Workshop on Semiclassical Theory of Eigenfunctions and PDEs

25 years after

Victor Ivrii,
Department of Mathematics,
University of Toronto

Toronto, June 10, 2004

Abstract

\begin{modestyoff}

25 years ago I proved Weyl conjecture.

\end{modestyoff}

I want to discuss some ideas leading to this proof and some developments during these 25 years.

1 1-67 years AW (After Weyl)

1.1 The First Blood

For Laplacian H.Weyl (1911) proved that

$$N(\lambda) = c_0\lambda^{\frac{d}{2}} + o(\lambda^{\frac{d}{2}}) \tag{1}$$

as $\lambda \to +\infty$ and conjectured that

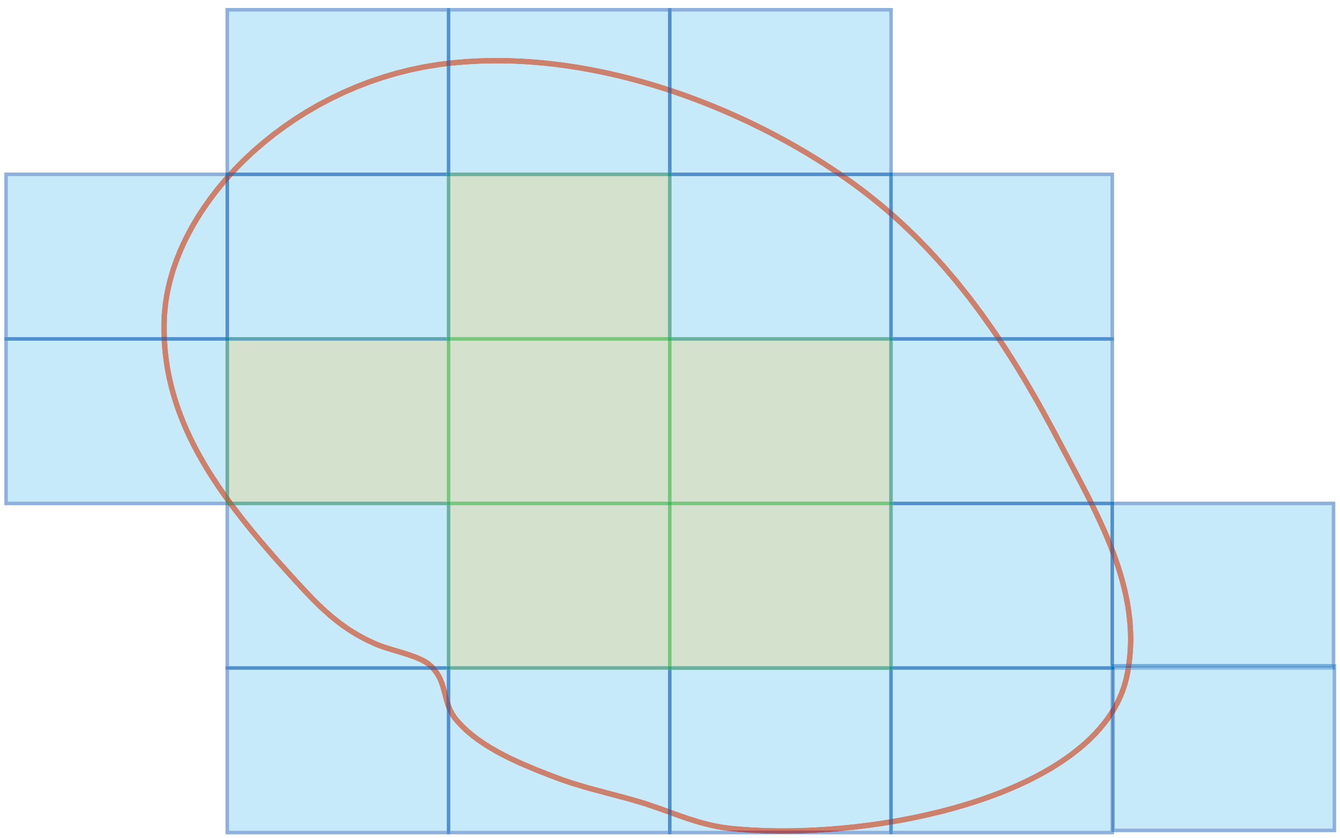
$$N(\lambda) = c_0\lambda^{\frac{d}{2}} + c_1\lambda^{\frac{d-1}{2}} + o(\lambda^{\frac{d-1}{2}}) \tag{2}$$

(1) was Debye's conjecture who derived it considering $N(\lambda)$ for Laplacian in rectangular $a_1 \times a_2 \times \cdots \times a_d$-box (and then $N(\lambda)$ equals to the number of integer points in the domain $\{(m_1,\ldots,m_d) \in \mathbf{Z}^{+\,d}, \frac{m_1^2}{a_1^2} + \cdots + \frac{m_d^2}{a_d^2} \le \frac{\lambda}{\pi^2}\}$

and as a Real Physicist decided that (*) must be true for any domain.

Weyl conjecture (2) was the result of a more precise analysis of the same problem by Weyl.

The proof of (1) by Weyl was based on this formula for boxes and variational arguments he invented. Covering domain by boxes



Weyl proved that

$$N(\lambda) \geq N_{\mathsf{new}}(\lambda) = \sum_\iota N_{\iota,D}(\lambda) \geq$$

$$\sum_\iota c_0 \operatorname{mes} B_\iota \, \lambda^{d/2} - o(\lambda^{d/2}) \geq c_0(\operatorname{mes} X - \epsilon)\lambda^{d/2}$$

with arbitrarily small $\epsilon > 0$; here ι runs inner boxes only and

$$N(\lambda) \leq N_{\mathsf{new}}(\lambda) = \sum_\iota N_{\iota,N}(\lambda) \leq$$

$$\sum_\iota c_0 \operatorname{mes} B_\iota \, \lambda^{d/2} + o(\lambda^{d/2}) \leq c_0(\operatorname{mes} X + \epsilon)\lambda^{d/2}$$

with arbitrarily small $\epsilon > 0$; here ι runs inner and boundary boxes. Combining these two inequalities Weyl got (1).

Richard Courant (1924) pushing Weyl approach to its limit proved

$$N(\lambda) = c_0\lambda^{\frac{d}{2}} + O(\lambda^{\frac{d-1}{2}} \log \lambda) \tag{3}$$

Note O and pesky $\log \lambda$.

Then generalizations , . . . generalizations , . . . generalizations , . . . generalizations , . . . generalizations but no improvement until 1952

1.2 Going Tauberian

B.Levitan (1952) and V.Avakumovič (1956) proved

$$N(\lambda) = c_0\lambda^{\frac{d}{2}} + O(\lambda^{\frac{d-1}{2}}) \tag{3}$$

but they considered Laplace-Beltrami on manifolds without boundary!

L.Hörmander (1968) generalized (LA) (still no boundary and only O).

J.J.Duistermaat-V.Guillemin (1975) proved Weyl conjecture
. . . but when there is no boundary! Geometric condition:
Periodic geodesic trajectories have measure 0.
Counter-example: sphere.

The approach in all these papers was Tauberian Fourier method. Namely consider

$$\sigma(t) = \operatorname{Tr}\cos(t\Delta^{\frac{1}{2}}) = \int \cos(\lambda t)\, d_\lambda N(\lambda^2) \tag{4}$$

where Δ is positive Laplacian.

On the other hand,

$$\sigma(t) = \int u(x,x,t)\, dx \tag{5}$$

where $u(x,y,t)$ solves

$$(D_t^2 - \Delta)u = 0, \tag{6}$$

$$u|_{t=0} = \delta(x-y), \qquad u_t|_{t=0} = 0. \tag{7}$$

So idea was to construct $u(x,y,t)$ and then $\sigma(t)$ by PDE methods and restore $N(\lambda)$ from (4).

Hörmander's calculus of Fourier integral operators was very useful.

$\sigma(t)$ was constructed modulo smooth functions.

It appeared that

- singularity of $\sigma(t)$at $t=0$ was isolated and
- $\operatorname{sing\,supp}\sigma(t)\subset\Pi$ where Π is the set of periods of periodic geodesics.

While Levitan, Avakumovič and Hörmander considered only singularity of $\sigma(t)$ at 0, Duistermaat and Guillemin considered all other singularities as well.

But Fourier Integral Operators served well far from the boundary while near the boundary tangent trajectories made life often difficult and often really miserable! Grazing rays, gliding rays, rays, which touch boundary but are neither grazing nor gliding.

It looked really bad until R.Seeley (1978) came with a new approach!

1.3 Almost There

Seeley's idea: it had been known (and used by Duistermaat-Guillemin) that if we know that there are no periodic trajectories with period less than T then

the remainder estimate would be $O(T^{-1}\lambda^{\frac{d-1}{2}})$.

Duistermaat-Guillemin looked at large T but Seeley looked at small T!

If we consider $\sigma(t) = \operatorname{Tr}\cos(t\Delta^{\frac{1}{2}})\psi$ where ψ is a nice cut-off function supported in the ball $B(x,\gamma(x))$ with $\gamma(x) = \frac{1}{2}\operatorname{dist}(x,\partial X)$ where X is our domain and ∂X is it's boundary then for time $T \asymp \gamma$ propagation from this ball does not know about boundary and contribution of this ball to the remainder estimate will be $O(T^{-1}\gamma^{d}\lambda^{\frac{d-1}{2}})$ which is $O(\gamma^{d-1}\lambda^{\frac{d-1}{2}})$.

Then the total remainder estimate will be

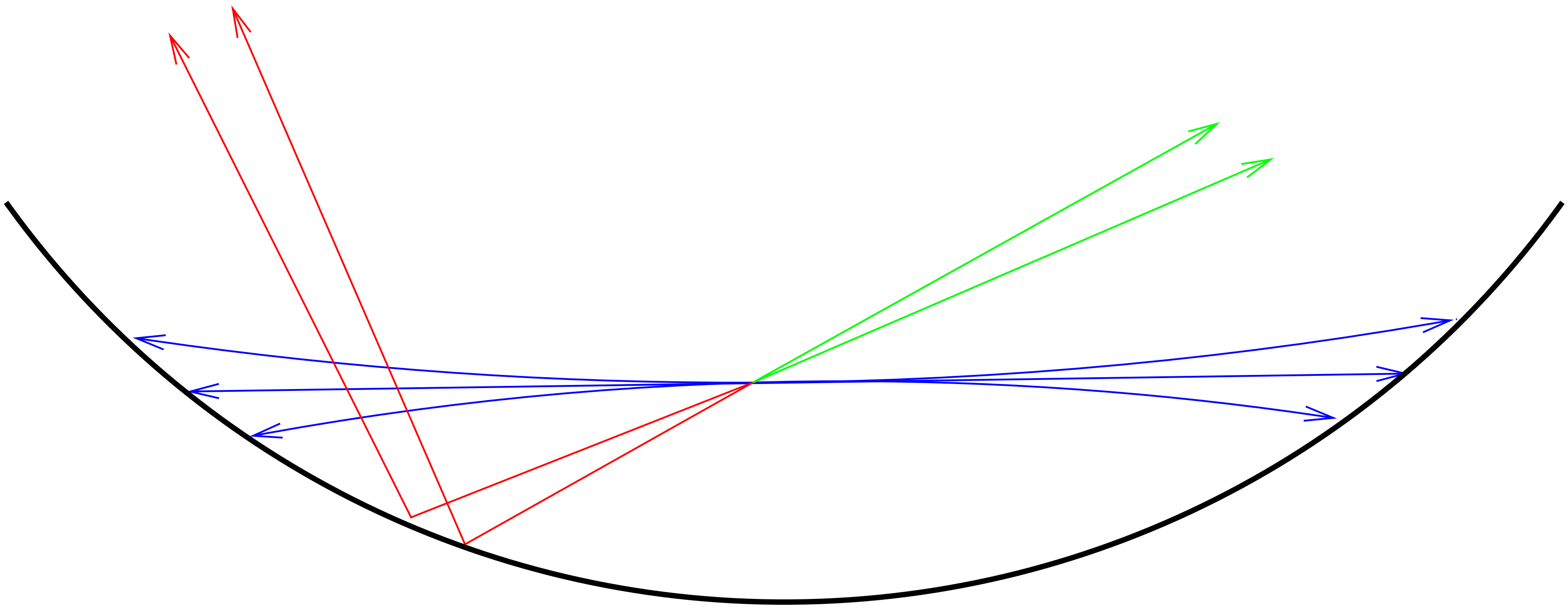
$$\lambda^{\frac{d-1}{2}}\int \gamma(x)^{-1}\,dx \tag{8}$$

where integral is actually taken over $\{x : \gamma(x) \geq \bar{\gamma} = \lambda^{-\frac{1}{2}}\}$

while contribution of the boundary strip $\{x : \gamma(x) \le \bar{\gamma}\}$ is $O(\lambda^{\frac{d}{2}} \times \bar{\gamma}) = O(\lambda^{\frac{d-1}{2}})$.

As ∂X is smooth integral $\int \gamma^{-1}\, dx \asymp \int \gamma^{-1}\, d\gamma$ mildly diverges and one gets Courant estimate $O(\lambda^{\frac{d-1}{2}} \log \lambda)$. **Oops!**

Seeley: increase T! Consider point (x, ξ) in the phase space. Then either trajectory launched from it is rather transversal to the boundary or almost parallel to it:



Outgoing (green), reflected (red) and tangent (blue) rays.

In the transversal case the trajectory launched in one time direction does not hit boundary for a while, and in the opposite time direction it hits boundary rather transversally and then again does not hit it for a while.

In the parallel case trajectory does not hit the boundary for a while. So we trace everything for time $T(x)$ which is of the same magnitude as a length of a blue line on the picture (green rays keep away from boundary even longer).

Then the total contribution of the inner part $\{x : \gamma(x) \ge \bar\gamma$ does not exceed

$$\lambda^{\frac{d-1}{2}} \int_{\{x:\gamma(x)>\bar\gamma\}} T(x)^{-1}\, dx \asymp \lambda^{\frac{d-1}{2}} \tag{9}$$

because Seeley considered smooth case when $T(x) = \epsilon\gamma(x)^{\frac{1}{2}}$.

So, Seeley (1978): (3) with the boundary!!!!

You forgot what was (3)? Shame on you:

$$N(\lambda) = c_0\lambda^{\frac{d}{2}} + O(\lambda^{\frac{d-1}{2}}) \tag{4}$$

2 Done!

2.1 New kid on the block

It was Winter '78-'79 when M.Shubin and B.Levitan suggested me to prove Weyl conjecture. I did not messed up with spectral asymptotics before.

My idea: to invent a new approach because (I thought) if Seeley's method worked for Weyl conjecture then Seeley would prove it!

I thought wrong!

Couple of years later D.Vassiliev who was секретный физик at that time gave the proof of Weyl conjecture using Seeley' method.

And much later I combined Seeley' approach with my own.

But it was the best mistake I ever made!

Because the method I invented worked in many situations Seeley' method did not, f.e. for general systems.

2.2 Normal Singularity

First, I conjectured that singularity of $\sigma(t)$ at 0 is *normal* i.e. $(tD_t)^n\sigma(t)$ at 0 have the same order of singularity for any n exactly as for manifolds without boundary.

I proved this conjecture by implicit method of propagation of singularities in Spring 1979.

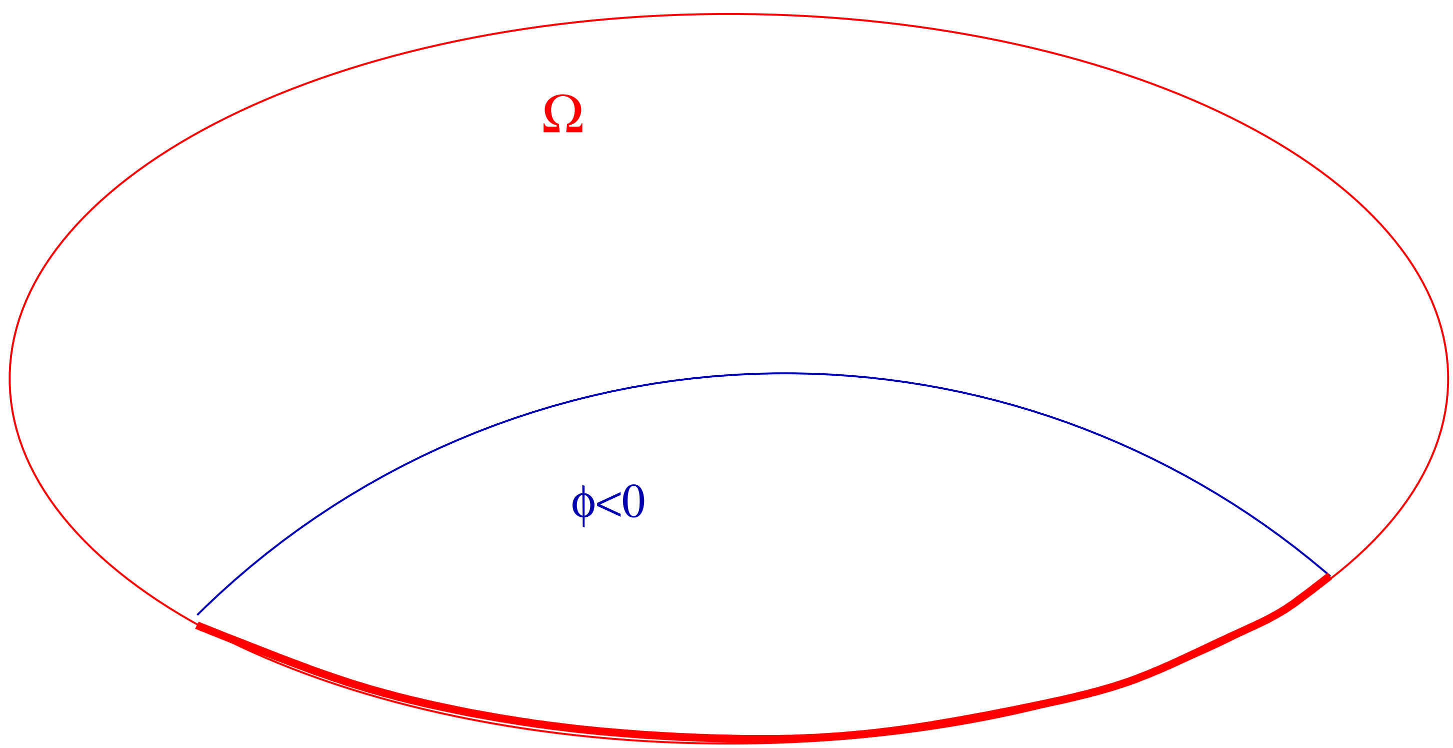
At this moment I had a powerful method of energy estimates with theorem stated in the form: if Ω is a domain in the phase space and ϕ satisfies microhyperbolicity condition,

and Pu is smooth in $\Omega \cap \{\phi < 0\}$,

and u is smooth in $\partial\Omega \cap \{\phi < 0\}$,

then u is smooth in $\Omega \cap \{\phi < 0\}$.

Looks like Holmgren uniqueness theorem! Look at the picture:



Using this technique plus rescaling arguments I proved that singularity of $\sigma(t)$ at $t = 0$ is normal.

2.3 Successive Approximations

My next idea was rather crazy: to calculate $u(x, y, t)$ using successive approximations

first going in coordinates where boundary was planar and getting operator with variable coefficients even if it originally was not

then to the problem

$$(D_t^2 - \Delta)u = 0, \tag{7}$$

$$u|_{t=0} = \delta(x-y), \quad u_t|_{t=0} = 0, \tag{8}$$

$$u|_{\partial X} = 0$$

apply successive approximation method freezing coefficients of Δ in point y.

This looks stupid because perturbation decreases smoothness by 2 while parametrix to the problem

$$\begin{aligned}
&(D_t^2 - \Delta)u = f,\\
&u|_{t=0} = u_t|_{t=0} = 0,\\
&u|_{\partial X} = 0
\end{aligned}$$

increases it only by 1, so each next term in this approximation approach is more singular than the previous one!

However, perturbation contains factors $(x_j - y_j)$ which are of magnitude t due to finite propagation speed and each parametrix contains factor t due to Duhamel integral, so in fact each next term in successive approximations acquires an extra factor $t^2 D_t$. Big deal, so what?

But then the same is true for $\sigma(t)$ as well, but for $\sigma(t)$ I knew already that near 0 each multiplication by t compensates one differentiation and this allowed me to justify the successive approximation method for $\sigma(t)$ near 0 without justification it for u.

So, complete asymptotics (with respect to smoothness) of $\sigma(t)$ near 0 was done. This would imply Seeley' result.

2.4 Other Singularities

It was early August 1979 and I wrote to M.Shubin about my progress.

His answer came two weeks later (no email at that time!): So what? Singularities at $t \neq 0$ are much more difficult!

But at this moment I already had a solution! Actually other singularities were easier.

Theorem 1 *If the set of periodic geodesic billiards has measure* 0 *then two-term (Weyl) asymptotics holds.*

To tackle "other" singularities of $\sigma(t)$ I analyzed Duistermaat-Guillemin method purging all irrelevant FIO stuff.

As a result arguments became very simple:

Fix any $T > 0$. Set of geodesic billiards periodic with periods $\leq T$ is closed nowhere dense set of measure 0.

And the set of all dead-end billiards is also of this type. Dead-end billiards are those which become tangent to the boundary or behaving badly.

Let $I = Q_1 + Q_2$ where Q_1 is a pdo with symbol in the small vicinity of the set Λ_T of all both types of bad billiards and the vicinity of boundary and Q_2 has a symbol vanishing in the vicinity of this set.

Then $\sigma(t) = \sigma_{Q_1}(t) + \sigma_{Q_2}(t)$ with $\sigma_Q(t) = \mathsf{Tr}(\cos(t\Delta^{\frac{1}{2}})Q)$.

Here $\sigma_{Q_2}(t)$ has no "other" singularities on $[-T, T]$ and Tauberian methods let me to recover asymptotics of $\mathsf{Tr}\, E(\lambda)Q_2$ with the remainder estimate $\frac{C}{T}\lambda^{\frac{d-1}{2}} + C'_T$.

Here and below $E(\lambda)$ is the spectral projector of Δ, C does not depend on T.

On the other hand, I recovered asymptotics of $\mathsf{Tr}\, E(\lambda)Q_1$ with the remainder estimate $C\epsilon\lambda^{\frac{d-1}{2}} + C'_{T,\epsilon}$ where $\epsilon = \mathsf{mes}\,\mathsf{supp}\, Q_1$ is arbitrarily small.

So, I recovered asymptotics of $N(\lambda) = \operatorname{Tr} E(\lambda)$ with the remainder estimate

$$C\Bigl(\epsilon + \frac{1}{T}\Bigr)\lambda^{\frac{d-1}{2}} + C'_{T,\epsilon}$$

and the rest was a second-year calculus exercise!

3 Aftermath (68-88 AW)

Using this method I instantly proved asymptotics with the remainder estimate $O(\lambda^{\frac{d-1}{m}})$ for m-th order elliptic systems on manifolds without boundaries,

and later on manifolds with the boundaries.

When I was peacefully exploiting my method and harvesting results and even published a book but two events happened:

3.1 Corners, edges etc

I got from РЖ Математика (Russian Math. Reviews but better) some weird paper deriving Weyl asymptotics for Euclidean Laplacian in polygons.

Author was spending a lot of efforts to consider wave equation near vertices and I realized that this was a completely unnecessary job!

So I decided to do a proper job and I with my student Sveta Fedorova proved (1984) Weyl formula for Laplace-Beltrami operators in domains with edges, vertices, conical points, cuts etc. But what is more important: rescaling technique was invented!

Somehow I lost the paper which inspired meand later I tried to find it with no success. **Spooky!**

3.2 Going Semiclassic

I decided to go cheap and prove some semiclassic asymptotics

I said cheap because at this time I believed as many did that really GREAT mathematicians like Weyl, Courant, Hörmander, Guillemin, Seeley and myself study classic: $N(\lambda)$ on compact manifolds

while less GREAT mathematicians study $N(\lambda)$ for Schrödinger with growing potential, semiclassics, etc

So I proved some semiclassical results and told M.Solomyak about them.

He asked: Why you just do not deduct it from classics by a cheap trick (Birman-Schwinger principle)?

I tried to follow this advice with rather surprising result. Semiclassical results derived this way were less general than I had already but working the opposite way I derived more general classical results than I had

It was an eye-opener:

$\boxed{\textbf{Semiclassical Asymptotics are most important!}}$

So, I began to study semiclassical asymptotics as a prime target.

3.3 Going Ballistic

First, I discovered that rescaling applied to semiclassic produces bunch of new results.

In particular eliminating $|V| + |\nabla V| \ne 0$ (as $d \ge 2$) as a precondition for sharp semiclassical asymptotics for Schrödinger operator

which using cheap trick got instantly new results for classical asymptotics.

Further I studied degenerations and singularities of different kinds, horns and cusps, other degenerations and singularities.

I derived classical asymptotics, asymptotics of eigenvalues for operators generalizing Schrödinger with potential growing at infinity, and for operators generalizing Schrödinger with potential slowly decaying at infinity, and these operators had their degenerations and singularities too.

It was a complete Results Explosion! And it was easy! I was like a prospector who found a place with native ores of gold lying just on the surface, ready to be picked. Look at my ICM-1986 talk!

But often remainder estimates were not as good as in the non-degenerate or non-singular cases and I felt that the reason was not my lack of skill but a more profound one.

3.4 Going Deeper

I modified my method to treat operators with symbols which were operators in auxiliary spaces and treated only some variables as Weylian and other as non-Weylian and I reexamined some singularities and degenerations and cusps and derived more sharp asymptotics than before. But these asymptotics either contained non-Weylian correction terms of the magnitude of the remainder estimates obtained on the previous stage or even were non-Weylian in their main parts.

I also considered Schrödinger and Dirac with strong magnetic field.

This was longer, slower and much more difficult process than before and its results are in my book.

3.5 Applications

This approach was applied in my and M.Sigal paper deriving Scott correction for asymptotics of the ground state energy for molecules consisting of very large atoms

and in my papers deriving Dirac-Schwinger correction for such molecules

and investigating this problem if there is a strong external magnetic field.

4 New Dawn (88-93 AW)

Last few years I have been working on sharp spectral asymptotics for operators with not very regular coefficients.

The major tool is the same kind of analysis as before, applied to mollified operator. Mollification scale depends on h and equals to $Ch|\log h|$ in the simplest case. This is related to Logarithmic Uncertainty Principle.

But it is a subject of my other talks.

One can download this and my other talks (in pdf) from web page

http://www.math.toronto.edu/ivrii/Research/Preprints.html