Oct 28, 2003

review:

$$\min_{u(x)+v(y)\le c(x,y)} \int c\, d\gamma \geq \max_{u(x)+v(y)\le c(x,y)} \int u\, d\mu + \int v\, d\nu$$

- holds in large generality. (in fact we expect =)
- Sufficient: c - lower semi continuous,
  $\exists\, u_0, v_0$ Borel st $\int u_0\, d\mu > -\infty$
  $\int v_0\, d\nu > -\infty$
  $u_0(x) + v_0(y) \le c(x,y)$

equivalently

$$\min_{\gamma\in\Gamma(\mu,\nu)} \int c(x,y) - u_0(x) - v_0(y)\, d\gamma(x \times y) \overset{?}{<} \infty$$

Assume

① $c(x,y) = h(x-y)$ h - strictly convex

& for simplicity ② $\operatorname{spt}\mu \subseteq X$ - compact
$\operatorname{spt}\nu \subseteq Y$ - compact

③ $\mu \ll H^n$ (ie μ is a.c. wrt Lebesgue meas.)

Last time we checked:

0) $u_c(y) = \inf_{x\in X} c(x,y) - u(x)$ is Lipschitz

$$\|Du_c\|_\infty \le \|Dh\|_{L^\infty(X-Y)}$$

1) maximum is attained by a pair $(u,v) = (v_c, u_c)$

2) $c(x,y) - u(x) - u_c(y) \geq 0$ on $X \times Y$

Moreover if $x \in \operatorname{dom} \nabla u$, then equality holds iff $\nabla h(x-y) - \nabla u(x) = 0$

i.e. $y = F_u(x) = x - \nabla h^{-1}(\nabla u(x))$

3) (Thm) If $(u,v) = (v_c, u_c)$ maximizes and $\mu << \mathcal{H}^n$ then $F_u : \operatorname{dom} \nabla u \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$(F_u)_{\#}\mu = \nu.$$

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today
optimality
4) If $(F_u)_{\#}\mu = \nu$ and $u = u_{cc}$

let $\gamma = (id \times F_u)_{\#}\mu$ minimizes Kantorovich problem then F_u solves Monge's problem and duality holds $I = J$

5) Uniqueness: Monges problem has only one sol.

6) If $F_u \# \mu = F_{\tilde{u}} \# \mu$ with $u = u_{cc}$, $\tilde{u} = \tilde{u}_{cc}$ then $F_u = F_{\tilde{u}}$

Proof of 4) Given $\gamma \in \Pi(\mu,\nu)$ and $u = u_{cc} \not\equiv -\infty$

$$\underset{\text{Lip}}{\underset{\Uparrow}{c(x,y)}} \geq \underset{\substack{\text{Lip} \\ \text{since } u = u_{cc}}}{\underset{\Uparrow}{u(x)}} + \underset{\text{Lip}}{\underset{\Uparrow}{u_c(y)}} \quad \text{on } X \times Y$$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} c \, d\gamma \geq \int_{\mathbb{R}^n} u \, d\mu + \int_{\mathbb{R}^n} u_c \, d\nu$$

Take inf then sup

$$I = \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} c\, d\gamma \geq \sup_{\substack{u: X \to \mathbb{R}^n \\ u_{cc} = u}} \int_{\mathbb{R}^n} u\, d\mu + \int_{\mathbb{R}^n} u_c\, d\nu = \tilde{J}$$

taking $u = u_{cc}$ from 3) $\Rightarrow (F_u)\# \mu = \nu$
(also $\mu(\mathrm{dom}\, \nabla u) = 1$, $\mu(\mathrm{dom}\, F_u) = 1$) $\Rightarrow \gamma_u := (id \times F_u)\# \mu \in \Gamma(\mu,\nu)$

$I \leq$ Moreover $c(x, F_u(x)) = u(x) + u_c(F_u(x))$ on $\mathrm{dom}\, \nabla u$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} c(x,y)\, d\gamma_u = \int_{\mathbb{R}^n} c(x, F_u(x))\, d\mu \geq \int_{\mathbb{R}^n} u\, d\mu + \int u_c(F_u(x))\, d\mu(x)$$

$$= \int u\, d\mu + \int u_c(y)\, d\nu(y) = \tilde{J}$$

so $I \geq \tilde{J}$ & $I \leq \tilde{J}$ hence $I = \tilde{J}$.
and $\geq$ & $\leq$ are equalities

So γ_u minimizes Kantorovich

$\therefore$ F_u minimizes Monge

also note $I \geq J \geq \tilde{J}$ so $J = I = \tilde{J}$.

<u>Proof of 5)</u>: Suppose $G\# \mu = \nu$ also minimizes Monges problem.
Then $\gamma = (id \times G)_\# \mu$ has the same cost as $\gamma_u = (id \times F_u)\# \mu$

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} c(x,y)\,d\gamma = \int_{\mathbb{R}^n} c(x,G(x))\,d\mu(x) = I = J$$

$$= \int u(x)\,d\mu + \int u_c(y)\,d\nu(y) =$$

$$= \int u(x)\,d\mu + \int u_c(G(x))\,d\mu(x)$$

$$\int_{\mathbb{R}^n} \left(c(x,G(x)) - u(x) - u_c(G(x)) \right) d\mu(x) = 0$$

$\therefore$ $c(x,G(x)) = u(x) + u_c(G(x))$ on set Z with $\mu(Z)=1$

also $\mu(\operatorname{dom} \nabla u) = 1$

Let $A = Z \cap \operatorname{dom} \nabla u$ $\quad \mu(A)=1$

For $x \in A$ lemma 2) $\Rightarrow$ $y = G(x) = F_u(x)$

$\therefore$ $G = F_u$ $\quad \mu$ - a.e. $\blacksquare$

<u>Proof of 6)</u> According to 4) both F_u and $F_{\tilde{u}}$ minimize Monges problem therefore $F_u = F_{\tilde{u}}$ μ-a.e. according to 5)

7) If $\nu \ll H^n$ then $F_u(F_{u_c}(y)) = y$ ν a.e.

and $F_{u_c}(F_u(x)) = x$ μ a.e.

i.e. for $\forall y \in (\operatorname{dom} \nabla u_c) \cap F_{u_c}^{-1}(\operatorname{dom} \nabla u)$

$\forall x \in \operatorname{dom} \nabla u \cap F_u^{-1}(\operatorname{dom} \nabla u_c)$

$$\nu(\mathrm{dom}\, Du_c) = 1$$

Proof of 7): $F_u \# \mu = \nu \Rightarrow \mu(F_u^{-1}(\mathrm{dom}\, Du_c)) = 1$

$$\mu(\mathrm{dom}\, Du \cap F_u^{-1}(\mathrm{dom}\, Du_c)) = 1$$

for such x we have $c(x, F_u(x)) \overset{\text{lemma}}{=} u(x) + u_c(F_u(x))$

$$= u_{cc}(x) + u_c(F_u(x))$$

Lemma 2 is other direction $\Rightarrow$ $x = F_{u_c}(F_u(x))$

$$x = F_{u_c}(y)$$

the other ~~equality~~ follows by symmetry as soon as we deduce $(F_{u_c})_\# \nu = \mu$ from $(F_u)_\# \mu = \nu$. We use test fns ξ

$\xi \in C(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \xi(z)\, d(F_{u_c} \# \nu)(z) = \int \xi(F_{u_c}(y))\, d\nu(y) \quad \nu = F_u \# \mu$$

$$= \int \xi(F_{u_c}(F_u(x)))\, d\mu \quad \text{by } \not{b}$$

Riesz $\Rightarrow$ $(F_{u_c})_\# \nu = \mu$. ∎

Structure of $c(x,y)$ determines uniqueness or non uniqueness of optimal γ.

Does the same structural condition determine generic uniqueness (non-uniqueness) of $\gamma \in \Gamma\left(\frac{1}{N}\sum^{N}\delta_{x_i}, \frac{1}{N}\sum^{N}\delta_{y_i}\right)$ x_i, y_i - random loc.

Rmks:

① $c(x,y) = \frac{1}{2}|x-y|^2 =: h(x-y)$ $\qquad \nabla h(x) = x$

$$F_u(x) = x - \nabla h^{-1}(\nabla u(x)) = x - \nabla u(x)$$
$$= \nabla\left(\frac{1}{2}|x|^2 - u\right)$$

$$u = u_{cc}(x) = \inf_{y\in Y} \frac{1}{2}|x-y|^2 - u_c(y)$$

$$\Rightarrow u(x) - \frac{1}{2}|x|^2 = \inf_{y\in Y} -x\cdot y + \frac{1}{2}|y|^2 - u_c(y)$$

$$\Rightarrow \frac{1}{2}|x|^2 - u(x) = \sup_{y\in Y} x\cdot y - \left(\frac{1}{2}|y|^2 - u_c(y)\right)$$

is convex $= \left(\frac{1}{2}|y|^2 - u_c\right)_c$

Flexibility to choose $X \supset \operatorname{spt}\mu$, $Y \supset \operatorname{spt}\nu$

small $Y = \operatorname{spt}\nu \Rightarrow$ gives more information about possible slopes of u

better for existence

For uniqueness it is better that the set of fns where $u_{cc} = u$ is large.

Thus choose Y large

It follows as before that $u_c(y) = \inf_{x \in X} c(x,y) - u(x)$ is also 1-Lipschitz (wrt c)

Choosing $X = Y \Rightarrow$

$$u_c(y) = \inf_{x \in X} c(x,y) - u(x) \leq c(y,y) - u(y) \leq -u(y)$$

On the other hand $u(x) \geq u(z) - c(x,z) \quad \forall z$

$$-u(x) \leq c(z,x) - u(z) \quad \forall z$$

$$-u(x) \leq \inf_{z \in Y} c(z,x) - u(z) = u_c(x)$$

So $\boxed{u_c(x) = -u(x)}$ on $X = Y$

$$\sup_{u(x)+v(y) \leq c(x,y)} \int u(x)\,d\mu(x) + \int v(x)\,d\nu(x) =$$

$$= \sup_{\substack{u = u_{cc} \\ u - 1 \text{ Lip wrt } c}} \int u(x)\,d\mu(x) + \int u_c(y)\,d\nu(y)$$

$$= \sup_{\substack{u\text{-1 Lip} \\ \text{wrt } c}} \int u(x)\,d\mu(x) - \int u(y)\,d\nu(y) =$$

$$= \sup_{u \in \mathrm{Lip}_1(c)} \int u(x)\,d\mu(x) - d\nu(x)$$

if $c(x,y) = |x-y|$ $\quad = \inf_{\operatorname{div} \vec{w} = \mu - \nu} \int |\vec{w}|^{1}$

related to Beckmann problem

(3) Monge-Ampere equation

formally if $d\mu(x) = f(x)\,d^n x$

$d\nu(y) = g(y)\,d^n y$

then $f(x) = |\det DF_u(x)|\, g(F_u(x))$

follows from $F_u \# \mu = \nu$ & $u_{cc} = u$.

- Formal argument. Let $\xi \in C(\mathbb{R}^n)$

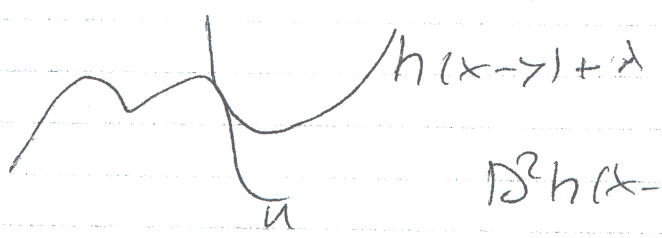
$$\int \xi(F_u(x)) f(x)\, d^n x = \int \xi(y) g(y)\, dy$$

if $y = F_u(x)$ is a C^1-diffeo or Lipschitz

$$= \int \xi(F_u(x))\, g(F_u(x))\, |\det DF_u(x)|\, dx$$

so $f(x) = g(F_u(x)) \, |\det DF_u(x)|$

$c(x,y) = h(x-y)$ ← strictly convex



$$D^2 h(x - F_u(x)) \geq D^2 u(x)$$

$$F_u(x) = x - Dh^{-1}(Du(x))$$

$$DF_u(x) = I - D^2 h^{-1}\big|_{Du(x)} D^2 u(x)$$

$$= D^2 h^*\big|_{Du(x)} \left(D^2 h\big|_{x - F_u(x)} - D^2 u(x) \right)$$ – product of non neg symm matrices

so $f(x) = \det DF_u(x)\ g(F_u(x))$

formally

$$f(x) = \det D^2h^*(Du(x))\ \det D^2h(Dh^{-1}(Du(x)) - D^2u(x)) \cdot g(x - Dh^{-1}(Du(x)))$$

an example of fully nonlinear 2-nd order deg. elliptic PDE.

Few results: Trudinger & Wang $c(x,y) = \log(1 \pm x \cdot y)$ or $\sqrt{1+|x-y|^2}$

Monge-Ampere when $c(x,y) = |x-y|^2$

$$f(x) = \det D^2\left(\tfrac{1}{2}|x|^2 - u(x)\right)\ g\left(D\left(\tfrac{1}{2}|x|^2 - u\right)\right)$$