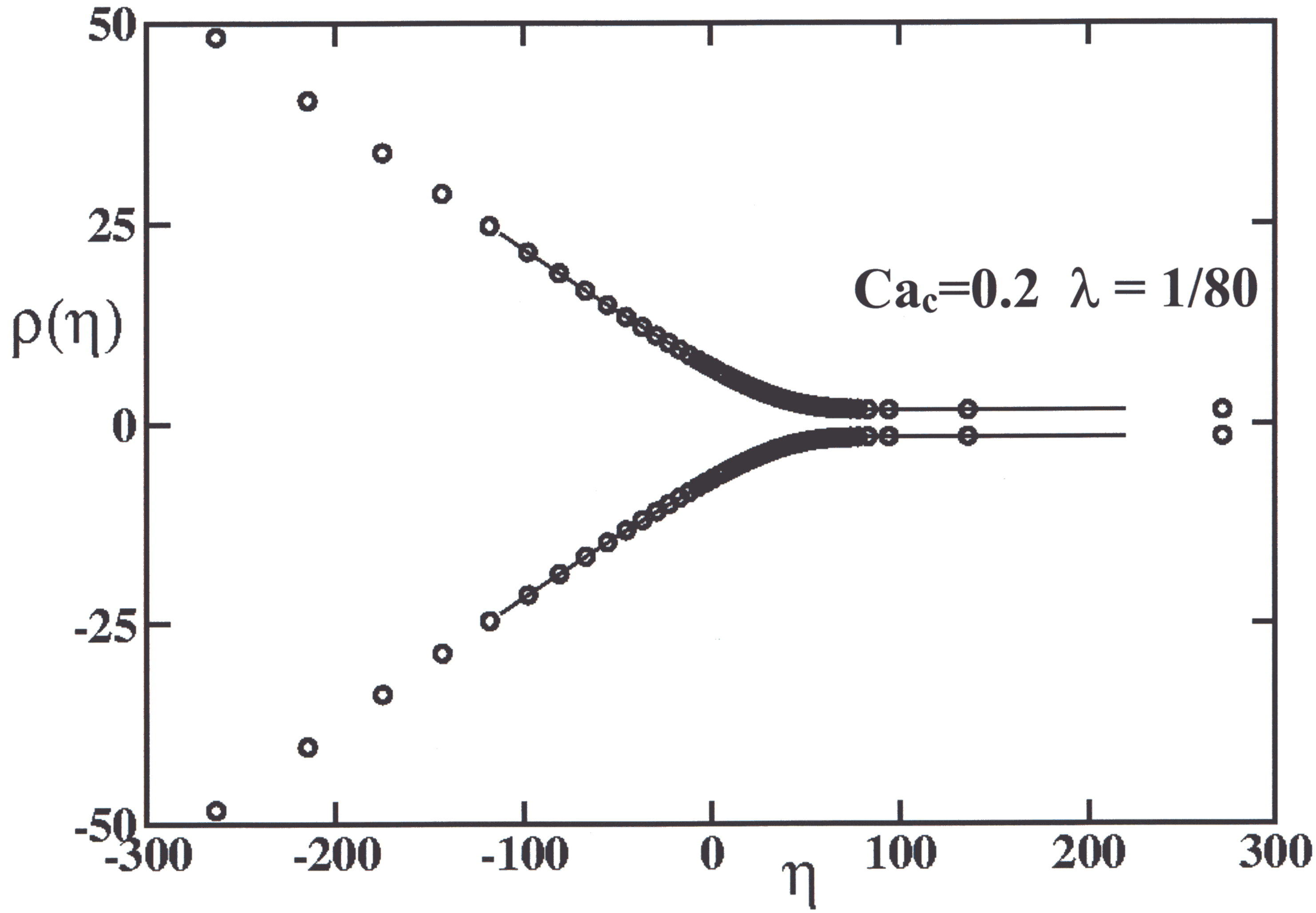
If $$R(z_N) = 1 - \left(\frac{z_N}{2\sqrt{(1/2Ca_c) - 1}}\right)^2 \qquad \left.\frac{dR}{dz}\right|_{z_N} = -\frac{z_N}{2\sqrt{(1/2Ca_c) - 1}}$$

then Qspout = 0 quadratic solution is a possible solution for R(z) when Ca = Ca_c

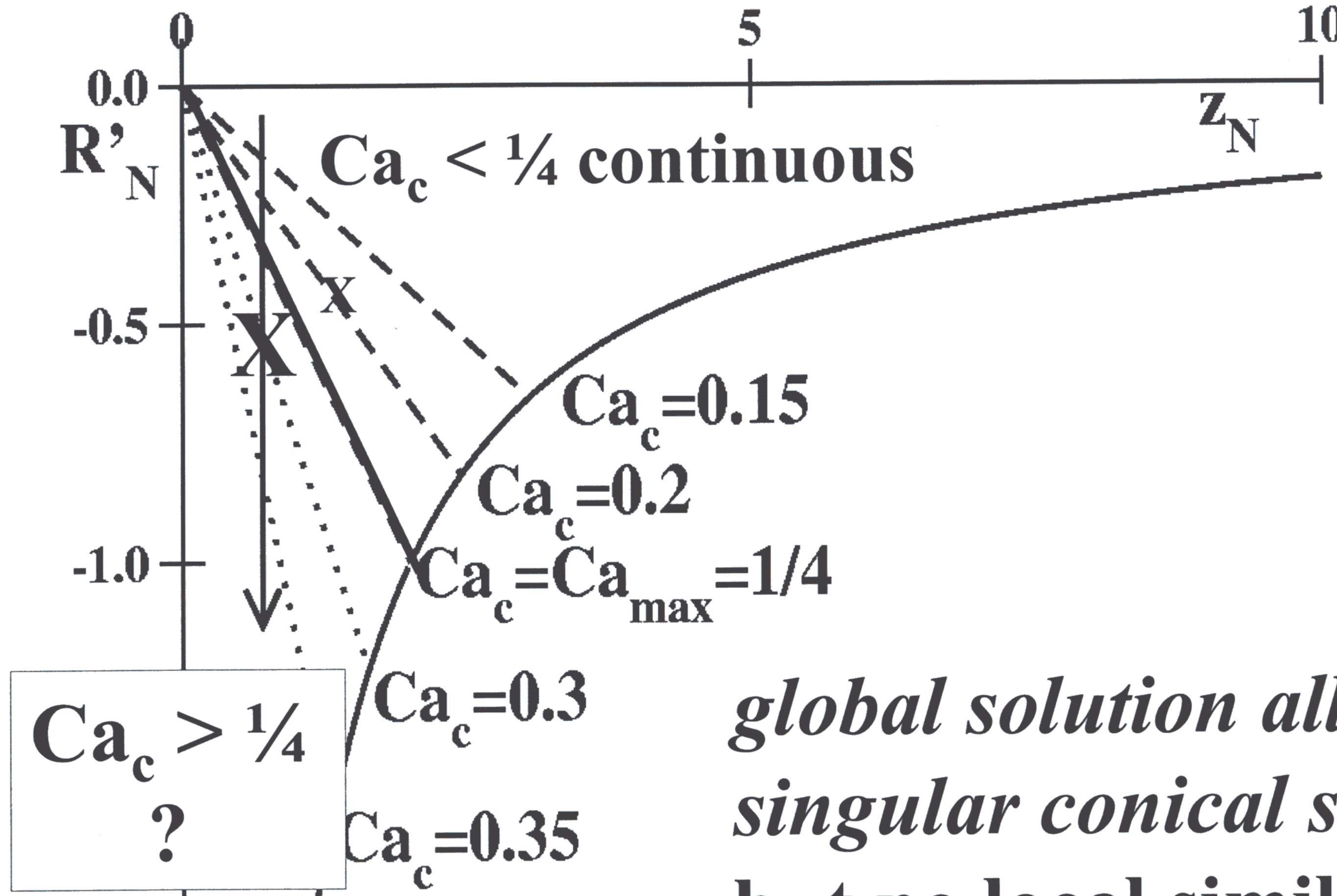


Ca=0.27　　Ca=0.21　　Ca→ Ca_c =0.2

rescaled full solution vs similarity solution

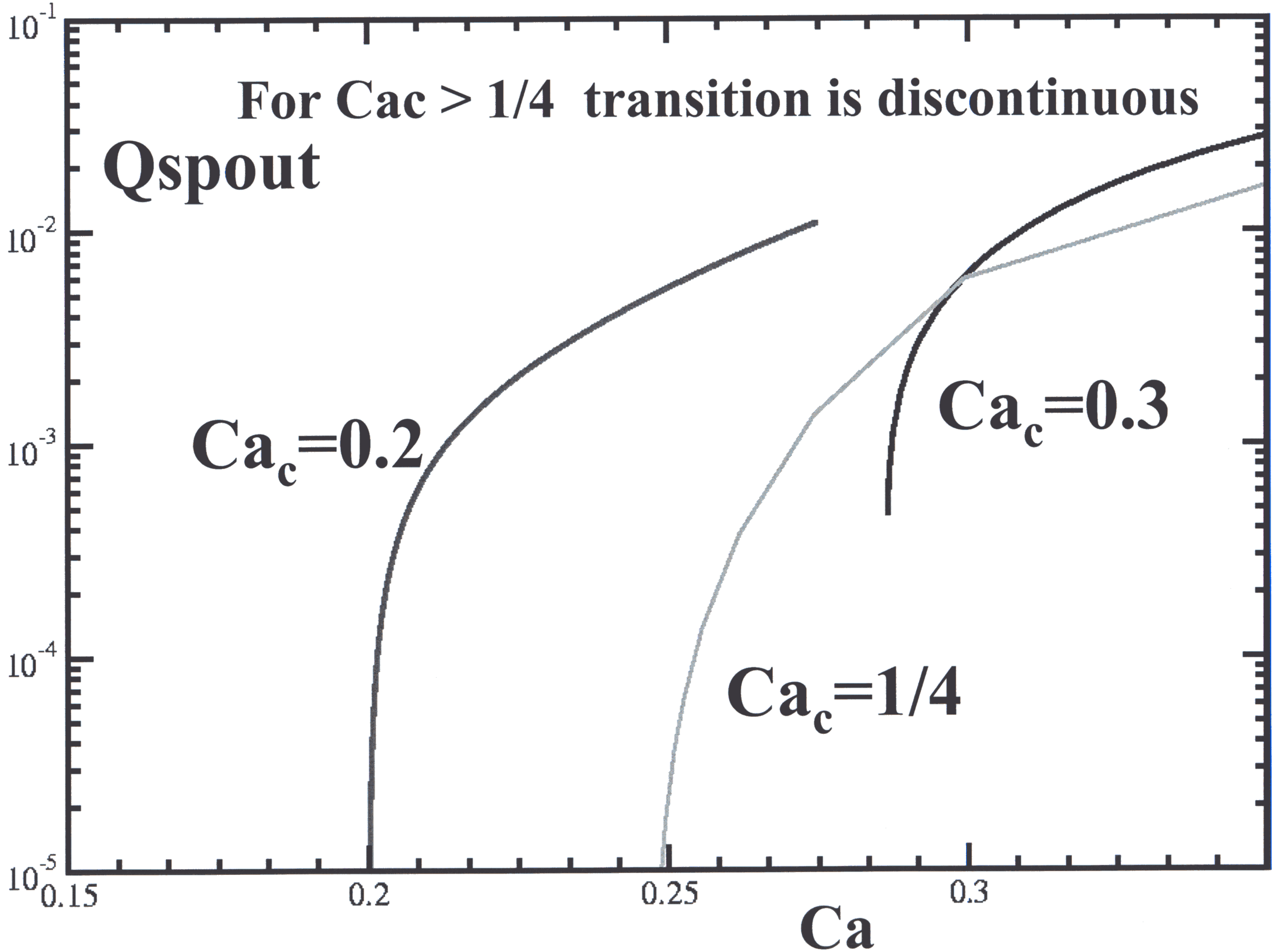


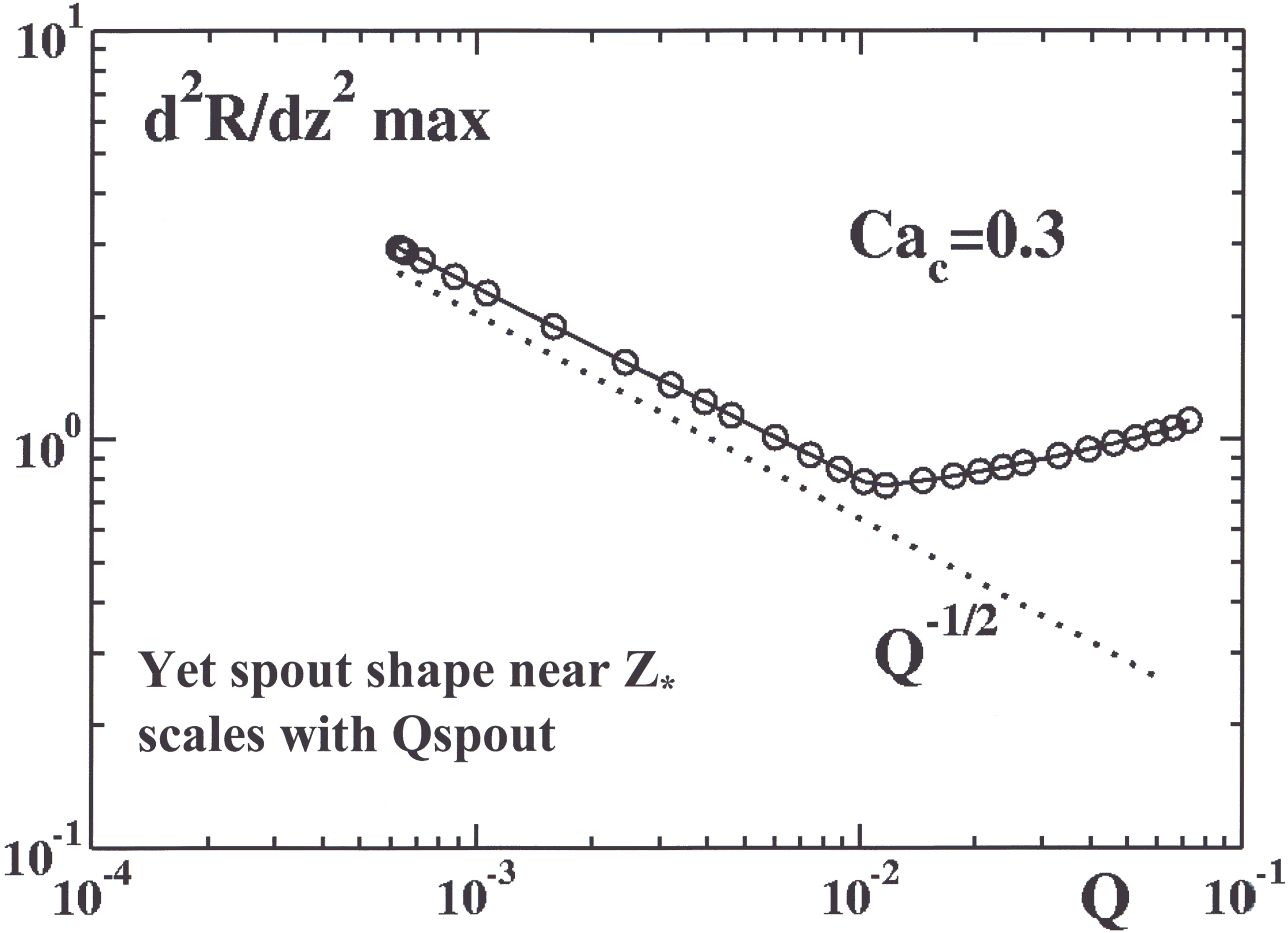
A continuous transition / singular spout created at $Ca_c = 0.2$

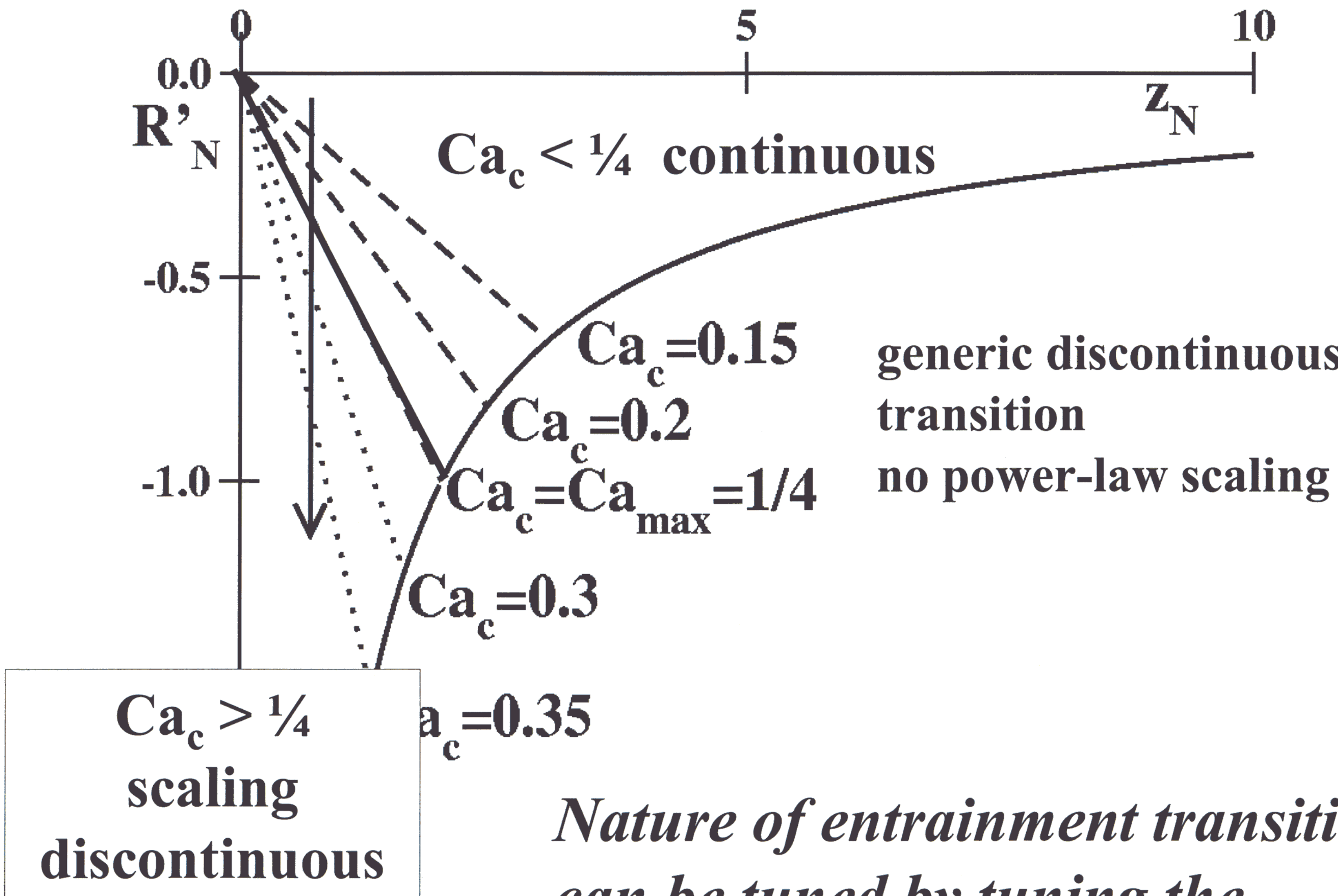


global solution allows singular conical shape

but no local similarity solution

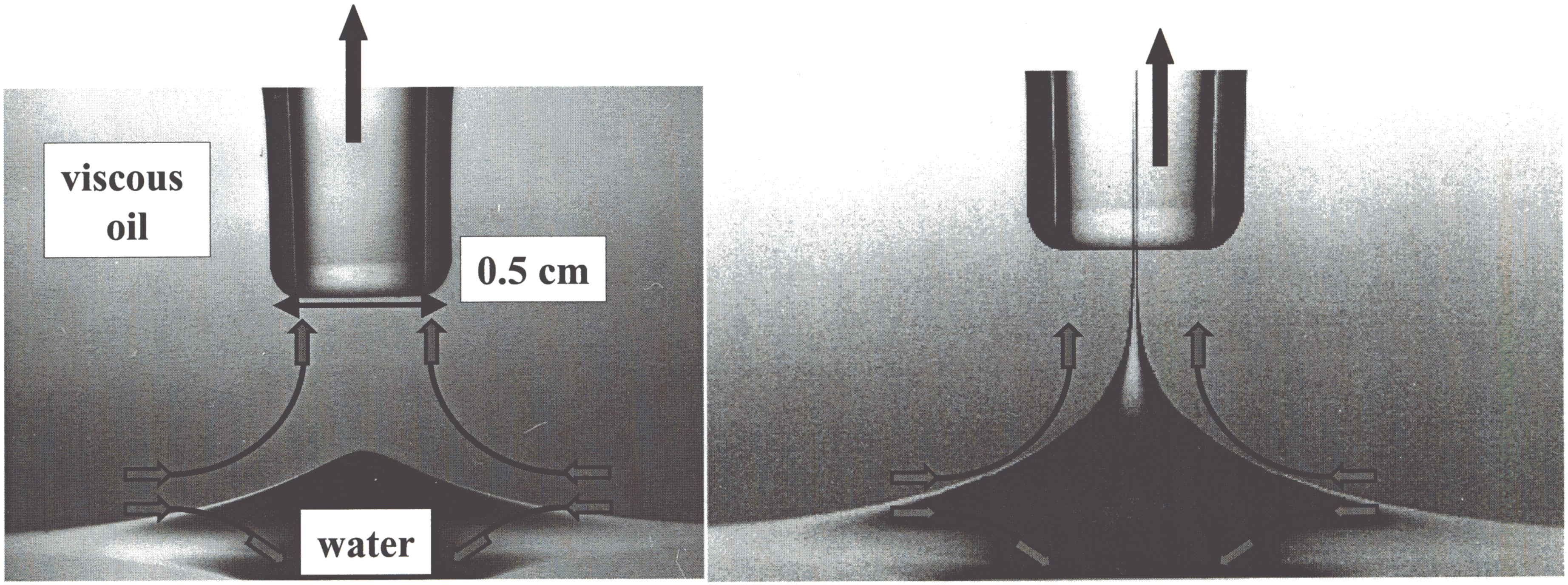

For Cac > 1/4 transition is discontinuous
Qspout
Ca_c=0.2
Ca_c=1/4
Ca_c=0.3
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
0.15
0.2
0.25
0.3
Ca


d^{2}R/dz^2 max
Ca$_c$=0.3
Q$^{-1/2}$
Yet spout shape near Z$_*$
scales with Qspout
10^1
10^0
10^{-1}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
Q



Nature of entrainment transition can be tuned by tuning the macroscopic boundary conditions

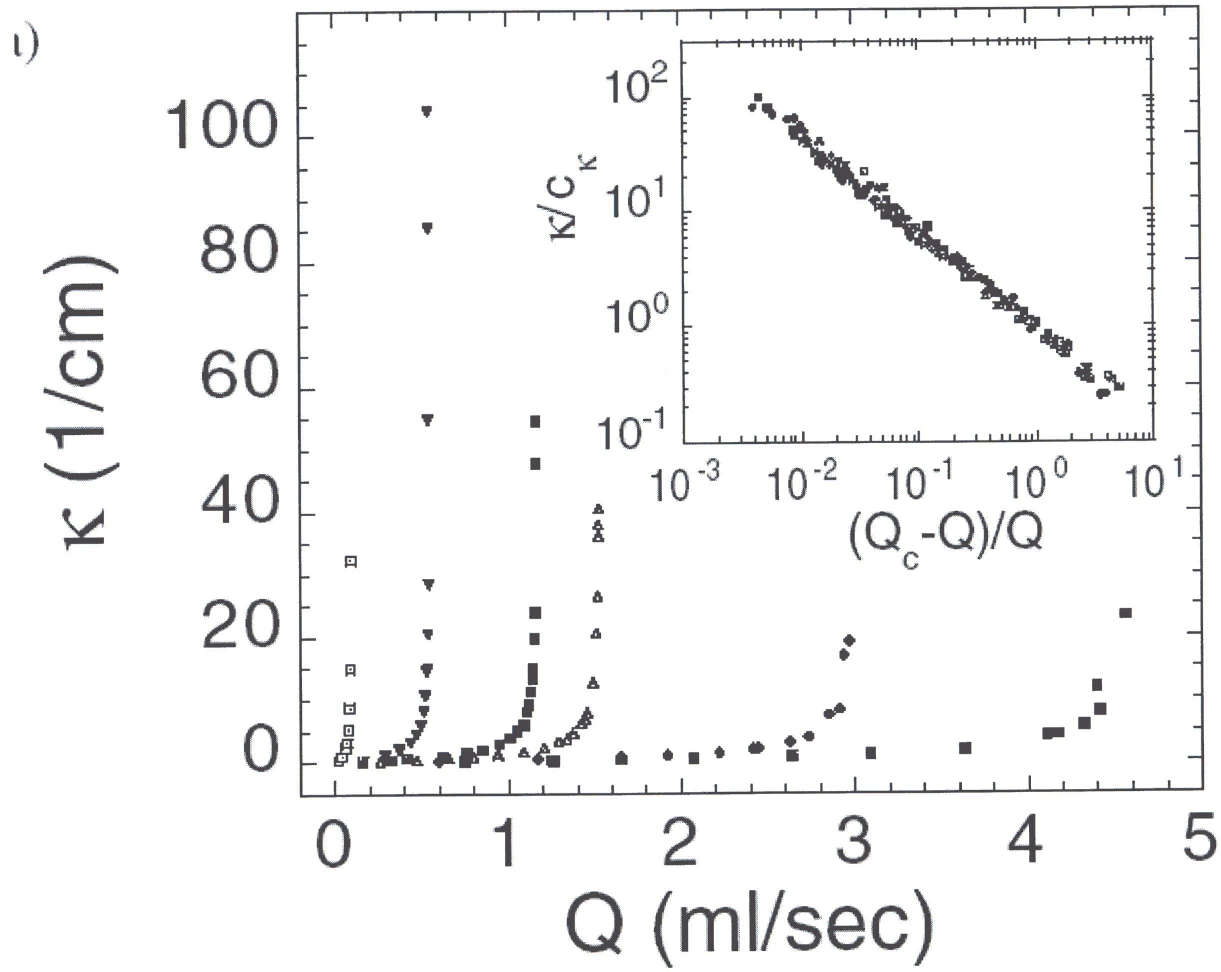
Selective withdrawal experiment



Cohen & Nagel PRL 2002

low
entrainment
rate

high
entrainment
rate



Cohen & Nagel PRL 2002

observed scaling & collapse of shape

but also

a small cutoff lengthscale:

R_hump ~ 10 μm