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Working Seminar on Nonlinear Evolution Equations
Nonlinear Heat Flow: Steepest Descent into a Convex Valley

Oct 2, 2002

Geodesics!

e.g. fluids

Lagrangian? friction
Conservative Hamiltonian

friction dominated
Dissipative; Gradient Flows

$\mu u\left(\frac{\partial v_i}{\partial t} + v_k \partial_k v_i\right) = -\partial_i P(u(x))$

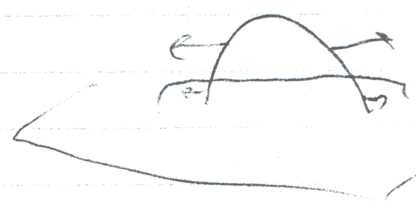
$\frac{\partial u}{\partial t} + \mathrm{div}[u \vec{v}_i] = 0$

$\frac{\partial u}{\partial t} = \Delta(P(u))$

$= \Delta(u^m)$

$\frac{\partial u}{\partial t} + \mathrm{div}[u^{m-1}(\nabla u)] = 0$

BUNNY RABBITS



$m > 1$

$m = 1$

$m < 1$

$\left[C_m + \frac{1-m}{2m} x^2\right]_+^{\frac{1}{1-m}}$ ⟸ minimizes $\|u\|_m$ at fixed mass

$u(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-|x|^2/2t} * u_0 \to 0$ as $t \to \infty$

$C_m > 0$

not any more dissipative

WHY FRICTION DOMINATED?
WHY FLUIDS?

Mechanical analogy

Behaviour of PDE:

if $m \geq 1 - \frac{2}{n}$

① Conserves 0th and 1st moments; fine CofM

② 2nd Moments grow $R^2 := \int x^2 u(x)$

③ Rate of convergence of shape: $u(x) = \frac{1}{R^n} \hat{u}\left(\frac{x}{R}\right)$

two solutions $u_1(t), u_2(t)$

Distance $d_2(\hat{u}_1(t), \hat{u}_2(t)) \leq \frac{d_2(\hat{u}_1(0), \hat{u}_2(0))}{(1 + k_n t)^{\frac{1}{2 - n(1-m)}}}$

if $m \geq 1 - \frac{1}{n}$

are adjusted $t(R(0) = 0)$

Fluid Dynamics: Friction Dominated Form

Compressible Euler's equations

$$\begin{cases} u\left(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x^k}\right) = -\frac{\partial}{\partial x^i} P(u(x)) - \beta v_i(x) u \\ \frac{\partial u(x)}{\partial t} + \operatorname{div}[u(x)\, \vec{v}(x)] = 0 \quad \text{dy} \\ P(u(t,x)) \quad \text{eq. of State} \end{cases}$$

Darcy's Law: neglect inertial term

$$v_i = -\frac{1}{\beta u} \frac{\partial}{\partial x^i} P(u(x))$$

$$\frac{\partial u}{\partial t} = \frac{1}{\beta} \operatorname{div}[P'(u(x))\, \nabla u(x)] = \Delta(u^m)$$

$$= \frac{1}{\beta} \operatorname{div}[\nabla P(u(x))]$$

if $P' = m\beta u^{m-1} \Rightarrow P(u) = \frac{m\beta}{m-1} u^{m-1} + \text{const}$

$$= \Delta(P(\rho))$$

increasing all $m>0$

~~decreasing~~

Internal Energy Associated

$$\int A(\rho(x))\, d^3x \quad \text{where} \quad A(\rho) = \rho \int_0^\rho \frac{P(s)}{s^2}\, ds$$

$$A' = \int_0^\rho \frac{P(s)}{s^2} + \frac{P(\rho)}{\rho} \qquad = \beta\rho \int_0^\rho s^{m-2}\, ds = \frac{\beta\rho}{m-1}\rho^{m-1} = \frac{\beta\rho^m}{m-1}$$

$$A'' = \frac{P'(\rho)}{\rho} \qquad P' = \rho A''$$

Proof: for a box $\boxed{m}\, V$ $\qquad U = \int_{m/\rho}^\infty P\left(\frac{m}{V}\right) dV$

$\rho = \frac{m}{V} \qquad dV = -\frac{m}{\rho^2}\, d\rho$

$$= m \int_0^\rho \frac{P(\rho)}{\rho^2}\, d\rho$$

$$U(\rho) = \int_{\mathbb{R}^3} \rho(x) \int_0^{\rho(x)} \frac{P(s)}{s^2}\, ds \; d^3x$$

? 2a

What is good about Wasserstein Distance? (Guessing exponents)
WARM UP PROBLEM $(m=1)$: 3 facts

1. translation: $d_2(u,v) = |z|$ if $u(x) = v(x+z)$
2. dilation: $d_2(u_1,u_2)^2 = (R_1-R_2)^2 + R_1R_2\, d_2(\hat{u}_1,\hat{u}_2)^2$

TOWAKF 3. contraction $d_2(u_1\circ f, u_2\circ f) \le d_2(u_1,u_2)$

Proof of 2: $|x_1-x_2|^2 = (|x_1|-|x_2|)^2 + |x_1||x_2|\left|\frac{x_1}{|x_1|} - \frac{x_2}{|x_2|}\right|^2$

$$(x,y) \xrightarrow{F} (R_1x, R_2y)$$

$$\hat{\gamma} \in \Gamma(\hat{u}_1,\hat{u}_2) \xrightarrow{F_\#} \gamma \in \Gamma(u_1,u_2)$$

$$\int |x-y|^2\, d\gamma = \int |R_1x - R_2y|^2\, d\hat{\gamma}$$

$$= R_1^2 \int |x|^2\, d\hat{u}_1 + R_2^2 \int |y|^2\, d\hat{u}_2 + R_1R_2 \int |x|^2 - 2\langle x,y\rangle + |y|^2\, d\hat{\gamma}$$

$$-2R_1R_2$$

$$= (R_1 - R_2)^2 + \int |x-y|^2\, d\hat{\gamma}(x,y)$$

$$\frac{d}{dt}(R(t)^2) = \int x^2\, \Delta u^m = 2n \int u^m = 2n \text{ if } m=1$$

$$\ge 2nR_{m}^{n(1-m)} \min_{i=1,2} \int \hat{u}_i^m$$

$$R(t)^2 = R(0)^2 + 2nt \ge 2nt$$

$$d_2(u_1(0),u_2(0)) \underset{③}{\ge} d_2(u_1(t),u_2(t))$$

$$\underset{②}{\ge} \sqrt{R_1(t)R_2(t)}\, d_2(\hat{u}_1(t),\hat{u}_2(t))$$

$$\ge \sqrt{2nt}\, d_2(\hat{u}_1(t),\hat{u}_2(t)).$$

Guess correct exponent. $m \neq 1$.

$$\frac{d}{dt}(R(t)^2) = 2R(t)\dot{R}(t) \geq 2n R^{n(1-m)}(t) \min_{[0,\infty)} \int \hat{u}_i^m$$

$$\frac{R^{2-n(1-m)}}{2-n(1-m)}\Big|_0^t \geq \dot{n} k_0 t$$

$$R(t) \geq \left(R(0)^\alpha + \alpha n k_0 t \right)^{1/\alpha}$$

$$\alpha = 2 - n(1-m)$$

$$d_2(\hat{u}_1(t), \hat{u}_2(t)) \leq \frac{1}{(\alpha n k_0 t)^{1/\alpha}} d_2(u_1(0), u_2(0))$$

if it still true that: $d_2(u_1(0), u_2(0)) \geq d_2(u_1(0), u_2(0))$

Friction Dominated Dynamics: Mechanical Analogy

Newton's 2nd Law

$$m\ddot{x} = -\nabla V(x) - \beta\dot{x}(t)$$

$\beta = 0 \Rightarrow$ conservative $\frac{m\dot{x}^2}{2} + V(x(t)) =$ const

rate of dissipation

$$\beta\dot{x} = -\nabla V(x)$$

$\Rightarrow$ rate of dissipation energy

~~$\beta \gg \min(m, \sqrt{m})$~~

friction dominated

either a.

$$\Rightarrow \beta\ddot{x} = -\nabla^2 V(x)\,\dot{x}(t)$$

$$m\ddot{x} = -\frac{m}{\beta^2}\nabla^2 V(x)\,\nabla V(x)$$

$\nabla^2 V(x) \ll \frac{\beta^2}{m}$

$\Rightarrow$ self consistent.

Global Contractions

LEMMA 1: If $V: \mathbb{R}^n \to \mathbb{R}$ is convex, then two solutions to $\frac{dx}{dt} = -\nabla V(x(t))$ satisfy $\frac{d}{dt}|x_1(t) - x_2(t)| \le 0$.

If $\nabla^2 V(x) \ge k$, then $|x_1(t) - x_2(t)| \le e^{-kt}|x_1(0) - x_2(0)|$.

Proof: Let $f(t) = |x_1(t) - x_2(t)|^2/2$. Then

$$
\begin{aligned}
f'(t) &= \langle x_1(t) - x_2(t), \dot{x}_1(t) - \dot{x}_2(t)\rangle \\
&= -\langle x_1(t) - x_2(t), \nabla V(x_1(t)) - \nabla V(x_2(t))\rangle \\
&= -\left[\frac{d}{ds} V((1-s)x_2(t) + s x_1(t))\right]_0^1 \le 0 \text{ by convexity} \\
&= -\frac{d^2}{ds^2} V((1-s^*)x_2(t) + s^* x_1(t))\,(1-0)\,\frac{|x_1 - x_2|^2}{|x_1 - x_2|^2} \\
&= -\langle x_1(t) - x_2(t), \text{Hess}_{x^*} V\,(x_1(t) - x_2(t))\rangle \\
&\le -2k\,f(t)
\end{aligned}
$$

$$\Rightarrow f(t) \le e^{-2kt} f(0)$$

$$\Rightarrow |x_1(t) - x_2(t)| \le e^{-kt}|x_1(0) - x_2(0)|$$

$\square$

where $d_2^2(u,v) = \inf_{\gamma\in\Gamma} \int |x-y|^2 \, d\gamma(x,y)$ ✓

$\Gamma = \{ 0 \le \gamma$ on $\mathbb{R}^n\times\mathbb{R}^n \mid \int f(x)u(x) = \int_{\mathbb{R}^n} f(x)\gamma(x,y)\,dx\,dy$, $\int g(y)\gamma(x,y) = \int g(y)v(y)\,dy$

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3 How to Prove: 3 central ingredients in the proof. (3)

(I) $\mathcal{M}_{2,0}(\mathbb{R}^n) := \{ 0 \le u$ on $\mathbb{R}^n \mid 1 = \int u \quad \int x^2 u < \infty \quad \int x u = 0 \}$

forms an infinite dimensional Riemannian manifold, on which d_2 gives the geodesic distance.

(II) Dynamics represent ~~steepest~~ descent of the (Newman-Renyi" energy on the manifold (M, d_2).

$$E(u) = \frac{1}{m-1}\int (u^m - u)\,dx$$

McCann (III) $E(u)$ is geodesically convex on the manifold M

$m \ge 1 - \frac{1}{n}$ $\Rightarrow$ gradient flows should be non-expansive

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MOTIVATION: finite dimensional convex gradient flows
(1) Global contraction proof.

PHYSICAL ENERGY

~~WHEN~~
WHERE DOES NEWMAN-RENYI ENTROPY COME FROM?

Nonlinear Heat Flow III: Steepest Descent into a Convex Valley
Oct 30, 2002

RECAP
$$\begin{cases} \frac{\partial u}{\partial t} = \Delta(u^m) \\ u(0,x) = u_0(x) \end{cases}$$

defines an evolution in $\mathcal{M}(\mathbb{R}^n) = \{ 0 \le u \text{ on } \mathbb{R}^n \mid \int u < \infty, \int x^2 u < \infty \}$
in fact in $\mathcal{M}_{1,0}(\mathbb{R}^n) = \{ u \in \mathcal{M}(\mathbb{R}^n) \mid 1 = \int u, \int \vec{x} u = 0 \}$

① $\mathcal{M}_{1,0}(\mathbb{R}^n)$ is a "Riemannian" manifold where geodesic dist is

$$d_2(u,v)^2 := \inf_{\gamma \in \Gamma(u,v)} \int |x-y|^2 \, d\gamma(x,y)$$

cut $\left\{ \Gamma(u,v) = \{ 0 \le \gamma \text{ on } \mathbb{R}^n \times \mathbb{R}^n \mid \int_{\mathbb{R}^{2n}} f(x) \, d\gamma(x,y) = \int_{\mathbb{R}^n} f(x) u(x), \quad \int f(y) \, d\gamma(x,y) = \int g(y) v(y) \} \right.$

② evolution is steepest descent of the "entropy"

$$E(u) = \frac{1}{m-1} \int u^m(x) - u(x) \quad \text{on } (\mathcal{M}, d_2)$$

③ $E(u)$ is geodesically convex i.e.

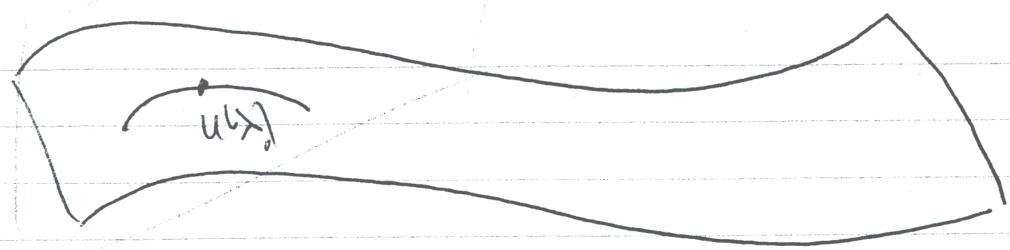
$$\left.\begin{array}{l} d(u_s, u_0) = s\, d(u_0, u_1) \\ d(u_s, u_1) = (1-s)\, d(u_0, u_1) \end{array}\right\} \Rightarrow E(u_s) \le (1-s) E(u_0) + s E(u_1)$$

①②③ $\Rightarrow d(u_1(t), u_2(t)) \le d(u_1(0), u_2(0))$ $\quad R_i(t) = \int x^2 u_i$

cut $\hat{u}_i = \frac{1}{R_i^n} u\left(\frac{\cdot}{R_i}\right) \Rightarrow d(\hat{u}_1(t), \hat{u}_2(t)) \le \cancel{C_0}\, O\left(\frac{1}{t^{\alpha m}}\right)$

$$\le O\left(t^{-\frac{1}{2-n(1-m)}}\right)$$

known



$M = \mathcal{P}_{2,ac}(\mathbb{R}^n)$

Object \ Representation	Abstract	Formal $u \ll H^n$ densities	Riemannian normal coor.
curve	$\lambda \in [0,1] \to u(\lambda) \in M$	$u(\lambda, x)\, d^n x$	
tangent vector	$\frac{du}{d\lambda} = \dot u(\lambda)$	$\frac{\partial u}{\partial \lambda}(\lambda_0, x)$	$\psi(x)\ \partial_n$; $\frac{\partial u}{\partial \lambda}\Big\vert_{\lambda_0} + \mathrm{div}[u \nabla \psi] = 0$; $\Delta_u \psi$
length²	$\lVert \dot u(\lambda) \rVert^2_{T_{u(\lambda)}}$	$\int \left\lvert u \nabla \Delta_u^{-1} \frac{\partial u}{\partial \lambda} \right\rvert^2$	$\int_{\mathbb{R}^n} u(\lambda, x)\, \lvert \nabla \psi(x) \rvert^2 = \lVert \psi \rVert^2_{W^{1,2}_u}$

cont.

CLAIM 1: $d_2^2(u_0, u_1) = \inf\limits_{\substack{\lambda \to u(\lambda) \in M \\ u(0) = u_0 \\ u(1) = u_1}} \int_0^1 \lVert \dot u(\lambda) \rVert^2_{u(\lambda)}\, d\lambda$

CLAIM 2: Gradient flow of $E(u)$ w.r.t. $\lVert \ \rVert^2_{T_M}$ is

$$\frac{\partial u}{\partial t} = \mathrm{div}\left[u \nabla \frac{\delta E}{\delta u}\right]$$

e.g. $\frac{\delta E}{\delta \rho} = \frac{m}{m-1} u^{m-1}$

$\nabla \frac{\delta E}{\delta \rho} = m\, u^{m-2}\, \nabla u$

$\mathrm{div}\left[u \nabla \frac{\delta E}{\delta \rho}\right] = \mathrm{div}\left[m\, u^{m-1} \nabla u\right] = \Delta u^m.$

DEF'N: (push-forward) Given measurable spaces (X, Σ) $(\tilde{X}, \tilde{\Sigma})$, and a mble map $T: X \to \tilde{X}$, any probability measure μ on X induces a measure $\tilde{\mu}$ on $\tilde{X}$ defined

$$\tilde{\mu}(\tilde{B}) := \mu(T^{-1}(\tilde{B})) \qquad \forall\, \tilde{B} \in \tilde{\Sigma}$$

LEMMA: (COV) $\displaystyle\int_{\tilde{X}} \tilde{f}(\tilde{x})\, d\tilde{\mu}(\tilde{x}) = \int_{X} \tilde{f}(T(x))\, d\mu(x)$

THM: (Brenier-McCann) Given $u_0, u_1 \in$ on $\mathbb{R}^n$ $\exists!\ T: \mathbb{R}^n \to$ $T_\# \mu_0 = u_1$ s.t. $T = \nabla \phi$ for some $\phi: \mathbb{R}^n \to \mathbb{R}$ convex

e.g. n=1 T nondecreasing and $\displaystyle\int_{-\infty}^{x} u_0 = \int_{-\infty}^{T(x)} u_1$

$$u_0(x) = T'(x)\, u_1(T(x)).$$

CLAIM: $\displaystyle d_2^2(u_0, u_1) = \int_{\mathbb{R}^n} u_0(x)\, |T(x) - x|^2$

Proof: $\leq$ take $\gamma_T = (id \times T)_\# u_0$

check $\gamma_T \in \Gamma(u_0, u_1)$ using COV and compute $\int |x-y|^2 d\gamma$

i.e. $\displaystyle\int_{\mathbb{R}^n \times \mathbb{R}^n} f(x,y)\, d\gamma_T(x,y) = \int_{\mathbb{R}^n} f(x, T(x))\, u_0(x)$

$$= \begin{cases} \int |y-x|^2 u_0 & \text{if } f(x,y) = |y-x|^2 \\ \int f\, u_0 & \text{if } f(x,y) = f(x) \\ \int f(T(x))\, u_0(x) = \int f \cdot T_\# u_0 = \int f u_1 & \text{if } f(x,y) = f(y) \end{cases}$$

Legendre Fenchel

Given any $\phi : \mathbb{R}^n \to \mathbb{R}$ $\qquad \phi^*(y) := \sup_{x \in \mathbb{R}^n} x\cdot y - \phi(x)$

if ϕ convex and $y = \nabla\phi(x) \Rightarrow \phi^*(\nabla\phi(x)) = x\cdot\nabla\phi(x) - \phi(x)$

$\forall \phi, x, y \qquad x\cdot y \le \phi(x) + \phi^*(y)$

$\forall \gamma \in \Gamma(\mu_0, \mu_1) \qquad \sup_\gamma \int x\cdot y \, d\gamma(x,y) \le \inf_\phi \int \phi\, \mu_0 + \int \phi^* \mu_1$

Recovering γ_T, $T = \nabla\phi$ $\qquad \int x\cdot y \, d\gamma_T = \int x\cdot\nabla\phi(x)\, \mu_0$

$$= \int \phi(x) + \phi^*(\nabla\phi(x))\, d\mu_0$$

$$= \int \phi\, \mu_0 + \phi^*\, \mu_1$$

$$\inf_{\gamma\in\Gamma(\mu_0,\mu_1)} \int |x-y|^2 \, d\gamma = \inf \int |x|^2 - 2x\cdot y + |y|^2$$

$$= R_0^2 + R_1^2 - 2\sup_{\gamma\in\Gamma(\mu_0,\mu_1)} \int x\cdot y \, d\gamma$$

Displacement interpolation: $\mu_s := F_s \# \gamma_T \qquad F_s(x,y) = (1-s)x + sy$

$$= F_s \# (\mathrm{id}\times T)_\# \mu_0$$

$$= [(1-s)x + sT(x)]_\# \mu_0$$

Claim μ_s is a minimal geodesic segment.

Pf: $d_2(\mu_0,\mu_s) \le s\sqrt{\int |T-x|^2 \, d\mu_0}$

$d_2(\mu_s,\mu_1) \le (1-s)\sqrt{\int |T-x|^2 \, d\mu_0}$

$\Rightarrow d_2(\mu_0,\mu_s) + d_2(\mu_s,\mu_1) \le d_2(\mu_0,\mu_1)$

triangle inequality $\Rightarrow$ [illegible] equality directly.

Proof 2: Along any curve $u(\lambda)$ with tangent $\chi(\lambda_0)$ in our case

$$\frac{d}{d\lambda} E(u(\lambda)) = \int_{\mathbb{R}^n} \frac{\delta E}{\delta u} \frac{\partial u}{\partial \lambda} = \int \frac{m}{m-1} u^{m-1} \frac{\partial u}{\partial \lambda}$$

$$= - \int_{\mathbb{R}} \frac{\delta E}{\delta u} \operatorname{div}[u \nabla \psi]$$

$$= \int_{\mathbb{R}^n} u \nabla \frac{\delta E}{\delta u} \cdot \nabla \psi \, dx$$

$= \text{iff } \frac{\delta E}{\delta u} = -c\psi$ $\quad \geq - \| \frac{\delta E}{\delta u} \|_{W^{1,2}_u} \| \psi \|_{W^{1,2}_u}$

$$= -c^2 \|\chi\|^2_{W^{1,2}_u}$$

Also thanks for a gradient flow $\frac{d}{dt} E(u(t)) = \langle \operatorname{grad}_u E \; \dot{u}(t) \rangle$
$= - |\operatorname{grad}_u E|^2$

$\psi \Leftrightarrow \operatorname{grad}_u E$ then $c = 1$ and $\psi = \frac{\delta E}{\delta u}$

After Brenier-McCann, before proof.

Monge-Ampere eq.

$$\int u_1(y) f(y) = \int u_0(x) f(\nabla\phi(x)) \, dx \quad \forall f \Rightarrow u_0(x) = u_1(\nabla\phi(x)) \det D^2\phi$$

$$= \int u_1(\nabla\phi(x)) f(\nabla\phi(x)) \det D^2\phi(x) \, dx$$

Also $u_0(x) = \det((1-s) I + s D^2\phi(x)) \; u_s((1-s)x + s\nabla\phi(x))$

$\frac{d}{ds}\Big|_{s=0} \Rightarrow 0 = \operatorname{tr}(D^2\phi(x) - I) u_0(x) + \nabla u_0 \cdot (\nabla\phi - x) + \frac{\partial u_s}{\partial s}$

$= \Delta_x \psi \quad - \frac{\partial u}{\partial s} \quad \psi = \phi(x) - \frac{1}{2}x^2$

Nonlinear Heat Flow IV: Steepest Descent into a Convex Valley
Nov 6 2002

RECAP

(*) $\frac{\partial u}{\partial t} = \Delta(u^m)$

$u(0,x) = u_0(x)$

$\in \mathcal{M}_{1,0}(\mathbb{R}^n) = \{u \in \mathcal{M}(\mathbb{R}^n) ; \ 1 = \int u \quad 0 = \int xu \quad \int x^2 u < \infty\}$

(1) a Riemannian manifold whose geodesic distance is given by

$d_2(u,v)^2 = \int_{\mathbb{R}^n} |x - \nabla\phi(x)|^2 u(x)$ where ϕ is convex

and $\nabla\phi : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $\nabla\phi_{\#} u = v$ (here $u \ll H^n$

i.e. $u(x) = \det D^2\phi(x) \, v(\nabla\phi(x))$ Monge-Ampère
formally
rigorously a.e.!

(2) evolution (*) is steepest descent of the entropy wrt d_2

$E(u) = \frac{1}{m-1}\int u^m(x) - u(x)$ More generally $\frac{\partial u}{\partial t} = \mathrm{div}\left[u \nabla \frac{\delta E}{\delta u}\right]$

(3) $m \geq 1 - \frac{1}{n} \Rightarrow$ geodesic convexity $E(u_s) \leq (1-s)E(u_0) + sE(u_1)$
i.e. d/ds^2
whenever $u_s := [(1-s)x + s\nabla\phi]_{\#} u$
with ϕ convex

$\Rightarrow \int g(z)\, du_s(z) = \int g((1-s)x + s\nabla\phi(x))\, u_0(x)$

$= \int g((1-s)x + s\nabla\phi(x)) \, \underbrace{u_s((1-s)x + s\nabla\phi(x)) \ \det [(1-s)I + sD^2\phi(x)]}_{u_0(x)} \, dx$

for g arbitrary.

I Local v. Global metric

RECALL: ① was based on tangent vector ψ to any curve $u(t,x)$ having length

$$\|\psi\|_{W^{1,2}_u} = \left(\int_{\mathbb{R}^n} |D\psi(x)|^2 u(x)\, dx \right)^{1/2}$$

Here $\frac{\partial u}{\partial t} + \operatorname{div}[u D\psi]$

Connection to $d_2(u,v)$? Consider our geodesic curves

a.e.
$$u_s((1-s)x + s D\phi(x)) \det((1-s)I + sD^2\phi(x)) = u(x)$$

$\frac{d}{ds}\Big|_{s=0}$
$$\rightarrow \frac{\partial u}{\partial s}\Big|_{s=0}(x) + \nabla u_0 \cdot (D\phi(x) - x) + u_0(x)\, \operatorname{tr}(D^2\phi - I) = 0$$

$$\frac{\partial u}{\partial s}(0,x) + \operatorname{div}\left[u_0(0,x)\, D\left(\phi - \frac{|x|^2}{2}\right)\right] = 0$$

"tangent vector" $\psi(x) = \phi(x) - \frac{1}{2}|x|^2$

$$d_2^2(u_0,u_1) = \int |D\psi(x)|^2\, du_0(x) = \|\psi\|^2_{W^{1,2}_u}$$

II. Displacement Convexity:

$$E(u_s) = \frac{1}{m-1}\int_{\mathbb{R}^n} u_s(z)^m\, dz = \frac{1}{m-1}\int_{\mathbb{R}^n} u_s((1-s)x + sT(x))^m \det((1-s)x + sDT(x))\, dx$$

$$= \frac{1}{m-1}\int_{\mathbb{R}^n} \left(\frac{u_0(x)}{\det((1-s)x + sDT(x))}\right)^m \det \cdots\, dx$$

$$= \frac{1}{m-1}\int_{\mathbb{R}^n} u_0(x) \det^{1-m}((1-s)I_n + sDT(x))\, dx$$

Facts: ① $\det^{1/n}((1-s)I + sB)$ is concave of s provided $B^t = B \geq 0$.

② $\perp$ Sign not determines convexity

③ Brenier's Theorem $DT(x) = D^2\phi(x) \geq 0$

④ arithmetic geometric mean

Proof: CLAIM: $\left(\prod_{i=1}^{n}(1-s)\lambda + s b_i\right)^{1/n} \geq (1-s) + s \left(\prod b_i\right)^{1/n}$

Proof: $(1-s)\left(\prod^{n} \frac{1}{(1-s)\lambda + s b_i}\right)^{1/n} + s\left(\prod^{n} \frac{b_i}{(1-s)\lambda + s b_i}\right)$

$\leq (1-s) \frac{1}{n} \sum^{n} \frac{1}{(1-s) + s b_i} + \frac{s}{n} \sum \frac{b_i}{(1-s) + s b_i}$

$= 1$ $\square$

III Rescaled
Villani

$u(t,x) = \frac{1}{(1+\alpha t)^{n/2}} V\left(\log(1+\alpha t)^{1/\alpha}, \frac{x}{(1+\alpha t)^{1/\alpha}}\right)$

$u_0(y) = v(0, \cdot)$

$\frac{\partial u}{\partial t} = \Delta(u^m) \Leftrightarrow \frac{\partial v}{\partial \tau} = \Delta(v^m) + \operatorname{div}[y\, v(\tau, y)]$

gradient flow of $F(v) = E(v) + \frac{1}{2}\int_{\mathbb{R}^n} y^2 v(y)$
wrt d_2

$\frac{\delta F}{\delta v} = \frac{\delta E}{\delta v} + \frac{1}{2} y^2$

$\nabla \frac{\delta F}{\delta v} = \nabla \frac{\delta E}{\delta v} + y$

Convexity of F? $F(v_s) \leq$ ~~$(1-s) F(v_0) + s E(v_1)$~~ $+ \frac{1}{2}\int |(1-s)y + s \nabla\phi(y)|^2 v_0|$

$\leq (1-s) F(v_0)$

$F(v_s) = E(v_s) + \frac{1}{2}\int |y + s \nabla\alpha(y)|^2 v_0(y)\, dy$

$\frac{d^2 F}{ds^2}(v_s) = \underbrace{\frac{d^2 E}{ds^2}}_{\text{not unite} \atop \text{this} \ \text{Sobolev vectors}} + \frac{1}{2}\int |\nabla\alpha(y)|^2 v_0(y)\, dy \geq 0 + d_2^2(v_0, v_1)$

uniform $\Rightarrow d_2(v_\tau, \tilde{v}_\tau) \leq e^{-\tau} d_2(v_0, \tilde{v}_0) = \frac{d_2(v_0, \tilde{v}_0)}{(1+\alpha t)^{1/2}}$

$\Leftarrow d_2(u_t, \tilde{u}_t) \leq$ doesn't change ~~[illegible]~~

HWI: 3 Basic Consequences of Uniform Convexity

LEMMA: If $f:\mathbb{R}\to[0,\infty)$ satisfies $f(0)=0$ $\quad f''(x)\geq\lambda$
then

Parabola from below

$$\frac{\lambda}{2}|w|^2 \underset{①}{\leq} f(w) \underset{②}{\leq} \frac{1}{2\lambda}|f'(w)|^2 \qquad \text{log Sobolev}$$

and ③ $$f(w)\leq |w|\,|f'(w)| - \lambda\frac{w^2}{2} \qquad \text{HWI}$$

Proof: ① Let $g(w)=f(w)-\lambda\frac{w^2}{2}$. Then $g(w)$ is convex and $g'(0)=0$, whence $g(w)\geq g(0)=0$.

e.g. $$g'(w)=f'(w)-\lambda w$$
$$g''(w)=f''(w)-\lambda \geq 0$$

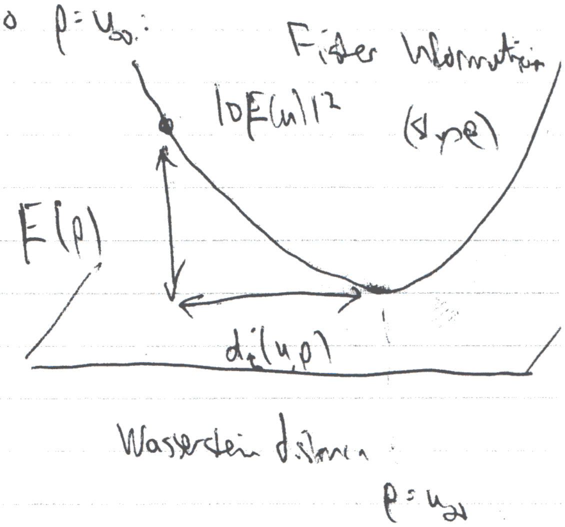
② Let $g(w)=f'(w)^2-2\lambda f(w)$. Then $g(w)$ has only one critical point and it is a local (hence global) minimum: $g(w)\geq g(0)=0$.

e.g. $$g'(w)=2f'(w)f''(w)-2\lambda f'(w)$$
$$=2f'(w)(f''(w)-\lambda) = 0 \Leftrightarrow w=0.$$
$$g''(0)=2f''(0)(f''(0)-\lambda)+2f'(0)\times(f'''(0))$$
$$\geq 2\lambda(f''(0)-\lambda)+0$$

③ Let $g_{\pm}(w)=\pm w f'(w)-\lambda\frac{w^2}{2}-f(w)$. Then $g_+(w)$ has a unique critical point: namely a local (hence global) minimum at $g_+(w)\geq g_+(0)=0$. ~~Also $g_+(-w)=-wf'(-w)-\lambda\frac{w^2}{2}-f(-w)$~~

e.g. $$g_+'(w)=f'(w)+wf''(w)-\lambda w-f'(w)=0 \Leftrightarrow w=0$$
$$g_+''(w)=f''(w)+wf'''(w)-\lambda\Big|_{w=0}=f''(0)-\lambda>0.$$

If only concerned about distance to $\rho = u_\infty$:
3 senses of convergence



Relative Entropy $\quad E(u) - E(\rho)$

Relationships
$\text{Hess } E \geq \lambda \Rightarrow \quad \frac{\lambda}{2} d_2^2(u, u_\infty) \leq E(u) - E(u_\infty) \leq \frac{1}{2\lambda} |\nabla E(u)|^2$

Talagrand's transportation $\neq$ $\qquad$ relative log Sobolev $\neq$

e.g. $\quad E(u) = \int u \log u + \frac{1}{2}\int x^2 u \qquad u_\infty = \rho = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$\lambda = 1 \qquad |\nabla E(u)|^2 = \int u \left|\nabla \frac{\delta E}{\delta \rho}\right|^2 = \int u \left|\nabla\left\{(\log u + 1) + \frac{1}{2}x^2\right\}\right|^2$

let $u = f u_\infty \qquad = \int \rho f \left|\nabla\left\{\log f + \log \rho + \frac{1}{2}x^2\right\}\right|^2$

$= \int \rho f \, |\nabla \log f|^2$

$E(f\rho) - E(\rho) = \int \rho f\left(\log f + \log \rho + \frac{1}{2}x^2\right) - \rho \log \rho - \frac{1}{2}x^2$

$= \int \rho \left\{ f \log f - \cancel{\frac{f x^2}{2}} - f \log\sqrt{2\pi} + \cancel{\frac{f x^2}{2}} + \cancel{\frac{\rho x^2}{2}} - \rho \log\sqrt{2\pi} \ldots \right.$

$= \int \rho f \log f$

$\therefore \qquad \leq \frac{1}{2} \int \rho \frac{|\nabla f|^2}{f}$

let $f = g^2$

$\int_{\mathbb{R}^n} e^{-x^2/2} g^2 \log g^2 \leq 2 \int \rho \frac{g^2}{g^2} |\nabla g|^2 = 2 \int \frac{e^{-x^2/2}}{\sqrt{2\pi}} |\nabla$

Bakry-Emery: Convergence to minimum only. NO GLOBAL CONTRACTION RESULT.

LEMMA: $D^2 V \geq k \Rightarrow |\nabla V(x(t))| \leq e^{-kt} |\nabla V(x(0))|$

COROLLARY: $V(x(t)) - V_{min} \leq |\nabla V(x(0))|^2 \frac{e^{-2kt}}{2k}$

Proof of Lemma: Let $f(t) = V(x(t))$ and $g(t) = |\nabla V(x(t))|^2$ then

$$f'(t) = \langle \nabla V(x(t)), \dot{x}(t) \rangle$$
$$= -|\nabla V(x(t))|^2 = -g(t)$$

$$f''(t) = -2 \langle \nabla V(x(t)), D^2 V(x(t)) \dot{x}(t) \rangle$$
$$= +2 \langle \nabla V, D^2 V \nabla V \rangle$$
$$\geq 2k |\nabla V(x(t))|^2 = 2k g(t)$$

$\therefore \quad -g'(t) \geq 2k g(t)$

$\Rightarrow \quad g(t) \leq g(0) e^{-2kt}.$ $\square$

Proof 1 of Corollary: Apply log Sobolev $V(x) - V_{min} \leq \frac{|\nabla V(x)|^2}{2k}$

$$\leq g(0) \frac{e^{-2kt}}{2k} \quad]$$

Proof 2 of Corollary: $g(t) \to 0$ as $t \to \infty$ whence $x(t) \to x_\infty$ crit pt

$$-f'(t) = g(t) \leq g(0) e^{-2kt}$$

$$\Rightarrow -f(t)\Big|_T^\infty \leq g(0) \frac{e^{-2kt}}{-2k}\Big|_T^\infty$$

$$f(T) - f(\infty) \leq \frac{g(0)}{2k} e^{-2kT}$$

$$V(x(T)) - V(x_\infty)$$

$\square$